Color Degree Sum Conditions for Properly Colored Spanning Trees in Edge-Colored Graphs

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Abstract

For a vertex v of an edge-colored graph, the color degree of v is the number of colors appeared in edges incident with v. An edge-colored graph is called properly colored if no two adjacent edges have the same color. In this paper, we prove that if the minimum color degree sum of two adjacent vertices of an edge-colored connected graph G is at least $|G|$, then G has a properly colored spanning tree. This is a generalization of the result proved by Cheng, Kano and Wang. We also show the sharpness of this lower bound of the color degree sum.

Keywords: edge-colored graph, spanning tree, properly colored, rainbow

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1 Introduction

In this paper, we consider finite graphs which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For two vertices x and y of G, an edge joining them is denoted by xy or yx. A *star* is a complete bipartite graph $K_{1,m}$ for some $m \geq 1$, and the vertex of a star with degree m is called its *center*.

In this paper, we deal with edge-colored graphs, in which two adjacent edges may have the same color. Let G be an edge-colored graph. For an edge e of G, let $col(e)$ denote the color of e. For a vertex v of G, the *color degree* of v, denoted by $\deg_G^c(v)$, is the number of colors appeared in edges incident with v. The *minimum color degree* of G is defined by

$$\delta^{col}(G) = \min\{\deg_G^c(v) : v \in V(G)\}.$$

An edge-colored graph G is called *properly colored* if no two adjacent edges of G have the same color. Moreover, G is called *rainbow* or *heterochromatic* if no two edges of G have the same color. Cheng, Kano and Wang [2] gave a minimum color degree condition for an edge-colored graph to have a properly colored spanning tree.

Theorem 1 (Cheng, Kano and Wang [2])**.** *Let G be an edge-colored connected graph. If $\delta^{col}(G) \geq |G|/2$, then G has a properly colored spanning tree.*

In order to prove Theorem 1, they proved the following theorem.

Theorem 2 (Cheng, Kano and Wang [2])**.** *Let G be an edge-colored connected graph. Suppose that for each color c, the set of edges colored with c forms a star. If $\delta^{col}(G) \geq |G|/2$, then G has a rainbow spanning tree.*

They also showed the sharpness of these lower bounds of the minimum color degree.

In this paper, we generalize the above two theorems by using a *minimum color degree sum*. Let

$$\bar{\sigma}_2^{col}(G) = \min\{\deg_G^c(u) + \deg_G^c(v) : u, v \in V(G) \text{ and } uv \in E(G)\}.$$

Note that for a graph, which is not an edge-colored graph, the minimum degree sum of two non-adjacent vertices is often used. However, here we consider the minimum color degree sum of two adjacent vertices. We will explain later the reason why we do not use the minimum color degree sum condition of two non-adjacent vertices.

Li proved the following theorem, in which a rainbow triangle is the same as a properly colored triangle.

Theorem 3 (Li [4]). *Let G be an edge-colored connected graph. If $\delta^{col}(G) \geq (|G|+1)/2$, then G has a rainbow triangle.*

Li, Ning and Zhang generalized the above theorem by using the minimum color degree sum $\bar{\sigma}_2^{col}(G)$.

Theorem 4 (Li, Ning and Zhang [5]). *Let G be an edge-colored connected graph. If $\bar{\sigma}_2^{col}(G) \geq |G|+1$, then G has a rainbow triangle.*

Let us remark that Ore [6] generalized Dirac's minimum degree condition for a graph to have a hamiltonian cycle [3] to the condition of minimum degree sum of two non-adjacent vertices. Therefore one can think that it is natural to use the minimum color degree sum of two non-adjacent vertices for edge-colored graphs. However the following example given in [5] shows that we can not generalize Theorem 3 by using color degree sum of two non-adjacent vertices. Let k and n be integers such that $(n+1)/2 \leq k \leq n-2$. Let G_1 be the edge-colored connected graph such that $V(G_1) = \{v_1, v_2, \ldots, v_n\}$, $E(G_1) = \{v_iv_j : 1 \leq i < j \leq n, 1 \leq i \leq k\}$ and $col(v_iv_j) = \min\{i, j\}$. Then for any two non-adjacent vertices u and v, $\deg^c_{G_1}(u) + \deg^c_{G_1}(v) = 2k \geq n+1$. On the other hand, G_1 has no rainbow triangle. Hence the condition of minimum color degree sum of two non-adjacent vertices is not available.

