Discrete Geometry on Colored Point Sets in the Plane – A Survey

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Abstract

Discrete geometry on colored point sets in the plane has a long history, but this research area has been extensively developed in the last two decades. In 2003, a short survey entitled "Discrete geometry on red and blue points in the plane – A survey" was published. Since then, many new and important results have been published, and thus the need of a new and up-to-date survey is evident.

Keywords Discrete geometry in the plane, Colored point set, Balanced partition, Geometric graph, Empty polygon, Balanced line, Plane lattice, Measure

1 Introduction

In this paper, we deal with discrete geometry on colored point sets in the plane, and collect some related results in higher-dimensional space. On the other hand, we do not deal with computational geometry (algorithms) on this topic since there are many results on them and it needs another paper to explain them. We also omit some topics including "Graph embedding", "Coloring points" and others due to the page limitation and a few new results on these topics were obtained after the survey [79]. For these topics, the reader is referred to the survey [79], the books [106] and [107].

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1.1 Notation and Definitions

We begin with some definitions and notation, which are used in this paper. For a finite set X, the number of elements in X is denoted by $|X|$ or $\#X$. In this paper, a line is regarded as a directed line so as to distinguish easily between its left and right sides. Thus a *line* means a directed line. For a line l in the plane, the open half-plane to the left of l is denote by $\mathit{left}(l)$ and the open half-plane to the right of l is denoted by $\mathit{right}(l)$. In particular, the plane is partitioned into three disjoint subsets $\mathit{left}(l) \cup \mathit{right}(l) \cup l$ (see (1) of Fig. 1). For a line l, let l^* denote the line at the same position as l and with the opposite direction of l (see (1) of Fig. 1). Thus $\mathit{left}(l^*) = \mathit{right}(l)$ and $\mathit{right}(l^*) = \mathit{left}(l)$.

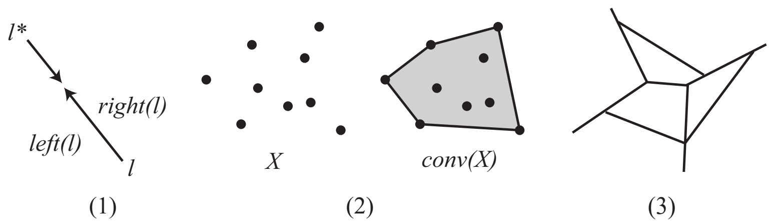

Figure 1: (1) A line l, l^*, $\mathit{left}(l)$ and $\mathit{right}(l)$; (2) A set X of points in the plane and its convex hull $\mathit{conv}(X)$; (3) A partition of the plane into 7 convex polygons.

For a set X of points in the plane, the smallest closed convex set containing X is called the *convex hull* of X and denoted by $\mathit{conv}(X)$ (see (2) of Fig. 1). For convenience, any region in the plane (whether closed or infinite) whose boundary consists of non-crossing straight-line segments is referred to as a *polygon* even if it is an infinite region. In other words, a polygon means a simple polygon, which may be an infinite region. If a point x is contained in the interior of a polygon P, then we will say that an *open polygon P contains x*. In (3) of Fig. 1, the plane is partitioned into seven convex polygons, four of which are infinite.

We consider discrete geometry on colored point sets in the plane. Throughout this paper, let R, B and G always denote *a set of red points, a set of blue points* and *a set of green points*, respectively. Moreover, we always assume that R, B and G are disjoint. If no three points of $R \cup B$ (or $R \cup B \cup G$) lie on the same line in the plane, we say that R and B (or R, B and G) are *in general position.* We always assume that R and B (or R, B and G) are in general position unless explicitly stated otherwise.

1.2 Borsuk-Ulam Theorem

Let $n \geq 2$ be an integer, and let $\mu : \mathbb{R}^n \to \mathbb{R}$ be a measure. In this paper, we always assume that every measure μ has the following properties: (i) μ is absolutely continuous with respect to the Lebesgue measure; and (ii) there is a bounded domain $D \subset \mathbb{R}^n$ such that $0 < \mu(D) = \mu(\mathbb{R}^n) < \infty$. Thus every open set X of $\mathbb{R}^n$ is measurable and $\mu(h) = 0$ for every hyperplane h of $\mathbb{R}^n$. Such a measure is called *a mass distribution*. Let

$$\mathbb{S}^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}$$

be the *unit sphere* in $\mathbb{R}^{n+1}$. For a vector $\boldsymbol{u} = (u_0, u_1, u_2) \in S^2$, where $u_0^2 + u_1^2 + u_2^2 = 1$, define two open half-planes $H^+(\boldsymbol{u})$ and $H^-(\boldsymbol{u})$ as

$$H^+(\boldsymbol{u}) = \{(x, y) \in \mathbb{R}^2 : u_1 x + u_2 y > u_0\},$$
$$H^-(\boldsymbol{u}) = \{(x, y) \in \mathbb{R}^2 : u_1 x + u_2 y < u_0\}.$$

A continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^d$ is said to be *antipodal* if

$$f(-\boldsymbol{u}) = -f(\boldsymbol{u}) \qquad \text{for all} \quad \boldsymbol{u} \in \mathbb{S}^n.$$

The next theorem has many applications in discrete geometry. For example, the continuous version of the Ham-sandwich Theorem can be shown by this theorem.

Theorem 1.1 (Borsuk-Ulam Theorem, [37], [100]) *Let $n \geq 2$ be an integer. Then for every antipodal continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$, there exists a vector $\boldsymbol{v} \in \mathbb{S}^n$ such that $f(\boldsymbol{v}) = \mathbf{0} = (0, 0, \dots, 0)$.*

2 Balanced Partitions

In this section, we consider balanced partition problems of colored point sets in the plane, on a line, or on a circle. Namely, we want to partition the plane into some convex polygons so that each open polygon contains a prescribed number of points colored with c for each color c. For lines or circles, we want to find an interval having the same property, i.e., containing a prescribed number of points of each color. Recall that R, B and G always denote *a set of red points, a set of blue points* and *a set of green points*, respectively. Moreover, we always assume that R and B (or R, B and G) are in general position.

2.1 Intervals on Lines and Circles

We first consider some problems on colored point sets on a line in general position, that is, no two points are in the same position.

Proposition 2.1 *Assume that R and B are on a line. Then there exists an open interval I that contains exactly $\lfloor \frac{|R|}{2} \rfloor$ red points and $\lfloor \frac{|B|}{2} \rfloor$ blue points (see (1) of Fig. 2).*

This proposition can be generalized to d-colored point sets, as in the next result proved by Goldberg and West [59] and Alon and West [18]. The proof of this result uses the Ham-sandwich Theorem (Theorem 2.7).

Theorem 2.2 (Necklace Theorem [59], [18]) *Let $d \geq 2$ be an integer. Assume that d pairwise disjoint point sets $X_1, X_2, \ldots, X_d$ are on a line. Then the line can be divided into two parts A and B, which consist of some intervals of the line obtained by at most d cuts of the line, so that each of A and B contains $\lfloor \frac{|X_i|}{2} \rfloor$ or $\lceil \frac{|X_i|}{2} \rceil$ points of each X_i, $1 \leq i \leq d$ (see (2) of Fig. 2).*

Sketch of proof. We may assume that the points are on the real number line $\mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}^d$ be a mapping defined by $f(t) = (t, t^2, \ldots, t^d)$ for every $t \in \mathbb{R}$. Thus, all points in $X_1 \cup X_2 \cup \cdots \cup X_d$ on the line are mapped onto the *moment curve* in $\mathbb{R}^d$. Then the images of points are in general position in $\mathbb{R}^d$. By Theorem 7.1, there exists a hyperplane h that bisects every X_i, $1 \leq i \leq d$. Namely, each of two open half-spaces defined by h contains exactly $\lfloor \frac{|X_i|}{2} \rfloor$ or $\lceil \frac{|X_i|}{2} \rceil$ points of each X_i. Moreover, h intersects the moment curve at most d points, which corresponds to cuts of the line. Hence the two parts of the line included in two open half-spaces are the desired two parts. $\square$

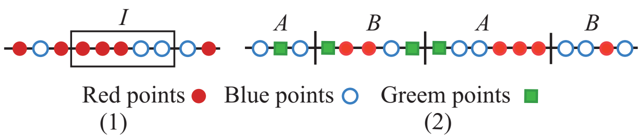

Figure 2: (1) An interval I containing exactly half of red points and of blue points; (2) Three cuts partition a necklace consisting of 3-colored points into two parts A and B so that A and B each contain the same number of points of each color.

