

Spanning trees with bounded degrees and leaves

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Abstract

Rivera-Campo provided a degree sum condition for a graph to have a spanning tree with bounded degrees and leaves. In this paper, we give an independence number condition for a graph to have a spanning tree with bounded degrees and leaves, which also partially solves the conjecture made by Enomoto and Ozeki (*J. Graph Theory*, 65, (2010) 173-184).

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G , let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G , respectively. We write $|G|$ for the *order* of G (i.e., $|G| = |V(G)|$). For a vertex v of G , we denote by $\deg_G(v)$ the degree of v and by $N_G(v)$ the neighborhood of v . Thus $\deg_G(v) = |N_G(v)|$. An edge joining two vertices u and v is denoted by uv or vu . The *independence number* and the *connectivity*

*This work was in part supported by JSPS KAKENHI(25400187)

†This work was in part supported by the NSFC(61273179), Young talent fund from Hubei EDU(Q20151311), the Yangtze Youth Fund(70107021) This work was done while the second author was in Ibaraki University.

of G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. Let T be a tree. A vertex of T with degree one is often called a *leaf*, and the set of leaves of T is denoted by $Leaf(T)$. For a set X , the cardinality of X is denoted by $|X|$ or $\#X$.

Chvátal and Erdős [2] showed that if $\alpha(G) \leq \kappa(G) + 1$, then G has a hamiltonian path. This result was generalized to a spanning k -tree as follows, where a k -tree is a tree with maximum degree at most k .

Theorem 1 (Neumann-Lara and Rivera-Campo [6]). *Let $k \geq 2$ be an integer and G be a connected graph. If $\alpha(G) \leq (k-1)\kappa(G)+1$, then G has a spanning k -tree.*

On the other hand, a hamiltonian path is a spanning tree having exactly two leaves. From this point of view, Win provided a sufficient condition for a graph to have a spanning tree having a small number of leaves.

Theorem 2 (Win [9]). *Let $k \geq 2$ be an integer and G be a connected graph. If $\alpha(G) \leq \kappa(G) + k - 1$, then G has a spanning tree having at most k leaves.*

Recently, Rivera-Campo obtained a degree sum condition for a graph to have a spanning tree with bounded degree as well as with a small number of leaves.

Theorem 3 (Rivera-Campo [7]). *Let p, n and d_1, d_2, \dots, d_p be integers such that $1 \leq n \leq p - 1$ and $2 \leq d_1 \leq d_2 \leq \dots \leq d_p \leq p - 1$. Let G be an n -connected graph of order p with vertex set $V(G) = \{v_1, v_2, \dots, v_p\}$. If*

$$\deg_G(x) + \deg_G(y) \geq p - 1 - \sum_{j=1}^n (d_j - 2)$$

for any non-adjacent vertices x and y of G , then G has a spanning tree T that has at most $\sum_{j=1}^n (d_j - 2) + 2$ leaves and satisfies $\deg_T(v_i) \leq d_i$ for all $i = 1, 2, \dots, p$.

For a function $f : V(G) \rightarrow \{1, 2, 3, \dots\}$, a spanning tree T of a graph G is called a *spanning f -tree* if $\deg_T(v) \leq f(v)$ for all vertices v of G . Here, we give a sufficient condition using independence number for a graph to have a spanning f -tree with a small number of leaves. The following is our result.

Theorem 4. *Let $n \geq 1$ be an integer. Let G be an n -connected graph and $f : V(G) \rightarrow \{2, 3, 4, \dots\}$ be a function. If*

$$\alpha(G) \leq \min_X \left\{ \sum_{x \in X} (f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n \right\} + 1, \quad (1)$$

then G has a spanning f -tree that has at most $\min_X \left\{ \sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n \right\} + 2$ leaves.

We now give some other results and a conjecture related to our theorem, and explain the relation between our theorem and them. The following theorem gives a sufficient condition for a graph to have a spanning tree that contains specified vertices as leaves.

