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Generalized Balanced Partitions of Two Sets of Points in the Plane

Atsushi Kaneko[1] and M.Kano[2]

[1] Department of Computer Science and Communication Engineering, Kogakuin University, Shinjuku-ku, Tokyo 1563-8677, Japan
[2] Department of Computer and Information Sciences, Ibaraki University, Hitachi 316-8511, Japan

Abstract. We consider the following problem. Let $n \geq 2$, $b \geq 1$ and $q \geq 2$ be integers. Let R and B be two disjoint sets of n red points and bn blue points in the plane, respectively, such that no three points of $R \cup B$ lie on the same line. Let $n = n_1 + n_2 + \cdots + n_q$ be an integer-partition of n such that $1 \leq n_i$ for every $1 \leq i \leq q$. Then we want to partition $R \cup B$ into q disjoint subsets $P_1 \cup P_2 \cup \cdots \cup P_q$ that satisfy the following two conditions: (i) conv $(P_i) \cap$ conv $(P_j) = \emptyset$ for all $1 \leq i < j \leq q$, where conv(P_i) denotes the convex hull of P_i; and (ii) each P_i contains exactly n_i red points and bn_i blue points for every $1 \leq i \leq q$.
We shall prove that the above partition exists in the case where (i) $2 \leq n \leq 8$ and $1 \leq n_i \leq n/2$ for every $1 \leq i \leq q$, and (ii) $n_1 = n_2 = \cdots = n_{q-1} = 2$ and $n_q = 1$.

1 Introduction

For a set X of points in the plane, we denote by conv(X) the convex hull of X, which is the smallest convex set containing X. We shall consider the following problem:

Problem 1. Let R and B be any two disjoint sets of n red points and bn blue points in the plane, respectively, such that no three points of $R \cup B$ lie on the same line. Find positive integer-partitions $n = n_1 + n_2 + \cdots + n_q$ for which $R \cup B$ can be partitioned into q disjoint subsets $P_1 \cup P_2 \cup \cdots \cup P_q$ that satisfy the following two conditions:

(1) conv $(P_i) \cap$ conv $(P_j) = \emptyset$ for all $1 \leq i < j \leq q$; and
(2) each P_i contains exactly n_i red points and bn_i blue points.

If $R \cup B$ can be partitioned into q subsets in the above way, then we say that $R \cup B$ is partitioned into $(n_1, n_2, \ldots, n_k)$-*balanced subsets*, or $R \cup B$ has a $(n_1, n_2, \ldots, n_k)$-*balanced partition*.

Figure 1 (a) gives a $(3, 2, 2, 1)$-balanced partition. Figure 1 (b) shows configurations $R \cup B$ having no $(2, 1)$-balanced partition or no $(3, 2)$-balanced partition, respectively. Non-existence of such balanced partitions can be shown by the fact that if such balanced partitions exist, then there exist lines that partition $R \cup B$

into two subsets $P_1 \cup P_2$ satisfying the above conditions (1) and (2). However there exist no such lines.

By a similar argument given above, if $n/2 < n_1 < n$, then we can easily give configurations $R \cup B$ having no (n_1, n_2)-balanced partitions as follows: Let C_1 and C_2 be two circles with the same center in the plane such that the radius of C_1 is much smaller than that of C_2, and b be a sufficient large integer. Then we uniformly place n red points and bn blue points on the boundaries of C_1 and C_2, respectively (see Figure 1(c)).

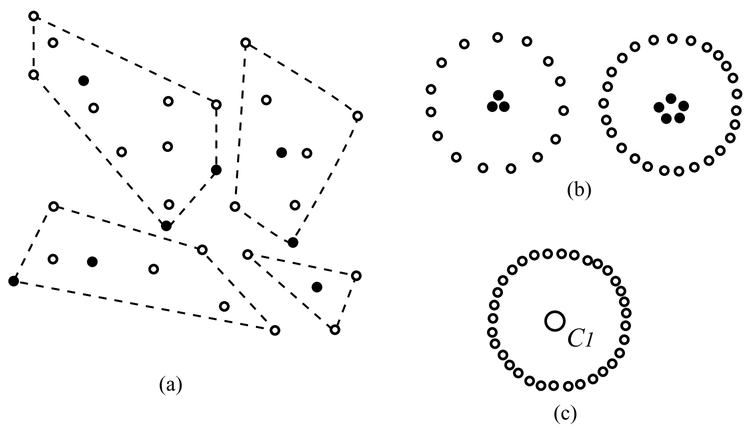

Fig. 1. (a) A $(3, 2, 2, 1)$-balanced partition. (b) Configurations having no $(2, 1)$-balanced partition and no $(3, 2)$-balanced partition, respectively. (c) A configuration having no (n_1, n_2)-balanced partition with $n/2 < n_1 < n$.

