A SHORT PROOF OF LOVASZ’S FACTOR THEOREM

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1 Introduction

We consider finite graphs which may have multiple edges and loops. We denote by \( V(G) \) and \( E(G) \) the set of vertices and the set of edges of a graph \( G \), respectively. If an edge \( e \) has end-vertices \( x \) and \( y \), then we write \( e = xy \). For a subset \( S \) of \( V(G) \), \( G - S \) denotes the subgraph of \( G \) obtained from \( G \) by deleting the vertices in \( S \) together with the edges incident to vertices in \( S \). If \( S \) and \( T \) are disjoint subsets of \( V(G) \), we write \( e_G(S,T) \) for the number of edges joining \( S \) and \( T \). For a vertex \( x \) of a subgraph \( H \) of \( G \), we denote by \( d_H(x) \) the degree of \( x \) in \( H \), in particular, \( d_G(x) \) is the degree of \( x \). Let \( g \) and \( f \) be two integer-valued function defined on \( V(G) \) such that \( 0 \leq g(x) \leq f(x) \leq d_G(x) \) for all \( x \in V(G) \). Then a spanning subgraph \( F \) of \( G \) is called a \((g,f)\)-factor of \( G \) if \( g(x) \leq d_F(x) \leq f(x) \) for all \( x \in V(G) \).

L. Lovasz found the following criterion for the existence of a \((g,f)\)-factor, and recently Tutte[5, Theorem 7.2] gave a short proof to it by using his \( f \)-factor theorem.

**Theorem 1.** (Lovasz[4]) Let \( G \) be a graph and \( g \) and \( f \) two integer valued functions defined on \( V(G) \) such that \( 0 \leq g(x) \leq f(x) \leq d_G(x) \) for all \( x \in V(G) \). Then \( G \) has a \((g,f)\)-factor if and only if

\[
\delta(S,T) = \sum_{t \in T} (d_G(t) - g(t)) + \sum_{s \in S} f(s) - e_G(S,T) - h(S,T) \geq 0
\]

for all disjoint subsets \( S \) and \( T \) of \( V(G) \) where \( h(S,T) \) is the number of components \( C \) of \( G - (S \cup T) \) such that \( g(c) = f(c) \) for all \( c \in V(C) \) and

\[
e_G(T,V(C)) + \sum_{c \in V(C)} f(c) \equiv 1 (mod 2) .
\]
Theorem 1 is called Lovasz’s \((g, f)\)-factor theorem. If two functions \(g\) and \(f\) in Theorem 1 satisfy \(0 \leq g(x) < f(x) \leq d_G(x)\) for all \(x \in V(G)\), then \(h(S, T) = 0\), and so the condition (1) becomes simple. In this paper, we shall give a quite elementary proof, in which alternating trails are used to Lovasz’s \((g, f)\)-factor theorem under the condition that \(g(x) < f(x)\) for all \(x \in V(G)\)(see Theorem 2). The idea of its proof technique can be found in [1] and [2]. La Vergnar [4] gave another form of \((g, f)\)-factor theorem under the assumption that \(0 \leq g(x) \leq 1 \leq f(x)\) for all \(x \in V(G)\).

2. \((g, f)\)-factor theorem with \(g < f\)

We shall give a short elementary proof to the \((g, f)\)-factor theorem in the case that \(g(x) < f(x)\) for all \(x \in V(G)\). However, we assume neither \(g(x) \geq 0\) nor \(f(x) \leq d_G(x)\). Note that Theorem 1 also holds under the weaker condition that \(g(x) \leq f(x)\) for all \(x \in V(G)\), and this result can be proved by the same way in [5].

**Theorem 2.** Let \(G\) be a graph and let \(g\) and \(f\) be two integer-valued functions defined on \(V(G)\) such that \(g(x) < f(x)\) for all \(x \in V(G)\). Then \(G\) has a \((g, f)\)-factor if and only if

\[
\delta(S, T) = \sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e_G(S, T) \geq 0
\]

for all disjoint subsets \(S\) and \(T\) of \(V(G)\).