Assume that there is a necklace consisting of d kind of beads, each with even number elements, and one wants to divide it into two persons with the same value. By Theorem 2.2, we can divided the necklace into two groups of intervals by cutting the necklace into at most $d+1$ intervals, in such a way that each group of intervals contains the same number of beads of each kind. So this theorem is called the Necklace Theorem.

The Necklace Theorem was generalized as follows.

Theorem 2.3 (Necklace Theorem (for k people), Alon [17]) *Let $d \geq 2$, $k \geq 2$ and $c_i \geq 1, 1 \leq i \leq m$, be integers. Assume that a necklace has kc_i beads of each kind $1 \leq i \leq m$. Then we can divide the necklace into k groups by $(k-1)d$ cuts so that each group (i.e., each person) has exactly c_i beads of each kind $1 \leq i \leq d$.*

For the existence problem of an interval that contains a prescribed numbers of red points and blue points, we have the following result:

Theorem 2.4 (Kaneko and Kano [80]) *Assume that R and B are on a line and $|R| \leq |B|$. Let $1 \leq r \leq |R|$ and $1 \leq b \leq |B|$ be two integers. Then there always exists an interval that contains exactly r red points and b blue points if and only if*

$$\left(\left\lfloor \frac{|R|}{r+1} \right\rfloor + 1\right)(b-1) < |B| < \left\lfloor \frac{|R|-1}{r-1} \right\rfloor (b+1),$$

where the rightmost term is infinite when $r = 1$ (see Fig. 3).

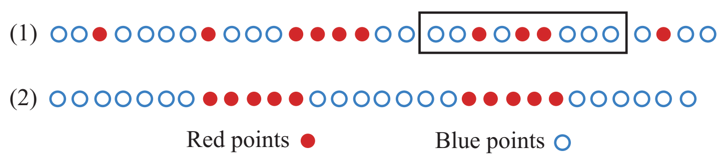

Figure 3: (1) A configuration on a line with $|R| = 10$ and $|B| = 20$, and an interval containing 3 red points and 6 blue points. (2) A configuration on a line with $|R| = 10$ and $|B| = 20$ that has no interval containing 4 red points and 8 blue points.

We next consider some problems of colored point sets on a circle.

Theorem 2.5 (Kaneko and Kano [80]) *Assume that R and B are on a circle and $|R| \leq |B|$. Let $1 \leq r \leq |R|$ and $1 \leq b \leq |B|$ be two integers. Then* *there always exists an interval on the circle that contains exactly r red points and b blue points if and only if*

$$\frac{|R|}{r+1}(b-1) < |B| < \frac{|R|}{r-1}(b+1),$$

where the rightmost term is infinite when $r = 1$ (see Fig. 4).

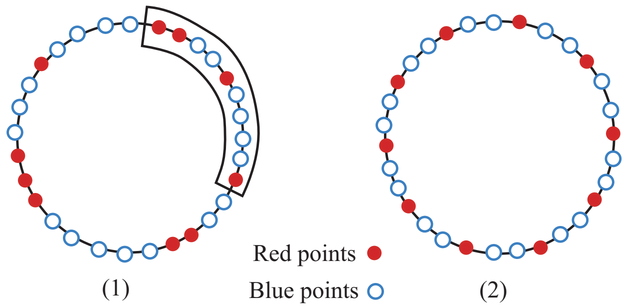

Figure 4: (1) A configuration on a circle with $|R| = 10$ and $|B| = 20$, and an interval containing 4 red points and 6 blue points; (2): A configuration with $|R| = 10$ and $|B| = 20$ that has no interval containing 4 red points and 5 blue points.

The following theorem deals with 3-colored point sets on a circle.

Theorem 2.6 (Bereg et al. [31]) *Assume that R, B and G are on a circle and $|R| = |B| = |G| = n$. Then for every integer $1 \leq k \leq n$, there exist two intervals I and J such that $I \cup J$ contains exactly k points of each color (see Fig. 5 (1)).*

Note that Fig. 5 (2) shows that the condition $|R| = |B| = |G| = n$ is necessary.

2.2 Bisectors in the Plane

Consider a ham-sandwich consisting of one slice of bread and one slice of ham, both with possibly irregular shapes (see (1) of Fig. 6). We want to cut this ham-sandwich using a kitchen knife so that each slice contains exactly the same amount bread and ham. The Ham-sandwich Theorem guarantees the existence of such a cut, which is also referred to as a bisector. From this theorem, a similar problem derives on red and blue points in the plane. We can show that if $2n$ red points and $2m$ blue points are given in the plane, then the plane can be partitioned into two half-planes by a line so that each open half-plane contains precisely n red points and m blue points.

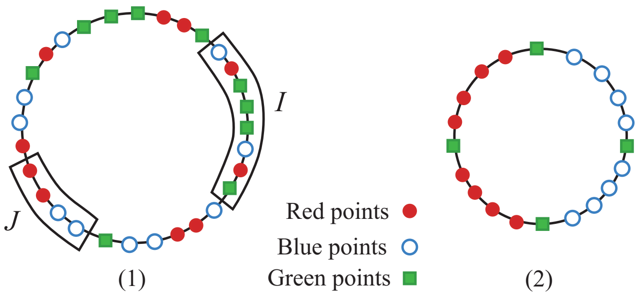

Figure 5: (1) A configuration on a circle with $|R| = |B| = |G| = 10$ and two intervals I and J such that $I \cup J$ contains 4 points of each color. (2) A configuration with $|R| = |B| = 8$ and $|G| = 4$ that has no intervals $I \cup J$ containing 3 points of each color.

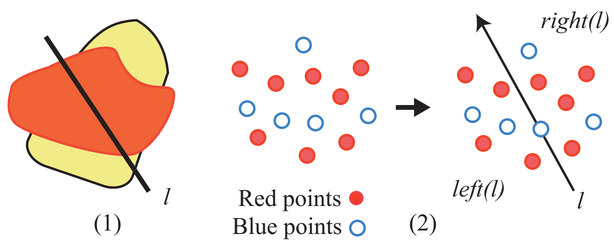

Figure 6: (1) A bisector of a ham-sandwich consisting of bread and ham; (2) A bisector l of R and B in the plane with $|R|$ even and $|B|$ odd.

Theorem 2.7 (Ham-sandwich Theorem (discrete)) *Assume that R and B are in the plane. Then there exists a line l that satisfies $|left(l) \cap R| = |right(l) \cap R|$, $|left(l) \cap B| = |right(l) \cap B|$, $|R \cap l| \leq 1$ and $|B \cap l| \leq 1$. In particular, if $|R|$ is even, then l passes through no red point. Otherwise, l passes through exactly one red point. A similar situation holds for B.*

The line l given in the above theorem is called a *bisector* or a *ham-sandwich cut* (see (2) of Fig. 6).

Let us give a remark on the Ham-sandwich Theorem (discrete). Fig. 7 (1) shows that the condition that R and B are in general position is necessary. Moreover, (2) and (3) of Fig. 7 shows that every bisector of red points is almost a bisector of blue points.