Throughout this paper, let R and B denote two disjoint sets of red points and blue points in the plane, respectively, such that no three points of $R \cup B$ lie on the same line. We shall give some parts of the following theorems since their complete proofs are quite long and rather tedious.

Theorem 1. *Let $2 \leq n \leq 8$, $1 \leq b$ and $2 \leq q$ be integers. Let R and B be disjoint sets of n red points and bn blue points in the plane respectively. Then for every integer-partition $n = n_1 + n_2 + \ldots + n_q$ such that $1 \leq n_i \leq n/2$ for every $1 \leq i \leq q$, $R \cup B$ has an $(n_1, n_2, \ldots, n_q)$-balanced partition.*

Theorem 2. *Let $5 \leq n$ be an odd integer, and $b \geq 1$ an integer. Let R and B be disjoint sets of n red points and bn blue points in the plane respectively. Then $R \cup B$ has a $(2, 2, \ldots, 2, 1)$-balanced partition.*

By the above results and by an example given latter, we propose the following conjecture.

Conjecture 1. Let $n \geq 3$, $a \geq 1$, $b \geq 1$ and $q \geq 2$ be integers. Let R and B be disjoint sets of an red points and bn blue points in the plane respectively. Let $n = n_1 + n_2 + \cdots + n_k$ be an integer-partition such that $1 \leq n_i \leq n/3$ for every $1 \leq i \leq q$. Then $R \cup B$ can be partitioned into q disjoint subsets $P_1 \cup P_2 \cup \cdots \cup P_q$ so that (i) conv $(P_i) \cap$ conv $(P_j) = \emptyset$ for all $1 \leq i < j \leq q$; and (ii) each P_i contains exactly an_i red points and bn_i blue points.

The conjecture is true if either $a = 1$ and $n \leq 8$ or $a = 1$, $n_1 = \cdots = n_{q-1} = 2$ and $n_q = 1$ by Theorems 1 and 2. Moreover the conjecture is true in the case where $n_1 = \cdots = n_q = 1$, that is, the following Theorem 3 holds. This theorem was partially proved by [5] and [6], and completely proved by Bespamyatnikh, Kirkpatrick and Snoeyink [2], Ito, Uehara and Yokoyama [4] and Sakai [8] independently.

Theorem 3. *Let R and B be two disjoint sets of an red points and bn blue points in the plane respectively, where $a \geq 1$ and $b \geq 1$. Then $R \cup B$ can be partitioned into n disjoint subsets $P_1 \cup P_2 \cup \ldots \cup P_n$ so that (i) $\mathrm{conv}(P_i) \cap \mathrm{conv}(P_j) = \emptyset$ for all $1 \leq i < j \leq n$, and (ii) every P_i contains exactly a red points and b blue points.*

Other related results can be found in [1] and [7], which deal with balanced partition of convex sets in the plane.

We conclude this section with an example which shows that the condition that $1 \leq n_i \leq n/3$ for every $1 \leq i \leq q$ in Conjecture 1 or other similar condition is necessary, and that Conjecture 1 does not hold under the condition that $1 \leq n_i \leq n/2$ for every $1 \leq i \leq q$. Namely, the configuration of $R \cup B$ whose rough sketch is given in Figure 2 has no $(8, 8, 4)$-balanced partition, and there might exist a similar configuration having no $(7, 7, 6)$-balanced partition.