Theorem 5 (Matsuda and Matsumura [5]). *Let n, k and s be integers such that $k \geq 2$, $0 \leq s \leq k$ and $s \leq n - 1$, and let G be an n -connected graph. If $\alpha(G) \leq (n - s)(k - 1) + 1$, then for any s vertices of G , G has a spanning k -tree that includes the s specified vertices as leaves.*

Enomoto and Ozeki made the following conjecture on a spanning f -tree from the above theorem, and partially solved it (Theorem 7).

Conjecture 6 (Enomoto and Ozeki [3]). *Let $n \geq 1$ be an integer, G be an n -connected graph and $f : V(G) \rightarrow \{1, 2, 3, \dots\}$ be a function. If $\sum_{x \in V(G)} f(x) \geq 2(|G| - 1)$ and*

$$\alpha(G) \leq \min_X \left\{ \sum_{x \in X} (f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n \right\} + 1,$$

then G has a spanning f -tree.

Theorem 7 (Enomoto and Ozeki [3]). *Let $n \geq 1$ be an integer, G be an n -connected graph and $f : V(G) \rightarrow \{1, 2, 3, \dots\}$ be a function. If $\#\{v \in V(G) : f(v) = 1 \text{ or } 2\} \leq n + 1$, $\sum_{x \in V(G)} f(x) \geq 2(|G| - 1)$ and*

$$\alpha(G) \leq \min_X \left\{ \sum_{x \in X} (f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n \right\} + 1,$$

then G has a spanning f -tree.

Note that Conjecture 6 is a generalization of Theorem 5 since by setting $f(u) = 1$ for s specified vertices u and $f(v) = k$ for the other vertices v , Conjecture 6 implies Theorem 5. Moreover, another conjecture related to Conjecture 6 was proposed by Ozeki and Yamashita [8] using a new notation $Cut(G, f)$ instead of the connectivity. Our Theorem 4 solves the conjecture under the assumption that $f(v) \geq 2$ for all vertices v , and the proof techniques in this paper are different from those in Enomoto and Ozeki [3].

Some other results on spanning trees related to our theorem are given in [1], and many current results on spanning trees can be found in [8].

2 Proof of Theorem 4

In order to prove Theorem 4, we need the following Lemmas.

Lemma 1 (Kouider [4]). *Let H be a subgraph of an n -connected graph G , where $n \geq 2$. Then either $V(H)$ is covered by a cycle of G , or there is a cycle C in G such that $\alpha(H - V(C)) \leq \alpha(H) - n$. In particular, if $\alpha(H) \leq n$, then $V(H)$ is covered by a cycle of G .*

By Lemma 1, we can easily obtain the following lemma.

Lemma 2. *Let H be a subgraph of an n -connected graph G , where $n \geq 1$. Then either $V(H)$ is covered by a path of G , or there is a path P in G such that $\alpha(H - V(P)) \leq \alpha(H) - (n + 1)$.*

Proof. Let w be a new vertex not contained in G , and let $G + w$ denote the graph obtained from G by joining w to every vertex of G . Then $G + w$ is $(n + 1)$ -connected and H is its subgraph. By Lemma 1, H is covered by a cycle D of $G + w$ or there exists a cycle C in $G + w$ such that $\alpha(H - V(C)) \leq \alpha(H) - (n + 1)$.

If D passes through w , then H is covered by the path $D - w$ of G ; otherwise, H is covered by a path obtained from D by removing an edge. If C passes through w , then the path $C - w$ of G satisfies the condition since $H - V(C - w) = H - V(C)$; otherwise, the path $C - e$ obtained from C by removing an edge e of C satisfies the condition since $H - V(C - e) = H - V(C)$. Hence the lemma holds. \square

The next lemma is well-known.

Lemma 3. *Let T be a tree and $W = \{v \in V(T) : \deg_T(v) \geq 2\}$. Then the number of leaves of T is $\sum_{x \in W} (\deg_T(x) - 2) + 2$.*

We are ready to prove Theorem 4.