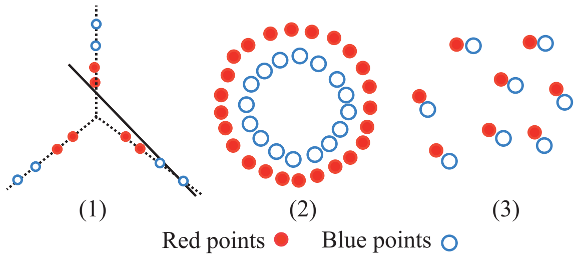

Figure 7: (1) A configuration of red points and blue points not being in general position in the plane, which has no bisector of red and blue points, and the line drawn in the figure is a bisector of blue points; (2) and (3) Configurations for which a bisector of red points is almost a bisector of blue points.

The following proposition is a variation of the Ham-sandwich Theorem, and is useful in some proofs of theorems.

Proposition 2.8 *Assume that R and B are in the plane such that $|R|$ is odd and $|B|$ is even. Then there exists a line l that passes through one red point and one blue point and satisfies $|left(l) \cap R| = |left(l) \cap R|$ and $|left(l) \cap B| = \frac{|B|}{2}$ and $|right(l) \cap B| = \frac{|B|}{2} - 1$.*

Ham-sandwich Theorem is easily shown by the following lemma by applying a bisector l of red points and l^* to l_1 and l_2, or l_2 and l_1, respectively, where l^* denotes the line at the same position as l and with the opposite direction of l. This lemma is also useful for other theorems. Notice that a point in $R \cup B$ is sometimes called a *data point*.

Lemma 2.9 (Intermediate Value Lemma, [33], [73]) *Given R and B in the plane, assume that there are two lines l_1 and l_2 which pass through no data point and satisfy $|left(l_1) \cap R| = |left(l_2) \cap R| = m$ and $|left(l_1) \cap B| \leq |left(l_2) \cap B|$. Then for every integer k, $|left(l_1) \cap B| \leq k \leq |left(l_2) \cap B|$, there exists a line l_3 that passes through no data point and satisfies $|left(l_3) \cap R| = m$ and $|left(l_3) \cap B| = k$.*

The following lemma is a variation of the Intermediate Value Lemma.

Lemma 2.10 ([3]) *Given R and B in the plane, assume that there are two lines l_1 and l_2 which pass through no data point and satisfy $|left(l_1) \cap (R \cup B)| = |left(l_2) \cap (R \cup B)| = m$. Then (i) for every integer $k, |left(l_1) \cap B| \leq k \leq |left(l_2) \cap B|$, there exists a line l_3 that passes through no data point and satisfies $|left(\ell_3) \cap (R \cup B)| = m$ and $|left(\ell_3) \cap B| = k$; and (ii) there exists a line l_4 that passes through a blue point and no red point and satisfies $|left(\ell_4) \cap (R \cup B)| = m - 1$ and $|left(\ell_4) \cap B| = |left(\ell_1) \cap B|$.*

A fast algorithm for finding a bisector of the Ham-sandwich Theorem was obtained after long and much research.

Theorem 2.11 (Lo, Matouśek and Steiger [99]) *Assume that R and B are given in the plane. Then there is a linear time algorithm for finding a bisector of R and B.*

The Ham-sandwich Theorem is an easy consequence of the Intermediate Value Lemma. We now sketch another proof of the Ham-sandwich Theorem using Borsuk-Ulam Theorem (Theorem 1.1). Notice that it is easy to derive the discrete version of the Ham-sandwich Theorem from the following continuous version.

Theorem 2.12 (Ham-sandwich Theorem (continuous)) *Assume that two mass distributions (or absolutely continuous finite measures) μ_1 and μ_2 are given in the plane. Then the plane can be partitioned into two open half-planes $left(l)$ and $right(l)$ by a line l so that $\mu_1(left(l)) = \mu_1(right(l))$ and $\mu_2(left(l)) = \mu_2(right(l))$.*

Sketch of Proof. For a vector $\boldsymbol{u} = (u_0, u_1, u_2) \in S^2$, where $u_0^2 + u_1^2 + u_2^2 = 1$, define two half-planes $H^+(\boldsymbol{u})$ and $H^-(\boldsymbol{u})$ as in Section 1.

Let $\mu_1(\mathbb{R}^2) = 2s$ and $\mu_2(\mathbb{R}^2) = 2t$, where s and t are positive real numbers. Then define a continuous function $f : \mathbb{S}^2 \to \mathbb{R}^2$ as

$$f(\boldsymbol{u}) = \big(s - \mu_1(H^+(\boldsymbol{u})), t - \mu_2(H^+(\boldsymbol{u}))\big).$$

Or without using s or t, we can define f as

$$f(\boldsymbol{u}) = \big(\mu_1(H^+(\boldsymbol{u})) - \mu_1(H^-(\boldsymbol{u})), \mu_2(H^+(\boldsymbol{u})) - \mu_2(H^-(\boldsymbol{u}))\big).$$

Then $f(\boldsymbol{u})$ is antipodal, and thus by Borsuk-Ulam Theorem, there exists a vector $\boldsymbol{v} = (v_0, v_1, v_2) \in \mathbb{S}^2$ such that $f(\boldsymbol{v}) = (0,0)$. This implies either $\mu_1(H^+(\boldsymbol{v})) = s$ and $\mu_2(H^+(\boldsymbol{v})) = t$ or $\mu_i(H^+(\boldsymbol{v})) = \mu_i(H^-(\boldsymbol{v}))$ for $i \in \{1,2\}$. Consequently Theorem 2.12 is proved. $\square$

We now explain an idea to derive discrete version of Ham-sandwich theorem from its continuous version. Assume that R and B are given in the plane in general position. We replace each point $x \in R \cup B$ by a disk with center x and with sufficiently small fixed diameter so that no line passes through three of these disks. A disk whose center point is red (blue) is called a red disk (blue disk, resp.). Then we put a thin cylinder on each disk with weight one (see Fig. 8), and the weight of every cylinder is uniformly distributed on the disk. For a region X in the plane, we define two measures μ_1 and μ_2 as follows:

$$\begin{aligned} \mu_1(X) = &\text{ the weight of red cylinders above } X, \quad \text{and} \\ \mu_2(X) = &\text{ the weight of blue cylinders above } X, \end{aligned}$$

where if a part of cylinder lies on X, then we include the weight of the part of cylinder on X.

Then μ_1 and μ_2 are mass distributions on the plane, and by Theorem 2.12, there exists a bisector l, which satisfies $\mu_1(\mathit{left}(l)) = \mu_1(\mathit{right}(l))$ and $\mu_2(\mathit{left}(l)) = \mu_2(\mathit{right}(l))$. For simplicity, here we assume that both $|R|$ and $|B|$ are even. If l does not passes through any disk, then l is a bisector of R and B. Otherwise, l passes through exactly two disks with the same color, and we can obtain the desired line by slightly moving l so that the resulting line passes through no data point (see (2) and (3) of Fig. 8).

The following proposition says that a monochromatic point set in the plane can be partitioned into 4 equal parts by two lines [102]. This result is an easy consequence of the Ham-sandwich Theorem. Namely, we first take a bisector l_1 of a set S, and color points in $\mathit{left}(l_1)$ red and those in $\mathit{right}(l_1)$ blue (see (1) of Fig. 9). Then applying Ham-sandwich theorem to these red and blue points, we obtain a bisector l_2. Then l_1 and l_2 are the desired two lines (see (2) of Fig. 9). Notice that it was proved by Buck and Buck [39] that for any convex set K in the plane, there exists three concurrent lines that partition K into 6 sectors with the same area.

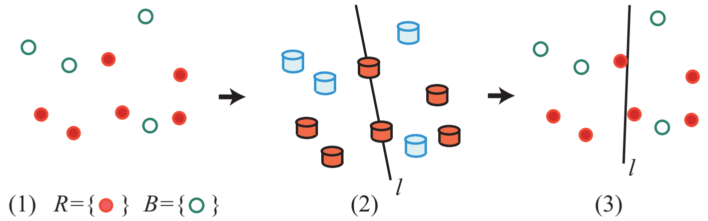

Figure 8: (1) Given red points and blue points: (2) The cylinders covering red and blue disks and a bisector l; (3) The bisector l of red and blue points obtained from the bisector in (2).