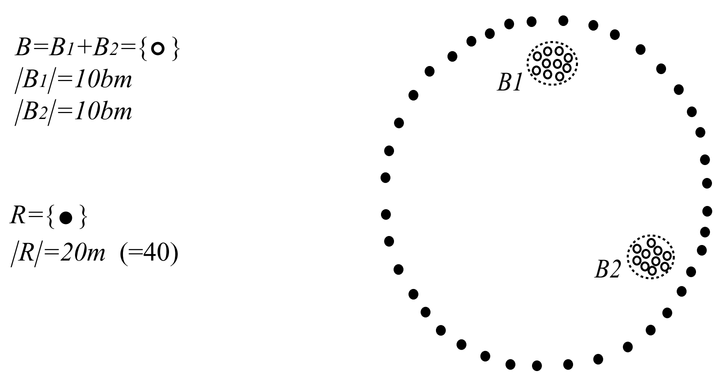

Fig. 2. A configuration having no $(8, 8, 4)$-balanced partition.

2 Proofs of Theorems

In this section we shall give some parts of proofs of Theorems as we mentioned before. We deal only with *directed lines* in order to define the right side of a line and the left side of it. Thus a *line* means a directed line. A line l dissects the plane into three pieces: l and two open half-planes $R(l)$ and $L(l)$, where $R(l)$ and $L(l)$ denote the *open half-planes* which are on the right side and on the left side of l, respectively (see Figure 3). Let r_1 and r_2 be two rays emanating from the same point p. Then we denote by $R(r_1) \cap L(r_2)$ the open region that is swept by the ray being rotated clockwise around p from r_1 to r_2, and does not contain the point p (see Figure 3). Similarly we define the open region $L(r_1) \cap R(r_2)$, which is swept by the ray being rotated counterclockwise around p from r_1 to r_2, and does not contain the point p. Then $r_1 \cup r_2$ dissects the plane into three pieces: $r_1 \cup r_2$ and two open regions $R(r_1) \cap L(r_2)$ and $L(r_1) \cap R(r_2)$.

If the internal angle $\angle r_1 p r_2 = \angle r_1 r_2$ of $R(r_1) \cap L(r_2)$ is less than π, then we call $R(r_1) \cap L(r_2)$ the *wedge* defined by r_1 and r_2, and denote it by $wdg(r_1 r_2)$ or $wdg(r_2 r_1)$.

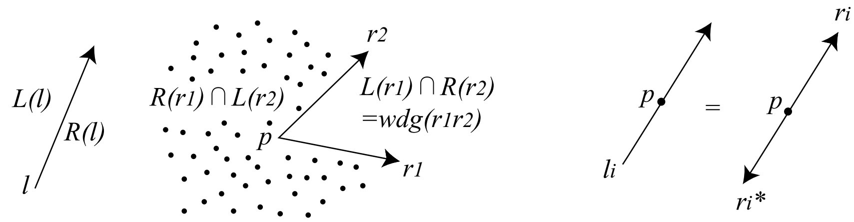

Fig. 3. Open regions $R(l)$, $L(l)$ and $L(r_1) \cap R(r_2)$ and a wedge $wdg(r_1 p r_2) = R(r_1) \cap L(r_2)$.

Let l_i be a line with suffix i, and p a point on l_i. We define l_i^* as the line lying on l_i and having the opposite direction of l_i. We next define the two rays r_i and r_i^* lying on the line l_i and having the same starting point p such that r_i has the same direction as l_i and r_i^* has the opposite direction of l_i. In particular, $l_i = r_i \cup r_i^*$ (see Figure 3). Conversely, given a ray r_i, we can similarly define the ray r_i^*, whose direction is opposite to r_i, and the line $l_i = r_i \cup r_i^*$, which has the same direction as r_i.

For a line l which passes through some points in $R \cup B$, there exist two lines l_1 and l_2 obtained from l by very small translations that pass through no points in $R \cup B$, and satisfy $L(l_1) \cap (R \cup B) = L(l) \cap (R \cup B)$, and $R(l_2) \cap (R \cup B) = R(l) \cap (R \cup B)$. We often use this fact without mentioning.

Lemma 1 (Ham-sandwich Theorem [3]). *Let R and B be disjoint sets of red points and blue points in the plane respectively. Then there exists a line l such that $|L(l) \cap R| = |R(l) \cap R|$, $|l \cap R| \leq 1$, $|L(l) \cap B| = |R(l) \cap B|$ and $|l \cap B| \leq 1$.*

If $l \cap R = \emptyset$ and $l \cap B = \emptyset$, that is, if l passes through no red point and no blue point, then we say that $R \cup B$ is partitioned into two balanced subsets by l.

The following Lemma 2 is known, and its proof can be found in [5] and [2].

