A SHORT PROOF OF LOVASZ'S FACTOR THEOREM

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1 Introduction

We consider finite graphs which may have multiple edges and loops. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively. If an edge e has end-vertices x and y, then we write $e = xy$. For a subset S of $V(G)$, $G - S$ denotes the subgraph of G obtained from G by deleting the vertices in S together with the edges incident to vertices in S. If S and T are disjoint subsets of $V(G)$. we write $e_G(S,T)$ for the number of edges joining S and T. For a vertex x of a subgraph H of G, we denote by $d_H(x)$ the degree of x in H, in particular, $d_G(x)$ is the degree of x. Let g and f be two integer-valued function defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then a spanning subgraph F of G is called a (g,f)-factor of G if $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$.

L.Lovasz found the following criterion for the existence of a (g,f)-factor, and recently Tutte[5. Theorem 7.2] gave a short proof to it by using his f-factor theorem.

Theorem 1.(lovasz[4]) *Let G be a graph and g and f two integer valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x) \leq d_G(x)$ for all $x \in V(G)$. Then G has a (g,f)-factor if and only if*

$$(1) \qquad \delta(S,T) = \sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e_G(S,T) - h(S,T) \geq 0$$

for all disjoint subsets S and T of $V(G)$ where $h(S,T)$ is the number of components C of $G - (S \cup T)$ such that $g(c) = f(c)$ for all $c \in V(C)$ and

$$e_G(T, V(C)) + \sum_{c \in V(C)} f(c) \equiv 1 (mod 2).$$

Theorem 1 is called Lovasz's (g, f)-factor theorem. If two functions g and f in Theorem 1 satisfy $0 \le g(x) < f(x) \le d_G(x)$ for all $x \in V(G)$, then $h(S,T) = 0$, and so the condition (1) becomes simple. In this paper, we shall give a quite elementary proof, in which alternating trails are used to Lovasz's (g, f)-factor theorem under the condition that $g(x) < f(x)$ for all $x \in V(G)$(see Theorem 2). The idea of its proof technique can be found in [1] and [2]. Las Vergnar [4] gave another form of (g, f)-factor theorem under the assumption that $0 \le g(x) \le 1 \le f(x)$ for all $x \in V(G)$.

2 (g, f)-factor theorem with $g < f$

We shall give a short elementary proof to the (g, f)-factor theorem in the case that $g(x) < f(x)$ for all $x \in V(G)$. However, we assume neither $g(x) \ge 0$ nor $f(x) \le d_G(x)$. Note that Theorem 1 also holds under the weaker condition that $g(x) \le f(x)$ for all $x \in V(G)$, and this result can be proved by the same way in [5].

Theorem 2. *Let G be a graph and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for all $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$(2) \quad \delta(S,T) = \sum_{t \in T} \{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e_G(S,T) \ge 0$$

for all disjoint subsets S and T of $V(G)$.

Proof of the necessity Suppose G has a (g, f)-factor F. Let $S, T \subset V(G)$ such that $S \cap T = \emptyset$. Then

$$\begin{aligned} \delta(S,T) &\ge \sum_{t \in T} \{d_G(t) - d_F(t)\} + \sum_{s \in S} d_F(s) - e_G(S,T) \\ &\ge e_{G-F}(T,S) + e_F(S,T) - e_G(S,T) = 0 \end{aligned}$$

where $G - F$ is the subgraph of G obtained from G by deleting all the edges of F. Hence (2) holds. q.e.d

In order to prove the sufficiency, we give some notation. Let G ba a graph. A sequence

$$[x_0, e_1, x_1, e_2, x_2, \cdots, e_n, x_n] \ \ (n \ge 1), \quad x_i \in V(G) \ \ \textit{and} \ \ e_i \in E(G_i)$$

is called a trail of length n if $e_i = x_{i-1}x_i$ for all i, $1 \le i \le n$, and $e_i \ne e_j$ for $1 \le i < j \le n$. For a subgraph H of G, an H-alternating trail of G is a trail whose edges are alternately in $E(H)$ and

$E(G)\backslash E(H)$. Let g and f be integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for all $x \in V(G)$. A spanning subgraph H of G is called a $(0, f)$-factor of G if $0 \le d_H(x) \le f(x)$ for all $x \in V(G)$. A vertex x of G is deficient in a $(0, f)$-factor F with respect to g if $d_F(x) < g(x)$.

Lemma 1. *Let G, g, f be the same as in Theorem 2. Suppose G satisfies the condition (2) in Theorem 2, and let K be a $(0, f)$-factor of G. If a vertex u of G is deficient in K with respect to g, then G has a $(0, f)$-factor H which satisfies that $d_K(u) < d_H(u)$ and $g(x) \le d_H(x)$ for every vertex x with $g(x) \le d_K(x)$.*