Proposition 2.13 (Megiddo [102]) *Assume that a set S of points is in the plane in general position and $|S| = 4n$ for some integer $n \geq 1$. Then there exist two lines such that each of four sectors determined by the two lines contains exactly n points of S. A similar result holds for a mass distribution (see (2) and (3) of Fig. 9).*

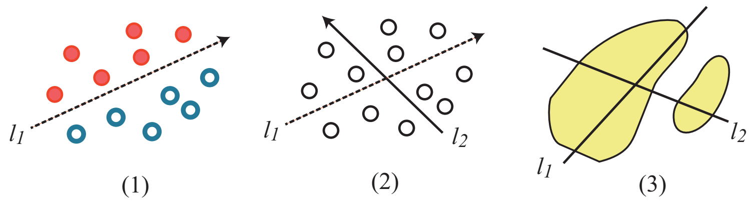

Figure 9: (1) Color points in *left*(l_1) with red and points in *right*(l_1) with blue; (2) Lines l_1 and l_2 partition a set of points into four equal sectors; (3) Breads can be partitioned into 4 parts with the same volume by two cuts.

We now consider some problems on 3-colored point sets and lines, where point sets possess some prescribed properties. Such lines are called *balanced lines.*

Theorem 2.14 (Bereg and Kano [32]) *Assume that R, B and G are in the plane and $|R| = |B| = |G| \geq 2$. If all the vertices of conv$(R \cup B \cup G)$ have the same color, then there exists an integer $1 \leq k \leq n-1$ and a line l such that left(l) contains exactly k points of each color, namely, there exists a balanced line (see (1) of Fig. 10).*

The next theorem shows another balanced line for 3-colored point sets.

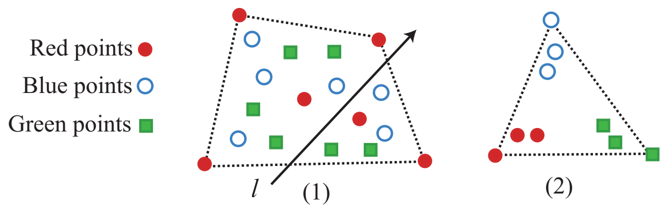

Figure 10: (1) A balanced line l of Theorem 2.14 such that *left*(l) contains exactly 4 points of each color; (2) A configuration of $R \cup B \cup G$ which has no balanced line.

Theorem 2.15 (Kano and Kynčl [84]) *Let $n \geq 2$ be an integer. Assume that R, B and G are in the plane, $|R \cup B \cup G| = 2n$ and $0 \leq |R|, |G|, |B| \leq n$. Then there exist an integer $1 \leq k \leq n-1$ and a line l such that left(l) contains exactly $2k$ points of $R \cup B \cup G$ and at most k points of each color and right(l) contains exactly $2(n-k)$ points of $R \cup B \cup G$ and at most $n-k$ points of each color (see Fig. 11).*

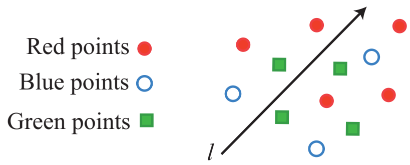

Figure 11: A balanced line l such that *left*(l) contains 4 data points and at most 2 points of each color and *right*(l) contains 8 data points and at most 4 points of each color.

A *double wedge* is a region consisting of two opposite wedges determined by two lines (see (1) of Fig. 12). If two lines are parallel, then two outer regions or one region between them forms a double wedge (see (3) of Fig. 12). If the plane is partitioned into two double wedges by two lines l_1 and l_2, then we say that $\{l_1, l_2\}$ partitions the plane. The following theorem shows the existence of a balanced double wedge for 3-colored point sets.

Theorem 2.16 (Bereg et al. [31]) *Let $n \geq 1$ be an integer. Assume that R, B and G are in the plane and $|R| = |B| = |G| = 2n$. Then there exists a set $\{l_1, l_2\}$ of two lines that simultaneously bisects R, B and G (see (3) of Fig. 12).*

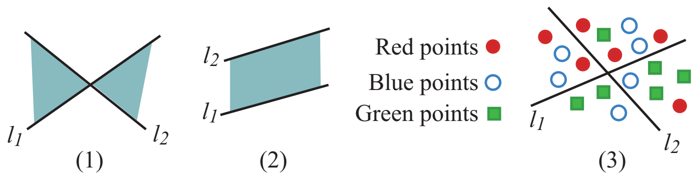

Figure 12: (1) Two double wedges determined by $\{l_1, l_2\}$, each of which consists of blue wedges or white wedges; (2) Two double wedges determined by two parallel lines $\{l_1, l_2\}$; (3) A set $\{l_1, l_2\}$ simultaneously bisects R, B and G, that is, each double wedge contains 3 points of each color.

The following theorem shows the existence of a pair of lines that bisect 4 mass distributions or 4-colored point sets, and so it is a generalization of Theorem 2.16. Notice that this is proved using Borsuk-Ulam Theorem.

Theorem 2.17 (Barba, Pilz and Schnider [24]) *Assume that 4 mass distributions μ_1, μ_2, μ_3, μ_4 are defined on the plane. Then there exists a set $\{l_1, l_2\}$ of two lines that simultaneously bisects $\mu_1, \mu_2, \mu_3, \mu_4$ (see Fig. 13)*

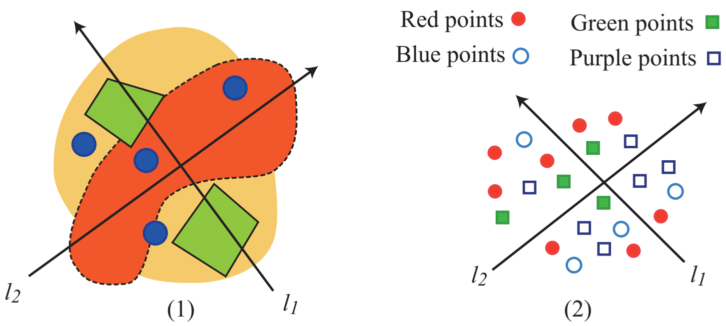

Figure 13: (1) A set $\{l_1, l_2\}$ simultaneously bisects $\mu_1, \mu_2, \mu_3, \mu_4$; (2) A set $\{l_1, l_2\}$ simultaneously bisects 4-colored point sets.

Theorem 2.18 (Barba, Pilz and Schnider [24]) *Assume that 3 mass distributions μ_1, μ_2, μ_3 are defined on the plane, and a line l is in the plane. Then there exists a set $\{l_1, l_2\}$ of lines that simultaneously bisects μ_1, μ_2, μ_3 and whose l_1 is parallel to l.*

Theorem 2.19 (Barba, Pilz and Schnider [24]) *Assume that 3 mass distributions μ_1, μ_2, μ_3 are defined on the plane, and a point p is in the plane. Then there exists a set $\{l_1, l_2\}$ of lines that simultaneously bisects μ_1, μ_2, μ_3 and whose l_1 passes through p.*

The following theorem was conjectured by Barba et al. [24] and proved by Hubard and Karasev [66].

Theorem 2.20 (Hubard and Karasev [66]) *Let $n \geq 1$ be an integer. Assume that $2n$ mass distributions $\mu_1, \mu_2, \ldots, \mu_{2n}$ are defined on the plane. Then there exist n lines $l_1, l_2, \ldots, l_n$ such that $\{l_1, l_2, \ldots, l_n\}$ simultaneously bisects $\mu_1, \mu_2, \ldots, \mu_{2n}$ (see Fig. 14).*

Note that Theorem 2.20 with $n = 1$ is the Ham-sandwich Theorem, and with $n = 2$ is Theorem 2.17. Moreover, Theorem 2.20 follows from Theorem 7.13.