Lemma 2. *Let R and B be disjoint sets of red points and blue points in the plane respectively. If there exist two lines l_1 and l_2 such that $|L(l_1) \cap R| = |L(l_2) \cap R|$, $|L(l_1) \cap B| < |L(l_2) \cap B|$, and they might pass through some points in $R \cup B$, then for every integer i, $|L(l_1) \cap B| \leq i \leq |L(l_2) \cap B|$, there exists a line l_3 such that $|L(l_3) \cap R| = |L(l_1) \cap R|$, $|L(l_3) \cap B| = i$ and l_3 passes through no point in $R \cup B$.*

The next Lemma 3 can be proved by the same argument as in the proof of the above Lemma 2, that is, we can continuously move a line l from l_1 to l_2 in such a way that $|L(l) \cap R| = |L(l_1) \cap R|$, $|L(l) \cap B|$ changes ± 1, and l passes through exactly one red point.

Lemma 3. *Let R and B be disjoint sets of red points and blue points in the plane respectively. If there exist two lines l_1 and l_2 such that $|L(l_1) \cap R| = |L(l_2) \cap R|$, $|L(l_1) \cap B| < |L(l_2) \cap B|$ and both l_1 and l_2 pass through exactly one red point, respectively, and might pass through blue points, then for every integer i, $|L(l_1) \cap B| \leq i \leq |L(l_2) \cap B|$, there exists a line l_3 such that $|L(l_3) \cap R| = |L(l_1) \cap R|$, $|L(l_3) \cap B| = i$ and l_3 passes through exactly one red point and no blue point.*

Lemma 4. *Let R and B be disjoint sets of n red points and bn blue points in the plane respectively. If $3 \leq |R| \leq 4$ and there exists a line l_1 such that $|L(l_1) \cap R| = 1$ and $|L(l_1) \cap B| \leq b$, then there exists a line l_2 such that $|L(l_2) \cap R| = 1$, $|L(l_2) \cap B| = b$ and l_2 passes through no point in $R \cup B$.*

Proof. Suppose first $|R| = 3$. Let l_3 be a line that passes through exactly one red point, and satisfies $|L(l_3) \cap R| = |R(l_3) \cap R| = 1$. Then at least one of $L(l_3) \cap B$ and $R(l_3) \cap B = L(l_3^*) \cap B$ contains at least b blue points. By applying Lemma 2 to l_1 and to either l_3 or l_3^*, we can obtain the desired line l_2.

We next consider the case $|R| = 4$. Let l_4 be a line that passes through two red points and satisfies $|L(l_4) \cap R| = |L(l_4) \cap R| = 1$. Then at least one of $L(l_4) \cap B$ and $R(l_4) \cap B$ contains at least b blue points, and so the lemma holds by Lemma 2.

Lemma 5. *Let R and B be disjoint sets of 5 red points and $5b$ blue points respectively. If there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_1) \cap B| \leq 2b$, then there exists a line l_2 such that $|L(l_2) \cap R| = 2$, $|L(l_2) \cap B| = 2b$ and l_2 passes through no point in $R \cup B$.*

Proof. Consider a line l_3 passing through exactly one red point and satisfying $|L(l_3) \cap R| = 2$ and $|R(l_3) \cap R| = 2$. Then $|L(l_3) \cap B| \geq 2b$ or $|R(l_3) \cap B| \geq 2b$. Thus by Lemma 2, there exists the desired line l_2.

Proposition 1. *Let R and B be disjoint sets of 5 red points and $5b$ blue points in the plane respectively. Then $R \cup B$ has a $(2, 2, 1)$-balanced partition.*

Proof. Unless otherwise stated, except when it moves, we always consider a line that passes through no point in $R \cup B$. We begin with two Claims.

Claim 1. If there exists a line l_1 such that $|L(l_1) \cap R| = 1$ and $|L(l_2) \cap B| \leq b$, then the proposition holds. Thus we may assume that for every line l with $|L(l) \cap R| = 1$, it follows that $|L(l) \cap B| > b$. By considering l^*, we may also assume that for every line l with $|R(l) \cap R| = |L(l^*) \cap R| = 1$, it follows that $|R(l) \cap B| = |L(l^*) \cap B| > b$.