Proof Setting $S = \emptyset$ and $T = x$ in (2), we have $g(x) \le d_G(x)$ for every $x \in V(G)$. Let $[u = x_0, e_1, x_1, e_2, x_2, \cdots, e_n, x_n]$ be a K-alternating trail with $e_{2i} \in E(K)$ and $e_{2i+1} \in E(G)\backslash E(K)$ for every i. If $d_K(x_{2r+1}) < f(x_{2r+1})$ for some r, then

$$H = K - \{e_2, e_4, \cdots, e_{2r}\} + \{e_1, e_3, \cdots, e_{2r+1}\}$$

is a $(0, f)$-factor of G, and satisfies the required conditions. Similarly, if $d_K(x_{2r}) > g(x_{2r})$ for some r, then

$$H = K - \{e_2, e_4, \cdots, e_{2r}\} + \{e_1, e_3, \cdots, e_{2r-1}\}$$

is a $(0, f)$-factor of G satisfying the desired conditions. So we may confine ourselves to the case that

$$d_K(x_{2r+1}) = f(x_{2r+1}) \quad and \quad d_K(x_{2r}) \le g(x_{2r}) \quad for\ every\ i$$

We now define two subsets OA and EA of $V(G)$ to be the sets of vertices z of G such that there exists a K-alternating trail from u to z of odd length and of even length, respectively, where we consider only trails of length at least one. Then it follows from the above assumption that

(3) $d_K(x) = f(x)$ if $x \in OA$, and $d_K(y) \le g(y)$ if $y \in EA$.

In particular, $OA \cap EA = \emptyset$ since $g(z) < f(z)$ for all $z \in V(G)$. Let $y \in EA$. If there is an edge $e = yz$ in $E(g)\backslash E(K)$, then it follows immediately that $z \in OA$. Moreover, there exists a vertex $x \in OA$ such that $yx \in E(K)$. Hence

(4) $d_{G-OA}(y) = d_{G-OA}(y)$ for all $y \in EA$.

Similarly, we can show that if $x \in OA$ and $e = xz \in E(K)$, then $z \in EA$. Hence we have

(5) $e_K(OA,\ EA) = \sum_{x \in OA} d_K(x).$

Put $S = OA$ and $T = EA \cup \{u\}$ or $T = EA$ according as $u \notin OA$ or $u \in EA$. Then we have

$$\begin{aligned} \delta(S,T) &= \sum_{t \in T}\{d_G(t) - g(t)\} + \sum_{x \in OA} f(x) - e_G(T, OA) \\ &= \sum_{t \in T}\{d_{G-OA}(t) - g(t)\} + \sum_{x \in OA} d_K(x) \qquad (by (3)) \end{aligned}$$

Since $g(u) > d_K(u) \geq d_{G-OA}(u)$, we obtain

$$\begin{aligned} \delta(S,T) &= \sum_{y \in EA}\{d_{K-OA}(y) - d_K(y)\} + e_K(OA, EA) \qquad (by (4), (5)) \\ &= -e_K(OA, EA) + e_K(OA, EA) = 0. \end{aligned}$$

This contradicts (2), and we conclude that the lemma is proved. q.e.d.

Proof of the sufficiency If $f(x) < 0$ for some vertex x of G, then we have a contradiction by setting $S = \{x\}$ and $T = \emptyset$ in (2). Then $f(x) \geq 0$ for all $x \in V(G)$, and so G has a $(0, f)$-factor. For example, a spanning subgraph K with $E(K) = \emptyset$ is a $(0, f)$-factor. We can easily obtain a (g, f)-factor by applying lemma 1 to a $(0, f)$-factor until it has no deficient vertices with respect to g. q.e.d.

A necessary and sufficient condition for the existence of a (g, f)-factor in a bipartite graph is given in the next theorem. Note that there may exists a vertex x such that $g(x) = f(x)$.

Theorem 3. *Let G be a bipartite graph with partite sets (X, Y) and let g and f be two integer-valued functions defined on $V(G) = X \cup Y$ such that $g(z) \leq f(z)$ for all $z \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\sum_{t \in T}\{d_G(t) - g(t)\} + \sum_{s \in S} f(s) - e_G(S, T) \geq 0$$

and

$$\sum_{s \in S}\{d_G(s) - g(s)\} + \sum_{t \in T} f(t) - e_G(T, S) \geq 0$$

for all $S \subset X$ and $T \subset Y$.

Proof The necessity can be proved easily by the same way as Theorem 2. So we prove only the sufficiency. Suppose G satisfies the conditions in the theorem. Let $U,\ W \subset V(G)$ such that

$U \cap W = \emptyset$. Put $A = U \cap X,\ B = U \cap Y,\ C = W \cap X$ and $D = W \cap Y$. Then we have

$$\begin{aligned} \delta(U,W) &= \sum_{u \in U} \{d_G(u) - g(u)\} + \sum_{w \in W} f(w) - e_G(U,W) \\ &= \sum_{a \in A} \{d_G(a) - g(a)\} + \sum_{d \in D} f(d) - e_G(A,D) + \sum_{b \in B} \{d_G(b) - g(b)\} + \sum_{c \in C} f(c) - e_G(B,C) \geq 0 \end{aligned}$$

Therefore

$$(6) \qquad \delta(U,W) = \sum_{u \in U} \{d_G(u) - g(u)\} + \sum_{w \in W} f(w) - e_G(U,W) \geq 0$$

for all disjoint subsets U and W of $V(G)$.

We next prove that G has a (g,f)-factor when G satisfies (6) even if $g(z) = f(z)$ for some $z \in V(G)$. It suffices to show that Lemma 1 holds for G, g and f in Theorem 3. Since G is a bipartitegraph, we have $EA \cap OA = \emptyset$, where EA and OA are subsets of $V(G)$ defined in the proof of Lemma 1. Hence the arguments in the proof of Lemma 1 can be applied to this case. Consequently, Theorem 3 follows. q.e.d.

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