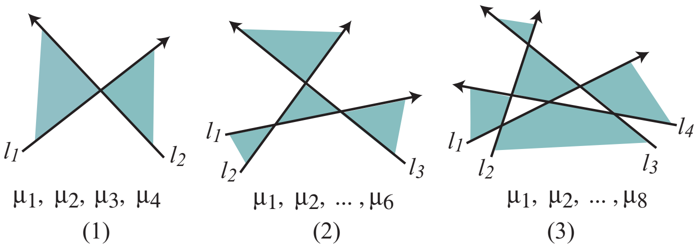

Figure 14: (1) Two double wedges determined by $\{l_1, l_2\}$, which simultaneously bisects 4 mass distributions $\mu_1, \mu_2, \mu_3, \mu_4$; (2) Two regions determined by $\{l_1, l_2, l_3\}$, which simultaneously bisects 6 mass distributions $\mu_1, \mu_2, \ldots, \mu_6$; (3) Two regions determined by $\{l_1, l_2, l_3, l_4\}$, which simultaneously bisects 8 mass distributions $\mu_1, \mu_2, \ldots, \mu_8$.

Theorem 2.21 (Stone and Tukey [116]) *Let $t \geq 2$ be an integer. Assume that t mass distributions $\mu_1, \mu_2, \ldots, \mu_t$ are defined on $\mathbb{R}^2$. Then there exists a polynomial $p(x)$ of degree at most $t-1$ that simultaneously bisects all μ_i, namely,*

$$\mu_i(\{(x, y) \in \mathbb{R}^2 : y > p(x)\}) = \frac{1}{2}\mu_i(\mathbb{R}^2) \qquad \textit{for all } 1 \leq i \leq t. \quad \textit{(see Fig. 15)}$$

The following theorem is a generalization of Theorem 6.2 given later.

Theorem 2.22 (Karasev, Roldán-Pensado and Soberón [93]) *Let $n \geq 2$ be an integer. Assume that n mass distributions $\mu_1, \mu_2, \ldots, \mu_n$ are defined on the plane. Then there exists a path formed by only horizontal and vertical line segments with at most $n-1$ turns that simultaneously bisects all μ_i. Moreover, the line is taken so that it is y-monotone and may go through infinity in the horizontal direction several times (see Fig. 16).*

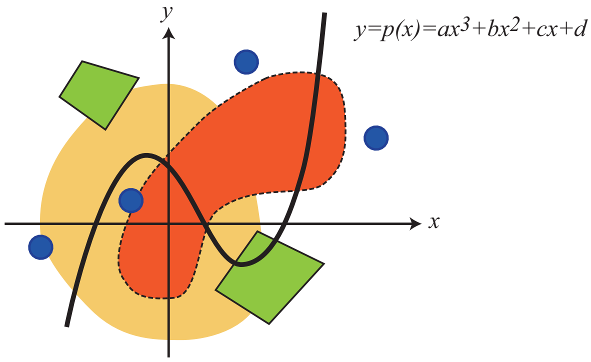

Figure 15: Four mass distributions defined on the plane are simultaneously bisected by a cubic polynomial $y = ax^3 + bx^2 + cx + d$.

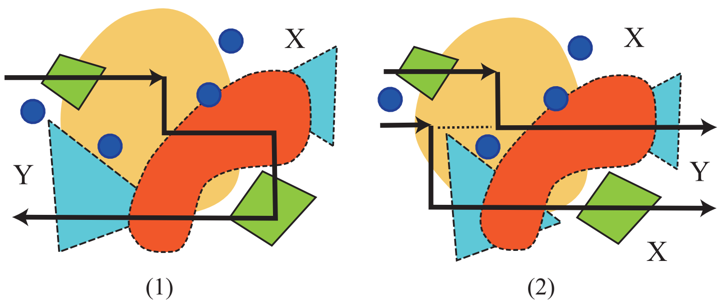

Figure 16: (1) 5 mass distributions $\mu_i, 1 \leq i \leq 5$, defined on the plane are simultaneously bisected by a path consisting of horizontal and vertical lines and having at most 4 turns, namely, $\mu_i(X) = \mu_i(Y)$ for all $1 \leq i \leq 5$; (2) Note that the path may go through infinity in the horizontal direction.

2.3 Balanced Partitions of the Plane

For R and B in the plane, if the plane is partitioned into n convex polygons so that every open polygon contains a fixed number of red points and of blue points, then we call such a partition a *balanced partition*, *equitable partition* or *equipartition*. The following theorem, easily obtained by induction and with the use of the Ham-sandwich Theorem, is a starting point of this subsection.

A graph drawn in the plane is called a *geometric graph* if every edge is a straight line segment. A matching each of whose edges joins a red point and a blue point is called a *bichromatic matching* or an *alternating matching*. A matching covering all given points is called a *perfect matching*.

Theorem 2.23 *Assume that R and B are in the plane and $|R| = |B| = n \geq 1$. Then (i) the plane can be partitioned into n convex polygons so that every open polygon contains exactly one red point and one blue point; and (ii) there exists a non-crossing geometric alternating perfect matching on $R \cup B$ (see (1) and (2) of Fig. 17).*

It is obvious that (i) of Theorem 2.23 implies (ii). On the other hand, (ii) does not imply (i) as shown in (4) of Fig. 17 [33]. We can also prove (ii) directly as follows. It is obvious that there exists a geometric bichromatic perfect matching on $R \cup B$, which may have some crossings (see (3) of Fig. 17). If we take such a matching with minimum total length of edges, then such a matching has no crossing (see (3) of Fig. 17).

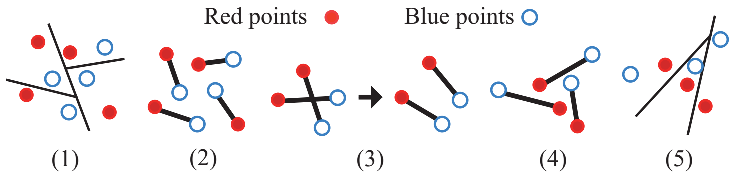

Figure 17: (1) A balanced partition; (2) A non-crossing geometric bichromatic perfect matching on $R \cup B$; (3) Two crossing edges and two non-crossing edges with a smaller sum of edge lengths; (4) A non-crossing geometric bichromatic perfect matching on $R \cup B$ for which there is no partition of the plane into three convex polygons such that each polygon contains one edge [33]; (5) A balanced convex partition of the red and blue points given in (4).

Theorem 2.23 can be generalized to Theorem 2.24, which is called *Balanced Partition Theorem* or *Equitable Partition Theorem*. Notice that Kaneko and Kano ([73] and [75]) proved the case where $a = 1, 2$ in Theorem 2.24 and conjectured that the theorem holds for $a \geq 3$. Later on, the said conjecture became a theorem with three independently written proofs (Bespamyatnikh, Kirkpatrick and Snoeyink [33], Ito et. al [71] and Sakai [112]). Note that in the following theorem, g stands for the number of groups (i.e., polygons).

Theorem 2.24 (Balanced Partition Theorem, [33], [71], [112]) *Let $a \geq 1$, $b \geq 1$ and $g \geq 1$ be integers. Assume that R and B are in the plane, and that $|R| = ag$ and $|B| = bg$. Then the plane can be partitioned into g convex polygons so that every open polygon contains exactly a red points and b blue points (see (1) of Fig. 18.)*

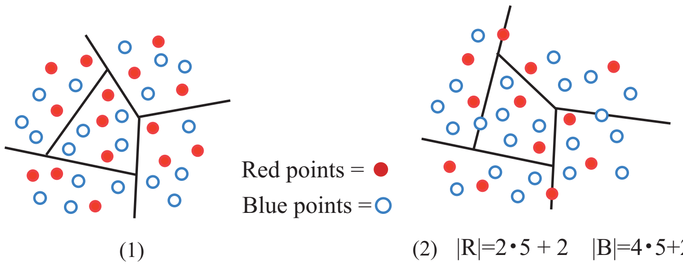

Figure 18: (1) A balanced partition with $a = 3$, $b = 4$ and $g = 5$; (2) A partition of Theorem 2.27 with $a = 2$, $b = 4$, $g = 5$, $|R| = 12$ and $|B| = 22$.