Suppose that there exists a line l_1 such that $|L(l_1) \cap R| = 1$ and $|L(l_1) \cap B| \leq b$. If there exists a line l_2 such that $|L(l_2) \cap R| = 1$ and $|L(l_2) \cap B| \geq b$, then there exists a line l_3 satisfying $|L(l_3) \cap R| = 1$ and $|L(l_3) \cap B| = b$ by Lemma 2. By Ham-sandwich Theorem, $R(l_3) \cap (R \cup B)$ can be partitioned into two balanced subsets. Thus the proposition holds. Therefore, we may assume that for every line l with $|L(l) \cap R| = 1$, it follows that $|L(l) \cap B| < b$. Similarly, we may assume that for every line l with $|R(l) \cap R| = 1$, it follows that $|R(l) \cap B| < b$.

Let l_4 and l_5 be two parallel lines with the same direction that pass through exactly one red point respectively, and satisfy $|L(l_4) \cap R| = 1$ and $|R(l_5) \cap R| = 1$. Then $|L(l_4) \cap B| < b$, $|R(l_5) \cap B| < b$ and $R(l_4) \cap L(l_5)$ contains exatly one red point. By the symmetry, we may assume that there exists a line l_6 that is parallel to l_4, lies between l_4 and l_5, and satisfies $|L(l_6) \cap R| = 2$ and $|L(l_6) \cap B| = 2b$.

By Lemma 4 and l_5^*, $R(l_6) \cap (R \cup B)$ has a $(2,1)$-balanced partition. Hence the proposition holds. Consequently Claim 1 is proved.

Claim 2. If there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_2) \cap B| \geq 3b$, then the proposition follows. Thus we may assume that for every line l with $|L(l_1) \cap R| = 2$, it follows that $|L(l) \cap B| < 3b$.

Suppose that there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_2) \cap B| \geq 3b$. Then there exists a line l_2 in $R(l_1)$ such that l_2 passes through exactly one red point and no blue point, $|L(l_2) \cap R| = |R(l_2) \cap R| = 2$, $|L(l_2) \cap B| \geq 3b$ and $|L(l_2^*) \cap B| = |R(l_2) \cap B| \leq 2b$. By applying Lemma 3 to l_2 and l_2^*, there exists a line l_3 that passes through exactly one red point, say x, and no blue point, and satisfies $|L(l_3) \cap R| = 2$ and $|L(l_3) \cap B| = 3b$.

We now consider $(L(l_3) \cap (R \cup B)) \cup \{x\}$, which contains three red points and $3b$ blue points. By Lemma 4, $(L(l_3) \cap (R \cup B)) \cup \{x\}$ has a $(2,1)$-balanced partition, and thus $R \cup B$ has the desired $(2,2,1)$-balanced partition. Therefore Claim 2 is proved.

Let l_1 be a line that passes through two red points, say x and y, and satisfies $|L(l_1) \cap R| = 1$ (see Figure 4). Put $L(l_1) \cap R = \{z_1\}$ and $R(l_1) \cap R = \{z_2, z_3\}$. By considering a line l_1' very closed to l_1 such that $L(l_1') \cap R = \{x, z_1\}$, $y \in R(l_1')$ and $L(l_1') \cap B = L(l_1) \cap B$, it follows from Claim 2 that $|L(l_1) \cap B| = |L(l_1') \cap B| < 3b$. By Claim 2, we have $|L(l_1^*) \cap B| = |R(l_1) \cap B| < 3b$, which implies $|L(l_1) \cap B| > 2b$. Therefore

$$2b < |L(l_1) \cap B| < 3b \quad \text{and} \quad |R(l_1) \cap B| < 3b. \tag{3}$$

Let p be any point on l_1, and r_2 a ray emanating from p such that $R(r_1^*) \cap L(r_2)$ contains exactly $2b$ blue points, where r_1^* denotes the ray on l_1 emanating from p and having the opposite direction of l_1. By (3), r_2 is contained in $L(l_1)$, and so $R(r_1^*) \cap L(r_2)$ is a wedge $wdg(r_1^* r_2)$. We next take a ray r_3 emanating from p such that $R(r_2) \cap L(r_3)$ contains exactly b blue points. By (3), r_3 is contained in $R(l_1)$, and if $R(l_2) \cup l_2$ contains a red point, then r_3 must be contained in $R(l_2)$ by Claim 1, and thus $R(r_2) \cap L(r_3)$ is a wedge $wdg(r_2 r_3)$.