Proof of Theorem 4. If G has a hamiltonian path, the theorem holds. So we may assume that G does not have a hamiltonian path. By applying Lemma 2 with $H = G$, we can take a path P in G such that

(P1) $\alpha(G - V(P)) \leq \alpha(G) - (n + 1)$.

(P2) P is as long as possible subject to (P1).

By the assumption, P is not a hamiltonian path of G . Let x_1 and x_2 be the endvertices of P . Then, by the choice (P2), we obtain the following claim.

Claim 1. x_1 and x_2 are not adjacent in G and $N_G(x_1) \cup N_G(x_2) \subseteq V(P)$.

Next, we choose a spanning tree T of G so that

(T1) T includes P .

(T2) $\sum_{x \in V_T^*} \deg_T(x)$ is as small as possible subject to (T1), where

$$V_T^* = \{x \in V(G) : \deg_T(x) > f(x)\}.$$

(T3) $|Leaf(T)|$ is as small as possible subject to (T2).

By the choice (T1), (T3) and Claim 1, we can obtain the following claim.

Claim 2. x_1 and x_2 are leaves of T , and $Leaf(T)$ is an independent set of G .

Claim 3. $|Leaf(T)| \leq \min_X \left\{ \sum_{x \in X} (f(x) - 2) : X \subseteq V(G), |X| = n \right\} + 2$.

Since $Leaf(T) - \{x_1, x_2\} \subseteq V(G) - V(P)$, by Claim 2, (P1) and by (1), we have

$$\begin{aligned} |Leaf(T)| - 2 &\leq \alpha(G - V(P)) \\ &\leq \alpha(G) - (n + 1) \\ &\leq \min_X \left\{ \sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n \right\}. \end{aligned}$$

Hence, $|Leaf(T)| \leq \min_X \left\{ \sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n \right\} + 2$.

Claim 4. $V_T^* = \emptyset$.

Assume that there exists a vertex $w \in V(G)$ such that $\deg_T(w) > f(w) \geq 2$. We regard T as a rooted tree with root x_1 . Let u be a child of w in T such that $u \notin V(P)$. Since G is a n -connected graph, there exist n internal disjoint paths P_1, P_2, \dots, P_n connecting x_1 and u in G . By Claim 1, x_1 and u are not adjacent in G , and so $V(P_i) - \{x_1, u\} \neq \emptyset$ for every $1 \leq i \leq n$.

Assume that for every path P_i , $1 \leq i \leq n$, there exists a vertex $w_i \in V(P_i) - \{x_1, u\}$ such that $\deg_T(w_i) \geq f(w_i)$. Let $W = \{w_1, w_2, \dots, w_n\} \cup \{w\}$. Then, by Lemma 3, we have

$$\begin{aligned} |Leaf(T)| &\geq \sum_{x \in W} (\deg_T(x) - 2) + 2 \geq \sum_{x \in W - \{w\}} (f(x) - 2) + 3 \\ &\geq \min_X \left\{ \sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n \right\} + 3. \end{aligned}$$

This contradicts Claim 3. Therefore there exists a path $P^* \in \{P_1, P_2, \dots, P_n\}$ such that $\deg_T(x) < f(x)$ for all $x \in V(P^*) - \{x_1, u\}$. Note that $\deg_T(x_1) = 1 < f(x_1)$.

Since x_1 and u are contained in the two distinct components of $T - wu$ and P^* connects x_1 and u , P^* contains an edge $e = u_1u_2$ such that u_1 and u_2 are contained in the two distinct components of $T - wu$. Obviously, $e \notin E(T)$. Let $T_1 = T - wu + e$. Then T_1 is a spanning tree of G that includes P and satisfies

$$\sum_{x \in V_{T_1}^*} \deg_{T_1}(x) < \sum_{x \in V_T^*} \deg_T(x).$$

This contradicts the choice (T2). Hence, Claim 4 holds.

Consequently, the theorem follows from Claim 3 and Claim 4. \square

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