**Proof of the necessity** Suppose \(G\) has a \((g, f)\)-factor \(F\). Let \(S, T \subset V(G)\) such that \(S \cap T = \emptyset\). Then

\[
\delta(S, T) \geq \sum_{t \in T} \{d_G(t) - d_F(t)\} + \sum_{s \in S} d_F(s) - e_G(S, T) \\
\geq e_{G-F}(T, S) + e_F(S, T) - e_G(S, T) = 0
\]

where \(G - F\) is the subgraph of \(G\) obtained from \(G\) by deleting all the edges of \(F\). Hence (2) holds. q.e.d

In order to prove the sufficiency, we give some notation. Let \(G\) ba a graph. A sequence

\[x_0, e_1, x_1, e_2, x_2, \ldots, e_n, x_n\quad (n \geq 1), \quad x_i \in V(G) \quad \text{and} \quad e_i \in E(G_i)\]

is called a trail of length \(n\) if \(e_i = x_{i-1}x_i\) for all \(i, 1 \leq i \leq n\), and \(e_i \neq e_j\) for \(1 \leq i < j \leq n\). For a subgraph \(H\) of \(G\), an \(H\)-alternating trail of \(G\) is a trail whose edges are alternately in \(E(H)\) and
Let $g$ and $f$ be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph $H$ of $G$ is called a $(0, f)$-factor of $G$ if $0 \leq d_H(x) \leq f(x)$ for all $x \in V(G)$. A vertex $x$ of $G$ is deficient in a $(0, f)$-factor $F$ with respect to $g$ if $d_F(x) < g(x)$.

**Lemma 1.** Let $G, g, f$ be the same as in Theorem 2. Suppose $G$ satisfies the condition (2) in Theorem 2, and let $K$ be a $(0, f)$-factor of $G$. If a vertex $u$ of $G$ is deficient in $K$ with respect to $g$, then $G$ has a $(0, f)$-factor $H$ which satisfies that $d_K(u) < d_H(u)$ and $g(x) \leq d_H(x)$ for every vertex $x$ with $g(x) \leq d_K(x)$.

**Proof** Setting $S = \emptyset$ and $T = x$ in (2), we have $g(x) \leq d_G(x)$ for every $x \in V(G)$. Let $[u = x_0, e_1, x_1, e_2, x_2, \ldots, e_n, x_n]$ be a $K$-alternating trail with $e_{2i} \in E(K)$ and $e_{2i+1} \in E(G) \setminus E(K)$ for every $i$. If $d_K(x_{2i+1}) < f(x_{2i+1})$ for some $r$, then

$$H = K - \{e_2, e_4, \ldots, e_{2r}\} + \{e_1, e_3, \ldots, e_{2r+1}\}$$

is a $(0, f)$-factor of $G$, and satisfies the required conditions. Similarly, if $d_K(x_{2r}) > g(x_{2r})$ for some $r$, then

$$H = K - \{e_2, e_4, \ldots, e_{2r}\} + \{e_1, e_3, \ldots, e_{2r-1}\}$$

is a $(0, f)$-factor of $G$ satisfying the desired conditions. So we may confine ourselves to the case that

$$d_K(x_{2i+1}) = f(x_{2i+1}) \quad \text{and} \quad d_K(x_{2r}) \leq g(x_{2r}) \quad \text{for every } i$$

We now define two subsets $OA$ and $EA$ of $V(G)$ to be the sets of vertices $z$ of $G$ such that there exists a $K$-alternating trail from $u$ to $z$ of odd length and of even length, respectively, where we consider only trails of length at least one. Then it follows from the above assumption that

(3) $d_K(x) = f(x)$ if $x \in OA$, and $d_K(y) \leq g(y)$ if $y \in EA$.

In particular, $OA \cap EA = \emptyset$ since $g(z) < f(z)$ for all $z \in V(G)$. Let $y \in EA$. If there is an edge $e = yz$ in $E(g) \setminus E(K)$, then it follows immediately that $z \in OA$. Moreover, there exists a vertex $x \in OA$ such that $yx \in E(K)$. Hence

(4) $d_{G-OA}(y) = d_{G-OA}(y)$ for all $y \in EA$. 