We first consider the case of $p = x$, that is, we take a point p on x. In this case, $R(r_2) \cap L(r_3)$ is a wedge $wdg(r_2 r_3)$. If $wdg(r_2 r_3)$ contains no red point, then z_1 is contained in $wdg(r_1^* r_2)$, and we get the desired balanced partition $wdg(r_1^* r_2) \cup \{y\}$, $wdg(r_2 r_3) \cup \{x\}$ and $wdg(r_1^* r_3)$, where $wdg(r_1^* r_3)$ contains the two red points z_2 and z_3. Hence we may assume that $wdg(r_2 r_3)$ contains at least one red point when $p = x$.

By moving a point p along l_1 in its direction from x to a point very far from x, we can find either (i) a point p_1 for which r_2 passes through the red point z_1 and $wdg(r_2 r_3)$ contains no red points, or (ii) a point p_2 for which r_3 passes through one red point, say z_2, and $wdg(r_2 r_3)$ contains no red point (see Figure 4). Since $wdg(r_1^* r_3)$ contains exactly $2b$ blue points, in each case we can easily obtain the desired $(2, 2, 1)$-balanced partition.

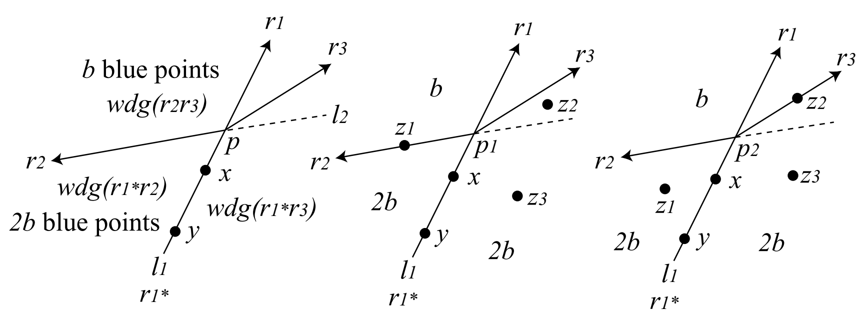

Fig. 4. A line l_1 and rays r_1, r_1^*, r_2 and r_3.

Proposition 2. *Let R and B be disjoint sets of 6 red points and $6b$ blue points respectively. Then $R \cup B$ has a $(3, 2, 1)$-balanced partition.*

Proof. Unless otherwise stated, we always consider a line that passes through no point in $R \cup B$. We begin with some Claims.

Claim 1. If there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_1) \cap B| \geq 3b$, then the proposition holds. Thus we may assume that for every line l with $|L(l) \cap R| = 2$, it follows that $|L(l) \cap B| < 3b$. By considering l^*, we may also assume that for every line l with $|R(l) \cap R| = 2$, it follows that $|R(l) \cap B| < 3b$.

Suppose that there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_2) \cap B| \geq 3b$. By a suitable very small rotation of l_1, we may assume that every line parallel to l_1 passes through at most one point in $R \cup B$. Then there exist two lines l_2 and l_3 in $R(l_1)$ such that they are parallel to l_1 and have the same direction as l_1, $|L(l_2) \cap R| = 2$, $|L(l_2) \cap B| \geq 3b$, $|R(l_3) \cap R| = 2$, $|R(l_3) \cap B| \leq 3b$ and such that each of l_2 and l_3 passes through exactly one red point, in particular, $R(l_2) \cap L(l_3)$ contains no red points.

By applying Lemma 3 to l_2 and l_3^*, there exists a line l_4 that passes through exactly one red point, say x, and satisfies $|L(l_4) \cap R| = 2$ and $|L(l_4) \cap B| = 3b$. Then $R(l_4)$ contains three red points and $3b$ blue points, and $Q = (L(l_4) \cap (R \cup B)) \cup \{x\}$ also contains three red points and $3b$ blue points. By Lemma 4, Q has a $(2, 1)$-balanced partition. Hence $R \cup B$ has the desired $(3, 2, 1)$-balanced partition. Therefore the claim is proved.

Claim 2. If there exists a line l_1 such that $|L(l_1) \cap R| = 1$ and $|L(l_2) \cap B| \leq b$, then the proposition holds. Thus we may assume that for every line l with $|L(l) \cap R| = 1$, it follows that $|L(l) \cap B| > b$.