In this paper, we generalize Theorems 1 and 2 by using $\bar{\sigma}_2^{col}(G)$ as follows.

Theorem 5. *Let G be an edge-colored connected graph. If $\bar{\sigma}_2^{col}(G) \geq |G|$, then G has a properly colored spanning tree.*

Theorem 6. *Let G be an edge-colored connected graph. Suppose that for each color c, the set of edges colored with c forms a star. If $\bar{\sigma}_2^{col}(G) \geq |G|$, then G has a rainbow spanning tree.*

We remark that the condition of minimum color degree sum of two non-adjacent vertices is not useful. Let G_1 be the edge-colored connected graph of order n defined above. Suppose that there exists a properly colored spanning tree T in G_1. Then each color is chosen at most once in T, in particular, T must be a rainbow spanning tree of G_1. On the other hand, G_1 has at most $n-2$ colors. Thus T does not exist. On the other hand, the color degree sum of two non-adjacent vertices is at least n as explained before. Therefore the condition of minimum color degree sum of two non-adjacent vertices is not available.

Since the lower bound of the minimum color degree sum in Theorems 1 and 2 are best possible, we can see that the lower bound of the color degree sum conditions in Theorems 5 and 6 are also best possible. For the convenience of readers, we explain an example given in [2]. Let K_m be a rainbow complete graph with $V(K_m) = \{u_1, u_2, \ldots, u_m\}$, all of whose edges have distinct colors. Let $v_1, v_2, \ldots, v_{m+1}$ be $m+1$ new vertices not contained in $V(K_m)$. Let $c_1, c_2, \ldots, c_m$ be m new colors not appearing in K_m. We construct the edge-colored connected graph G_2 from K_m and $\{v_1, v_2, \ldots, v_{m+1}\}$ by adding a new edge u_iv_j colored by c_i for all $1 \le i \le m$ and $1 \le j \le m+1$. Then for each color c, the set of edges colored with c forms a star, and $\bar{\sigma}_2^{col}(G_2) = 2m = |G_2| - 1$. However, G_2 has no properly colored spanning tree.

In Section 2, we prove Theorem 6. Our proof is completely different from the proof of Theorem 2. In Section 3, we prove Theorem 5 by using Theorem 6.

2 Proof of Theorem 6

We denote by $\omega(G)$ the number of components of a graph G. Given an edge-colored graph G and a color set R, we define $E_R(G) = \{e \in E(G) : col(e) \in R\}$. For a set X, the cardinality of X is denoted by $|X|$ or $\#X$.

Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. For an arc uv in D, u is its initial vertex and v is its terminal vertex. For an arc uv of D, we define the *common outdegree* $d_D^+(uv)$ of uv (resp. the *common indegree* $d_D^-(uv)$ of uv) to be the number of common out-neighbors (resp. common in-neighbors) of u and v. Namely, for an arc uv of D, we define

$$d_D^+(uv) = \#\{x \in V(D) : ux, vx \in A(D)\},$$

$$d_D^-(uv) = \#\{x \in V(D) : xu, xv \in A(D)\}.$$

In this section, we prove Theorem 6. In order to prove Theorem 6, we need the following theorem and lemma.

Theorem 7 (Akbari and Alipour [1], Suzuki [7])**.** *An edge-colored connected graph G has a rainbow spanning tree if and only if for every color set R with $1 \le |R| \le |G| - 2$, $\omega(G - E_R(G)) \le |R| + 1$.*

Lemma 1. *For every digraph D, $\sum_{uv \in A(D)} d_D^+(uv) = \sum_{uv \in A(D)} d_D^-(uv)$.*

Proof. Let D be a digraph. Define

$$\begin{aligned} O &= \{(uv, x) : uv \in A(D),\, x \in V(D),\, ux, vx \in A(D)\}, \\ I &= \{(uv, x) : uv \in A(D),\, x \in V(D),\, xu, xv \in A(D)\}, \quad \text{and} \\ \triangle &= \{(x, y, z) : x, y, z \in V(D),\, xy, yz, xz \in A(D)\}. \end{aligned}$$

Then there exist a bijection from O to $\triangle$, and a bijection from I to $\triangle$. This implies that $|O| = |\triangle| = |I|$. On the other hand, by the definitions of $d_D^+(uv)$ and $d_D^-(uv)$, we can see that $\sum_{uv \in A(D)} d_D^+(uv) = |O|$ and $\sum_{uv \in A(D)} d_D^-(uv) = |I|$. Hence the desired equality holds. $\square$