An algorithm for finding balanced partition is obtained as follows.

Theorem 2.25 (Bespamyatnikh, Kirkpatrick and Snoeyink [33]) *Let $a \geq 1$, $b \geq 1$ and $g \geq 1$ be integers. Assume that R and B are in the plane, and that $|R| = ag$ and $|B| = bg$. Let $n = |R| + |B|$. Then there is an algorithm for finding a balanced partition of $R \cup B$ in $O(n^{4/3} \log^3 n \log g)$ time.*

Balanced partition of two mass distributions (i.e., absolutely continuous Lebesgue measures) defined on the plane is also obtained. Note that a mass distribution μ with $\mu(\mathbb{R}^2) = 1$ is called a *probability measure.*

Theorem 2.26 (Balanced Partition Theorem (continuous), Bespamyatnikh, Kirkpatrick and Snoeyink [33], Sakai [112]) *Let $g \geq 1$ be an integer. Assume that two mass distributions μ_a and μ_b are defined on the plane $\mathbb{R}^2$. Then the plane can be partitioned into g convex polygons $X_1, X_2, \ldots, X_g$ so that*

$$\mu_a(X_i) = \frac{\mu_a(\mathbb{R}^2)}{g} \quad \text{and} \quad \mu_b(X_i) = \frac{\mu_b(\mathbb{R}^2)}{g} \qquad \text{for all } 1 \leq i \leq g.$$

In order to prove the Balanced Partition Theorem, they showed that if for every integer $1 \leq k \leq n-1$, there is no line l such that $\mathit{left}(l)$ contains exactly ak red points and bk blue points, then there exist three positive integers g_1, g_2, g_3 with $g_1 + g_2 + g_3 = g$ and three convex wedges W_1, W_2 and W_3 determined by three rays emanating from a common apex such that every open wedge W_i contains exactly ag_i red points and bg_i blue points. Moreover, it can be shown that $g_i \leq (2g/3)$ for every i.

In the case where $|R|$ and $|B|$ are not multiples of a and b, we can obtain a partition of the plane given in the following theorem.

Theorem 2.27 (Kano and Uno [90]) *Let $a \geq 1$ and $b \geq 1$ and $g \geq 2$ be integers. Assume that R and B are in the plane, and that $ag \leq |R| < (a+1)g$ and $bg \leq |B| < (b+1)g$. Then the plane can be partitioned into g convex polygons so that every open polygon contains exactly a red points and b blue points and that the remaining red and blue points, if any, lie on the boundary of the partition (see (2) of Fig. 18).*

Assume that $ag \leq |R| \leq (a+1)g$ and $bg \leq |B| \leq (b+1)g$. Without loss of generality, we may assume that $|R| - ag \leq |B| - bg$. Let $g_3 = |R| - ag$, $g_2 = |B| - bg - g_3$ and $g_1 = g - g_2 - g_3$. Then we can express $|R| = a(g_1 + g_2) + (a+1)g_3$ and $|B| = bg_1 + (b+1)(g_2 + g_3)$, where $g = g_1 + g_2 + g_3$, $g_1 \geq 0$, $g_2 \geq 0$ and $g_3 \geq 0$. In this case, we can find the following partition.

Theorem 2.28 (Kano and Uno [89]) *Let $a \geq 1$, $b \geq 1$, $g_1 \geq 0$, $g_2 \geq 0$ and $g_3 \geq 0$ be integers such that $g = g_1 + g_2 + g_3 \geq 1$. Assume that R and B are in the plane, and that $|R| = a(g_1 + g_2) + (a+1)g_3$ and $|B| = bg_1 + (b+1)(g_2 + g_3)$. Then the plane can be partitioned into $g_1 + g_2 + g_3$ convex polygons $X_1, \ldots, X_{g_1}, Y_1, \ldots, Y_{g_2}, Z_1, \ldots, Z_{g_3}$ so that every open X_i contains exactly a red points and b blue points, every open Y_j contains exactly a red points and $b+1$ blue points, and every open Z_k contains exactly $a+1$ red points and $b+1$ blue points (see (1) of Fig. 19).*

The following theorem gives another stronger result in a special case.

Theorem 2.29 (Kaneko, Kano and Suzuki [82]) *Let $c \geq 1$, $g \geq 0$ and $h \geq 0$ be integers such that $g + h \geq 1$. Assume that R and B are in the plane, and that $|R| = cg + (c+1)h$ and $|B| = (c+1)g + ch$. Then the plane can be partitioned into $g + h$ convex polygons $X_1, \cdots, X_g, Y_1, \ldots, Y_h$ so that every open X_i contains exactly c red points and $c+1$ blue points, and every open Y_j contains exactly $c+1$ red points and c blue points (see (2) of Fig. 19).*

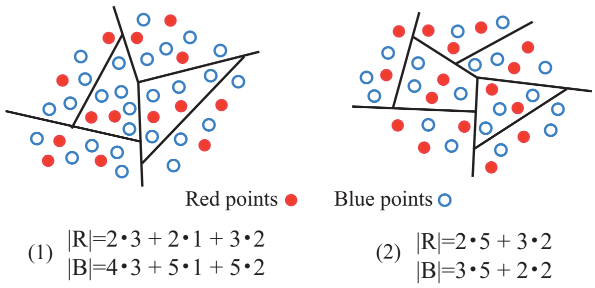

Figure 19: (1) A balanced partition given in Theorem 2.28 with $a = 2$, $b = 4$, $g_1 = 3$, $g_2 = 1$ and $g_3 = 2$; (2) A balanced partition of Theorem 2.29 with $c = 2$, $g = 5$ and $h = 2$.

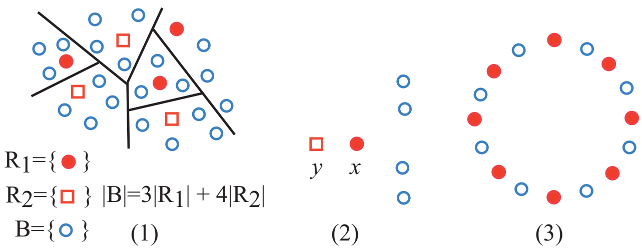

Figure 20: (1) A balanced partition of Theorem 2.30 with $|R_1| = 3$, $|R_2| = 3$, $b = 3$ and $|B| = 21$; (2) A configuration that has no partition of Theorem 2.30 with $R_1 = \{x\}$, $R_2 = \{y\}$, $b = 1$ and $b+2 = 3$; (3) A configuration that has no partition of Theorem 2.29 with $c = 3$, $c+2 = 5$, $g = h = 1$, $|R| = 3g+5h = 8$ and $|B| = 5g + 3h = 8$.

Fig. 20 (3) shows that Theorem 2.29 does not hold for c and $c+2$ instead of c and $c+1$. If $a = 1$ in Balanced Partition Theorem, the theorem also can be strengthened as follows.

Theorem 2.30 (Kaneko and Kano [81]) *Let $b \geq 1$, $g \geq 1$ and $h \geq 1$ be integers. Assume that R and B are in the plane, and that $R = R_1 \cup R_2$ (disjoint union), $|R_1| = g$, $|R_2| = h$ and $|B| = bg + (b+1)h$. Then the plane can be partitioned into $g + h$ convex polygons $X_1, \ldots, X_g, Y_1, \ldots, Y_h$ so that every open X_i contains exactly one red point of R_1 and b blue points and every open Y_j contains exactly one red point of R_2 and $b+1$ blue points (see (1) of Fig. 20) .*

Notice that Theorem 2.30 cannot be extended to b and $b+2$ as shown in (2) of. Fig. 20.