Suppose that there exists a line l_1 such that $|L(l_1) \cap R| = 1$ and $|L(l_1) \cap B| \leq b$. We now assume that there exists a line l_2 such that $|L(l_2) \cap R| = 1$ and $|L(l_2) \cap B| \geq b$. Then by Lemma 2, there exists a line l_3 such that $|L(l_3) \cap R| = 1$ and $|L(l_3) \cap B| = b$. By considering a very small rotation of l_3 if necessary, we may assume that every line parallel to l_3 passes through at most one point in $R \cup B$.

Let l_4 be a line that is parallel to l_3, pass through exactly one red point, and satisfy $|L(l_4) \cap R| = 2$. Then by Claim 1, $R(l_3) \cap L(l_4)$ contains at most $2b - 1$ blue points, which implies that a line l_5 very closed to l_4 and lying to the right of l_4 satisfies $|(L(l_5) \cap R(l_3)) \cap R| = 2$ and $|(L(l_5) \cap R(l_3)) \cap B| < 2b$. By applying Lemma 5 to $R(l_3) \cap (R \cup B)$ and l_5, we obtain that $R(l_3) \cap (R \cup B)$ has a $(3, 2)$-balanced partition, which implies that $R \cup B$ has the desired $(3, 2, 1)$-balanced partition.

Hence we may assume that for every line l with $|L(l) \cap R| = 1$, it follows that $|L(l) \cap B| < b$.

By Ham-sandwich Theorem, $R \cup B$ can be partitioned into two balanced subsets $Q_1 \cup Q_2$, each of which contains exactly three red points and $3b$ blue points. By Lemma 4 and by the above statement on a line l, the subset Q_1 has a $(2, 1)$-balanced partition. Hence $R \cup B$ has the desired $(3, 2, 1)$-balanced partition. Therefore the claim is proved.

Claim 3. If there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_2) \cap B| \leq 2b$, then the proposition holds. Thus we may assume that for every line l with $|L(l) \cap R| = 2$, it follows that $|L(l) \cap B| > 2b$.

Suppose that there exists a line l_1 such that $|L(l_1) \cap R| = 2$ and $|L(l_1) \cap B| \leq 2b$. By considering a line l_2 passing through two red points and satisfying $|L(l_2) \cap R| = |R(l_2) \cap R| = 2$, we may assume that $|L(l_2) \cap B| \geq 2b$ by symmetry. Thus there exists a line l_3 such that $|L(l_3) \cap R| = 2$ and $|L(l_3) \cap B| = 2b$. Let l_4 be a line that is parallel to l_3, passes through exactly one red point, and satisfies $|L(l_4) \cap R| = 2$. Then by Claim 1, the number of blue points lying between l_3 and l_4 is at most $b - 1$. Hence by Lemma 4, $R(l_3) \cap (R \cup B)$ has a $(3, 1)$-balanced partition. Thus $R \cup B$ has a $(3,2,1)$-balanced partition, and hence the claim is proved.

Let l_1 be a line that passes through two red points, say x and y, and satisfies $|L(l_1) \cap R| = 1$. Put $L(l_1) \cap R = \{z_1\}$ and $R(l_1) \cap R = \{z_2, z_3, z_4\}$. By considering a line l_1' very closed to l_1 such that $L(l_1') \cap R = \{x, z_1\}$ and $L(l_1') \cap B = L(l_1) \cap B$, it follows from Claims 1 and 2 that $2b < |L(l_1) \cap B| = |L(l_1') \cap B| < 3b$, that is, the following inequality holds:

$$2b < |L(l_1) \cap B| < 3b \quad \text{ and } \quad |R(l_1) \cap B| < 4b. \tag{4}$$

Let p be any point on l_1, and r_2 a ray emanating from p such that $R(r_1^*) \cap L(r_2)$ contains exactly $2b$ blue points, where r_1^* denotes the ray on l_1 emanating from p and having the opposite direction of l_1. By (4), r_2 is contained in $L(l_1)$, and so $R(r_1^*) \cap L(r_2)$ is a wedge $wdg(r_1^* r_2)$. We next take a ray r_3 emanating from p such that $R(r_2) \cap L(r_3)$ contains exactly b blue points. By (4), r_3 is contained in $R(l_1)$, and if $R(l_2) \cup l_2$ contains a red point, then r_3 must be contained in $R(l_2)$ by Claim 2, and so $R(r_2) \cap L(r_3)$ is a wedge $wdg(r_2 r_3)$.