Proof of Theorem 6. Assume that an edge-colored connected graph G satisfies the assumption of Theorem 6 but has no rainbow spanning tree. Then, by Theorem 7, there exists a color set R such that

$$1 \leq |R| \leq |G| - 2 \quad \text{and} \quad \omega(G - E_R(G)) \geq |R| + 2. \tag{1}$$

For every color $c \in R$, let $Star(c)$ denote the monochromatic star formed by the set of edges of G colored with c. For every $Star(c)$ isomorphic to $K_{1,1}$, we choose any vertex of it as its center, and fix it. Let us define $Cent(R)$ as

$$Cent(R) = \{x \in V(G) : x \text{ is the center of } Star(c) \text{ for some } c \in R\}.$$

The center of $Star(c)$ for some $c \in R$ is called a *color center*.

We first orient each edge of $E_R(G)$ from its color center to the other end-vertex. Then we define the digraph D as follows: the vertex set of D is $Cent(R)$ and the arc set of D is the set of oriented edges of $E_R(G)$ joining two vertices of $Cent(R)$ contained in distinct components of $G - E_R(G)$ (see Figure 1).

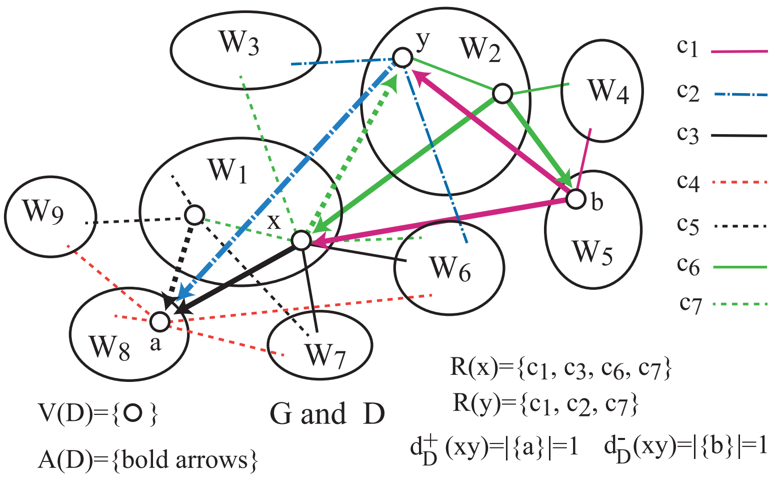

Figure 1: An edge-colored graph G, $R = \{c_1, c_2, \ldots, c_7\}$, $G - E_R(G)$ with components $W_1, W_2, \ldots, W_9$, and the digraph D with vertex set $V(D) = Cent(R) = \{\circ\}$ and arc set $A(D) = \{\text{bold arrows}\}$.

We consider the following two cases.

Case 1. $A(D) \neq \emptyset$.

By Lemma 1, there exists an arc $x_1x_2 \in A(D)$ such that

$$d_D^+(x_1x_2) \geq d_D^-(x_1x_2). \tag{2}$$

For $i = 1, 2$, let W_i be the component of $G - E_R(G)$ such that $x_i \in V(W_i)$, and let us define $R(x_i)$ as follows (see Figure 1):

$$R(x_i) = \{col(x_iu) : x_iu \in E(G) \text{ and } u \notin V(W_i)\}.$$

Then $R(x_i) \subseteq R$. It is easy to see that if $y \in V(D)$ satisfies $x_1y, x_2y \in A(D)$, then $R(x_1) \cup R(x_2)$ does not contain a color c such that the center of $Star(c)$ is y. This implies that

$$|R(x_1) \cup R(x_2)| \leq |R| - d_D^+(x_1x_2). \tag{3}$$

For every $c \in R(x_1) \cap R(x_2)$, it follows that either $c = col(x_1x_2)$ or there exists $y \in V(D)$ such that $yx_1, yx_2 \in A(D)$ and $c = col(yx_1) = col(yx_2)$. Hence we obtain,