Conjecture 2.31 (Kaneko and Kano [77]) *Let $a, b, n_1, n_2, \ldots, n_g$ be positive integers, $g \geq 3$ be integers, and let $N = n_1 + n_2 + \cdots + n_g$. Assume that R and B are in the plane, $|R| = aN$, $|B| = bN$, and that $1 \leq n_i \leq \frac{N}{3}$ for every $1 \leq i \leq g$. Then the plane can be partitioned into g convex polygons $X_1, X_2, \ldots, X_g$ so that open X_i contains exactly an_i red points and bn_i blue points for every $1 \leq i \leq g$ (see (1) of Fig. 21).*

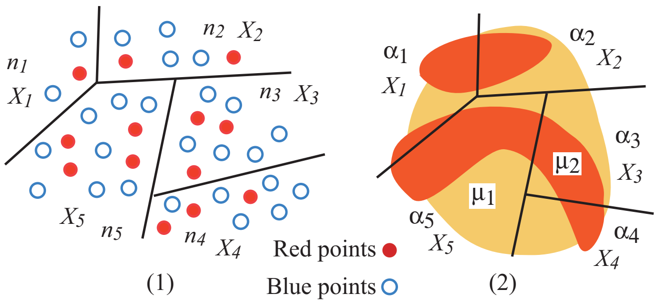

Figure 21: (1) A balanced partition of Conjecture 2.31 with $a = 1, b = 2, n_1 = 1, n_2 = 2, n_3 = n_4 = 3, n_5 = 4$ and $N = 13$; (2) A balanced partition of Conjecture 2.32.

The above conjecture is almost equivalent to this next conjecture, and if the conjecture is true, then the condition $1 \leq n_i \leq \frac{N}{3}$ is necessary and sharp [96].

Conjecture 2.32 *Let $g \geq 3$ be integer and $\alpha_1, \alpha_2, \ldots, \alpha_g$ be positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_g = 1$ and $0 < \alpha_i \leq \frac{1}{3}$ for every $1 \leq i \leq g$.*

Assume that two probability measures μ_a and μ_b are defined on the plane. Then the plane can be partitioned into g convex polygons $X_1, X_2, \cdots, X_g$ so that $\mu_a(X_i) = \mu_b(X_i) = \alpha_i$ for every $1 \leq i \leq g$ (see (2) of Fig. 21).

Theorem 2.33 (Holmsen, Kynčl and Valculescu [63]) *Let $k \geq 2$ and $n \geq 1$ be integers. Assume that R and B are in the plane, and that $|R|+|B| = kn$, $|R| \geq n$ and $|B| \geq n$. Then $R \cup B$ can be partitioned into n disjoint subsets $X_1, X_2, \ldots, X_n$ so that every X_i contains exactly k points including at least one red point and at least one blue point and $conv(X_i) \cap conv(X_j) = \emptyset$ for all $1 \leq i < j \leq n$ (see (1) of Fig. 22).*

Let S be a set of colored points in the plane. If a subset $X \subset S$ satisfies $conv(X) \cap S = X$, we say that $conv(X)$ is an *island spanned by S* or *S spans* $conv(X)$. Moreover, if $conv(X)$ is an island spanned by S and has exactly k points of S, then $conv(X)$ is called a *k-island*. If X contains at least j points with distinct colors, then X is called *j-colorful*. On the other hand, X is called an *m-colored point set* if the number of colors in X is at most m. So a 3-colored point set in the plane is a subset of $R \cup B \cup G$. By using these terminologies, Theorem 2.33 can be presented as follows, and a conjecture is given.

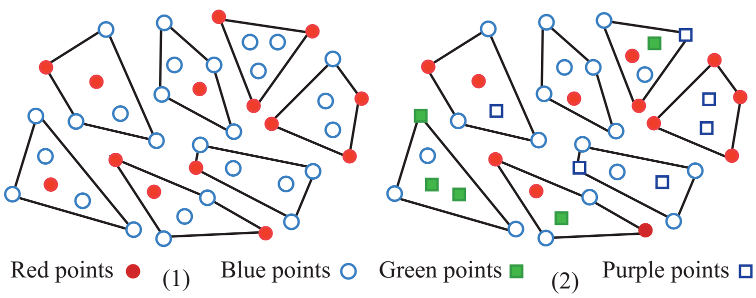

Figure 22: (1) A partition of Theorem 2.33, in other words, a 2-colored point set of 42 points that spans 7 pairwise disjoint 2-colorful 6-islands; (2) A partition of Conjecture 2.35, in other words, a 4-colored point set of 42 points that spans 7 pairwise disjoint 2-colorful 6-islands .

Theorem 2.34 (Holmsen, Kynčl and Valculescu [63]) *Let $k \geq 2$ and $n \geq 1$ be integers, and let S be a 2-colored point set in the plane in general position. If $|S| = kn$ and S contains at least n points of each color, then S spans n pairwise disjoint 2-colorful k-islands (see (1) of Fig. 22).*

Conjecture 2.35 (Holmsen, Kynčl and Valculescu [63]) *Let $k \geq 2$ and $m \geq 2$ be integers. Let S be an m-colored point set of kn points in the plane in general position. Suppose that S has a partition $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ such that every Y_i is 2-colorful and contains k points. Then X spans n pairwise disjoint 2-colorful k-islands (see (2) of Fig. 22).*

The above conjecture says that if S has a partition $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ having certain combinatorial properties (being 2-colorful and containing k points), then S can be partitioned into $Z_1 \cup Z_2 \cup \cdots \cup Z_n$ that has the same combinatorial properties as $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ together with a geometric property $conv(Z_i) \cap conv(Z_j) = \emptyset$ for all $i \neq j$.

A region X of a polygon P is said to be *relatively-convex* if for any two points in X, a geodesic path (shortest path) connecting them in P is contained in X (see Fig. 23).

Theorem 2.36 (Bose et. al [38]) *Let P be a polygon in the plane. Assume that R and B are in the interior of P. Then there exists a geodesic bisector in P. Namely, P can be partitioned into two relatively-convex regions by a geodesic bisector so that each region contains exactly $\lfloor \frac{|R|}{2} \rfloor$ red points and $\lfloor \frac{|B|}{2} \rfloor$ blue points in its interior (see (1) of Fig. 23).*

Theorem 2.37 (Bereg, Bose and Kirkpatrick [27]) *Let $a \geq 1$, $b \geq 1$ and $g \geq 2$ be integers. Let P be a polygon in the plane. Assume that R and B are in the interior of P and satisfies $|R| = ag$ and $|B| = bg$. Then P can be partitioned into g relatively-convex regions so that each region contains exactly a red points and b blue points in its interior (see (2) of Fig. 23).*

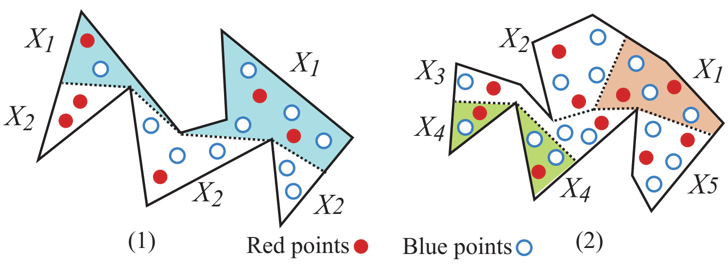

Figure 23: (1) A geodesic bisector of $R \cup B$ in a polygon, where two regions X_1 and X_2 are relatively-convex; (2) A balanced partition of a polygon into 5 relatively-convex regions each of which contains 2 red points and 3 blue points.

We now consider the following problem: how to divide a cake among the children attending a birthday party such that each child gets the same amount of cake and the same amount of icing. A cake is convex, has uniform height and is iced uniformly on the top and sides, and thus we can model the problem as follows: For a given convex set S in the plane and an integer $k \geq 2$, we want to partition S into k convex polygons so that each polygon has the same area and the same length of the perimeter of S. If $k = 2$, we can apply Ham-sandwich Theorem (continuous version) to S and the perimeter of S, and obtain the desired bisection. For $k \geq 3$, the following theorems give affirmative answer to this problem. Note that the perimeter of a convex set S is denoted by $\partial(S)$.