Hereafter, by similar arguments in the proof of Proposition 1, we can prove Proposition 2.

We omit the proof of the following Proposition 3, but in each case of Proposition 3, we can individually prove it by similar arguments in the proofs of the above two propositions.

Proposition 3. *Let R and B be two disjoint sets of n red points and bn blue points respectively. If $n = 7$, then $R \cup B$ has both a $(3,3,1)$-balanced partition and a $(3,2,2)$-balanced partition. If $n = 8$, then $R \cup B$ has both a $(4,3,1)$-balanced partition and a $(3,3,2)$-balanced partition.*

Theorem 1 can be easily proved by using Propositions 1, 2, and 3, Ham-sandwich Theorem and Theorem 3. For example, the existence of a $(2,2,2)$-balanced partition is guaranteed by Theorem 3, and a $(3,1,1,1)$-balanced partition is obtained from a $(3,2,1)$-balanced partition by applying Ham-sandwich Theorem to its subset containing two red points and $2b$ blue points.

We now give a sketch of the proof of Theorem 2 since it is quite long, but it is similar to the proof given in [5].

Let R and B be two disjoint sets of n red points and bn blue points respectively. Put $n = 2k + 1$. We prove the theorem by induction on n.

Claim 1. If $n = 5, 7, 9$, then $R \cup B$ has a $(2, 2, \ldots, 2, 1)$-balanced partition. Thus we may assume that $n \geq 11$.

Claim 2. If there exists a line l such that $5 \leq |L(l) \cap R| = t \leq n - 5$ and $|L(l) \cap B| = bt$, then $R \cup B$ has a $(2, 2, \cdots, 2, 1)$-balanced partition.

We give a proof of Claim 2 since it is short. If the integer t defined in the claim is odd, then $L(l) \cap (R \cup B)$ has a $(2, 2, \cdots, 2, 1)$-balanced partition by the induction hypothesis. Since $n-t$ is even, $R(l) \cap (R \cup B)$ has a $(2, 2, \ldots, 2)$-balanced partition by Ham-sandwich Theorem 3. Hence $R \cup B$ has the desired balanced partition. If t is even, then we apply the inductive hypothesis to $R(l) \cap (R \cup B)$, and we can similarly obtain the desired partition as above.

Claim 3. Let i be an integer such that $1 \leq i \leq k$ and $i \neq 3$. Then for every line l with $|L(l) \cap R| = i$, we may assume that $|L(l) \cap B| > bi$ since otherwise the theorem holds. Moreove if $|L(l) \cap R| = k$, then we may assume that $km < |L(l) \cap B| < k(m+1)$.

Let l_1 be a line passing through one red point, say x, such that $R(l_1)$ contains no red point, and l_2 be a line passes through x and one more red point, say y, such that $|R(l_2) \cap R| = k$ and $|L(l_2) \cap R| = k - 1$. Without loss of generality, we may assume that the direction of l_2 is downward. Let r_3 be a ray emanating from x so that $R(r_2) \cap L(r_3)$ contains exactly $2m$ blue points.

Claim 4. We may assume that $R(r_2) \cap L(r_3)$ contains at least two red points since otherwise the theorem holds.

Let r_4 be a ray emanating from x such that $R(r_2) \cap L(r_4)$ contains as many as points in $R \cup B$ subject to $R(r_2) \cap L(r_4)$ contains exactly $2i$ red points and at most $2ib$ blue points, and lies in $R(l_2)$.

The existence of the above r_4 is guaranteed by Claim 3. By applying similar arguments given in [5] to these two lines l_1 and l_2, rays r_3 and r_4, and other rays emanating from x, we can prove Theorem 2.

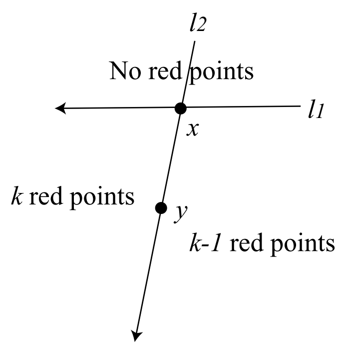

Fig. 5. Two lines l_1 and l_2.

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