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Similarly, we can show that if $x \in OA$ and $e = xz \in E(K)$, then $z \in EA$. Hence we have

(5) $e_K(OA, EA) = \sum_{x \in OA} d_K(x)$.

Put $S = OA$ and $T = EA \cup \{u\}$ or $T = EA$ according as $u \notin OA$ or $u \in EA$. Then we have

$$\delta(S, T) = \sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{x \in OA} f(x) - e_G(T, OA)$$

$$= \sum_{t \in T} \{d_G-OA(t) - g(t)\} + \sum_{x \in OA} d_K(x) \quad \text{(by (3))}$$

Since $g(u) > d_K(u) \geq d_G-OA(u)$, we obtain

$$\delta(S, T) = \sum_{y \in EA} \{d_K-OA(y) - d_K(y)\} + e_K(OA, EA) \quad \text{(by (4), (5))}$$

$$= -e_K(OA, EA) + e_K(OA, EA) = 0.$$ 

This contradicts (2), and we conclude that the lemma is proved. q.e.d.

**Proof of the sufficiency** If $f(x) < 0$ for some vertex $x$ of $G$, then we have a contradiction by setting $S = \{x\}$ and $T = \emptyset$ in (2). Then $f(x) \geq 0$ for all $x \in V(G)$, and so $G$ has a $(0, f)$-factor. For example, a spanning subgraph $K$ with $E(K) = \emptyset$ is a $(0, f)$-factor. We can easily obtain a $(g, f)$-factor by applying lemma 1 to a $(0, f)$-factor until it has no deficient vertices with respect to $g$. q.e.d.

A necessary and sufficient condition for the existence of a $(g, f)$-factor in a bipartite graph is given in the next theorem. Note that there may exist a vertex $x$ such that $g(x) = f(x)$.

**Theorem 3.** Let $G$ be a bipartite graph with partite sets $(X, Y)$ and let $g$ and $f$ be two integer-valued functions defined on $V(G) = X \cup Y$ such that $g(z) \leq f(z)$ for all $z \in V(G)$. Then $G$ has a $(g, f)$-factor if and only if

$$\sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e_G(S, T) \geq 0$$

and

$$\sum_{s \in S} \{d_G(s) - g(s)\} + \sum_{t \in T} f(t) - e_G(T, S) \geq 0$$

for all $S \subset X$ and $T \subset Y$.

**Proof** The necessity can be proved easily by the same way as Theorem 2. So we prove only the sufficiency. Suppose $G$ satisfies the conditions in the theorem. Let $U, W \subset V(G)$ such that
\[ U \cap W = \emptyset. \] Put \( A = U \cap X, \) \( B = U \cap Y, \) \( C = W \cap X \) and \( D = W \cap Y. \) Then we have

\[
\delta(U, W) = \sum_{u \in U} \{d_G(u) - g(u)\} + \sum_{w \in W} f(w) - e_G(U, W)
\]

\[
= \sum_{a \in A} \{d_G(a) - g(a)\} + \sum_{b \in B} f(b) - e_G(A, D) + \sum_{c \in C} \{d_G(b) - g(b)\} + \sum_{c \in C} f(c) - e_G(B, C) \geq 0
\]

Therefore

\[
(6) \quad \delta(U, W) = \sum_{u \in U} \{d_G(u) - g(u)\} + \sum_{w \in W} f(w) - e_G(U, W) \geq 0
\]

for all disjoint subsets \( U \) and \( W \) of \( V(G). \)

We next prove that \( G \) has a \((g, f)\)-factor when \( G \) satisfies (6) even if \( g(z) = f(z) \) for some \( z \in V(G). \) It suffices to show that Lemma 1 holds for \( G, g \) and \( f \) in Theorem 3. Since \( G \) is a bipartite graph, we have \( EA \cap OA = \emptyset, \) where \( EA \) and \( OA \) are subsets of \( V(G) \) defined in the proof of Lemma 1. Hence the arguments in the proof of Lemma 1 can be applied to this case. Consequently, Theorem 3 follows. q.e.d.

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References