$$|R(x_1) \cap R(x_2)| \leq d_D^-(x_1x_2) + 1. \tag{4}$$

By (1)–(4), we obtain,

$$\begin{aligned}
&\deg_G^c(x_1) + \deg_G^c(x_2) \\
&\leq (|W_1| - 1 + |R(x_1)|) + (|W_2| - 1 + |R(x_2)|) \\
&= |W_1| + |W_2| - 2 + |R(x_1) \cup R(x_2)| + |R(x_1) \cap R(x_2)| \\
&\leq |W_1| + |W_2| - 2 + (|R| - d_D^+(x_1x_2)) + (d_D^-(x_1x_2) + 1) \\
&\leq |W_1| + |W_2| - 1 + |R| \\
&\leq |W_1| + |W_2| + \big(\omega(G - E_R(G)) - 2\big) - 1 \\
&\leq |G| - 1.
\end{aligned}$$

This contradicts the color degree sum condition of Theorem 6.

Case 2. $A(D) = \emptyset$.

Since $\omega(G - E_R(G)) \geq 2$, there exists an edge $x_1x_2 \in E(G)$ such that $col(x_1x_2) \in R$ and x_1x_2 connects different components of $G - E_R(G)$. By symmetry and the assumtion of Case 2, we may assume that $x_1 \in Cent(R)$ and $x_2 \notin Cent(R)$. For $i = 1, 2$, let W_i be the component of $G - E_R(G)$ such that $x_i \in W_i$. Let

$$Col(x_1) = \{c \in R : x_1 \text{ is the center of } Star(c)\}.$$

By the assumption of Case 2, $\deg^c_G(x_1) \leq |W_1| - 1 + |Col(x_1)|$. Also, since x_2 is not incident with any edge of the color in $Col(x_1) \setminus \{col(x_1x_2)\}$, we have

$$\deg^c_G(x_2) \leq |W_2| - 1 + (|R| - |Col(x_1)| + 1) = |W_2| + |R| - |Col(x_1)|.$$

Hence, by (1), we have

$$\begin{aligned}
&\deg^c_G(x_1) + \deg^c_G(x_2) \\
&\leq (|W_1| - 1 + |Col(x_1)|) + (|W_2| + |R| - |Col(x_1)|) \\
&\leq |W_1| + |W_2| - 1 + |R| \\
&\leq |W_1| + |W_2| + \big(\omega(G - E_R(G)) - 2\big) - 1 \\
&\leq |G| - 1.
\end{aligned}$$

This is a contradiction. Therefore we complete the proof of Theorem 6. □

3 Proof of Theorem 5

In this section we prove Theorem 5. At first, we prepare the following lemma.

Lemma 2. *Let G be an edge-colored connected graph that satisfies $\bar{\sigma}_2^{col}(G) \geq |G|$. Suppose that G has a monochromatic path $(xyzu)$ of length 3, where $x, y, z, u \in V(G)$. Then $G - yz$ is connected and satisfies $\bar{\sigma}_2^{col}(G - yz) \geq |G|$.*

Proof. Suppose that $G - yz$ is not connected. Let H_y and H_z be components of $G - yz$ such that $y \in V(H_y)$ and $z \in V(H_z)$. Then $|G| \leq \bar{\sigma}_2^{col}(G) \leq \deg^c_G(y) + \deg^c_G(z) \leq |H_y| + |H_z| - 2 = |G| - 2$, a contradiction. Hence $G - yz$ is connected. It is easy to see that $G - yz$ satisfies $\bar{\sigma}_2^{col}(G - yz) \geq |G|$ since $col(xy) = col(yz) = col(zu)$. □

We are now ready to prove Theorem 5 in the same way as in the proof of Theorem 1.

Proof of Theorem 5. We prove Theorem 5 by induction on $|E(G)|$. Suppose that G satisfies the assumption of Theorem 5. By Lemma 2, we may assume that G has no monochromatic path of length 3 since otherwise we can apply the induction hypothesis to $G - yz$. Hence for each color c, the set of edges colored with c forms disjoint union of stars. We construct a new edge-colored graph G_3 from G by recoloring all monochromatic stars with distinct colors. Namely, for each monochromatic star of G, we recolor all its edges by a new color depending only on it, and denote the resulting edge-colored graph by G_3. Then G_3 satisfies the assumption of Theorem 6. Hence G_3 has a rainbow spanning tree T. By recoloring all the edges of T with their original colors in G, we obtain a properly colored spanning tree of G. □

Acknowledgment
The authors would like to thank the referees for their valuable comments and suggestions.

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