Theorem 2.38 (Akiyama et al. [14]) *Let S be a convex set in the plane with area* area(S) *and perimeter length* length(∂S). *Then for an integer $k \geq 3$, S can be partitioned into k convex subsets so that each subset has the same area* $\frac{1}{k}$area(S) *and the same perimeter length* $\frac{1}{k}$length(∂S) *of S (see (1) of Fig. 24).*

The partition given in the above theorem is called a *perfect k-partition* of a convex set S, and it has been generalized in the following result, which is also called a *general perfect partition*.

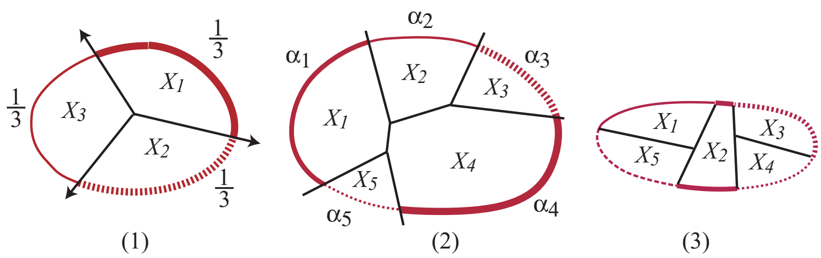

Figure 24: (1) A perfect 3-partition of a convex set given in Theorem 2.38; (2) A perfect partition of a convex set given in Theorem 2.39; (3) A partition that satisfies (i) and (ii) but not (iii) of Theorem 2.39.

Theorem 2.39 (Kaneko and Kano [78]) *Let S be a convex set in the plane with area* area(S) *and perimeter length* length(∂S). *Let $n \geq 2$ be an integer, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $0 < \alpha_i \leq \frac{1}{2}$ for all $1 \leq i \leq n$. Then S can be partitioned into n convex subsets $X_1, X_2, \ldots, X_n$ so that each X_i satisfies the following three conditions: (i) the area of X_i is $\alpha_i \cdot$* area(S)*; (ii) the length of $X_i \cap \partial(S)$ is*

$\alpha_i \cdot \text{length}(\partial S)$*; and (iii)* $X_i \cap \partial(S)$ *consists of one continuous curve (see (2) and (3) of Fig. 24)*

Assume that $R \cup B$ is in the plane, and every red point has weight α and every blue point has weight β, where α and β are positive integers such that $\alpha > \beta$. Then for any configuration of $R \cup B$ with weight $\alpha|R| + \beta|B| = 2\omega$, there exists a line that partitions the plane into two half-planes with weight ω if and only if either both $|R|$ and $|B|$ are even or $\alpha = 2\beta$ and $|B|$ is even [40] (see (1) of Fig. 25). Hence the case where $\alpha = 2\beta$ is important, which is essentially equivalent to the case where $\alpha = 2$ and $\beta = 1$.

Theorem 2.40 (Buot and Kano [40]) *Assume that* R *and* B *are in the plane, every red point has weight 2 and every blue point has weight 1. Let* $2|R| + |B| = n\omega$*, where both* n *and* ω *are positive integers (see Fig. 25).*
(i) If ω *is even, then for any* $R \cup B$ *with total weight* $2|R| + |B| = n\omega$*, there exists a partition of the plane into* n *convex polygons each with weight* ω*.*
(ii) If ω *is odd and* $|B| \geq n$*, then for any* $R \cup B$ *with total weight* $2|R| + |B| = n\omega$*, there exists a partition of the plane into* n *convex polygons each with weight* ω*. Note that if* ω *is odd, the condition* $|B| \geq n$ *is necessary for the existence of such a partition.*

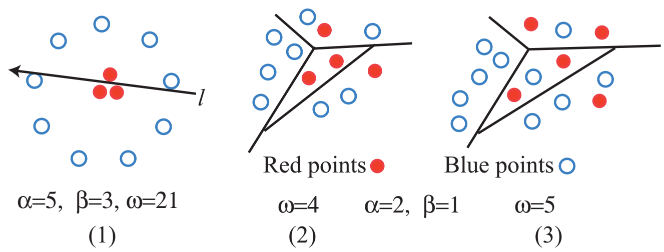

Figure 25: (1) A configuration with total weight 42 which has no bisector of weight; (2) A weight-equitable partition with even ω; (3) A weight-equitable partition with odd ω.

2.4 Balanced Partitions by Fans

For an integer $k \geq 2$, a *k-fan* is a partition of the plane into k open angular sectors $\sigma_1, \sigma_2, \ldots, \sigma_k$ by k rays $r_1, r_2, \ldots, r_k$ emanating from the same point p, called an *apex*, where the rays are labeled in clockwise order around the apex p, and the sector σ_i is determined by two consecutive rays r_i and r_{i+1}

for every $1 \leq i \leq k$ (σ_k is determined by r_k and $r_{k+1} = r_1$). If all the sectors σ_i are convex, then it is called a *convex k-fan.* (see (1) and (2) of Fig. 26). Hereafter, we deal with k-fans that are not necessarily convex. If an apex is at the infinity, then rays become parallel lines and one sector splits into two infinite regions and these two infinite regions form one sector (see (4) of Fig. 26).

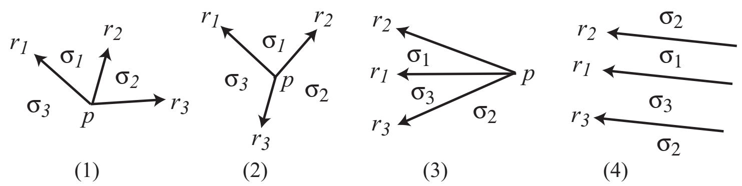

Figure 26: (1) A 3-fan; (2) A convex 3-fan: (3) A 3-fan with far apex; (4) A 3-fan with infinity apex, which consists of 3 parallel lines.

Recall that every measure μ on the plane $\mathbb{R}^2$ has the following properties: (i) μ is absolutely continuous with respect to the Lebesgue measure; and (ii) there is a bounded domain $D \subset \mathbb{R}^2$ such that $0 < \mu(D) = \mu(\mathbb{R}^2) < \infty$. Thus every open set X of $\mathbb{R}^2$ is measurable and $\mu(l) = 0$ for every line l. Such a measure is called *a mass distribution.* If $\mu(\mathbb{R}^2) = 1$, then such a μ is called a *probability measure.*

Let $n \geq 2$ be an integer, and let $\mu_1, \mu_2, \ldots, \mu_n$ be *probability measures* on the plane. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a vector with non-negative real components α_i such that $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$. We say that a k-fan with sectors $\sigma_1, \sigma_2, \ldots, \sigma_k$ *α-partitions* a probability measure μ_t if $\mu_t(\sigma_i) = \alpha_i$ for every $1 \leq i \leq k$.

Theorem 2.41 (Bárány and Matoušek [22]) *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a vector such that $\alpha_i \geq 0$ for every i and $\alpha_1 + \alpha_2 \cdots + \alpha_k = 1$. Then the following statements hold (see Fig. 27).*

1. *Any 2 probability measures can be simultaneously α-partitioned by a 2-fan for all α with $k = 2$. The apex of the 2-fan can be prescribed arbitrarily (see (1)).*

2. *Any 3 probability measures can be simultaneously α-partitioned by a 2-fan for $\alpha = (\frac{1}{2}, \frac{1}{2})$ and for $\alpha = (\frac{2}{3}, \frac{1}{3})$ (see(2), (3)).*

3. *Any 2 probability measures can be simultaneously α-partitioned by a 3-fan for $\alpha = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and for $\alpha = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ (see (4), (5)).*

4. *Any 2 probability measures can be simultaneously α-partitioned by a 4-fan for $\alpha = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. In particular, any 2 measures can be simultaneously α-partitioned by a 3-fan for $\alpha = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$ and for $\alpha = (\frac{3}{5}, \frac{1}{5}, \frac{1}{5})$ (see (6),(7), (8)).*

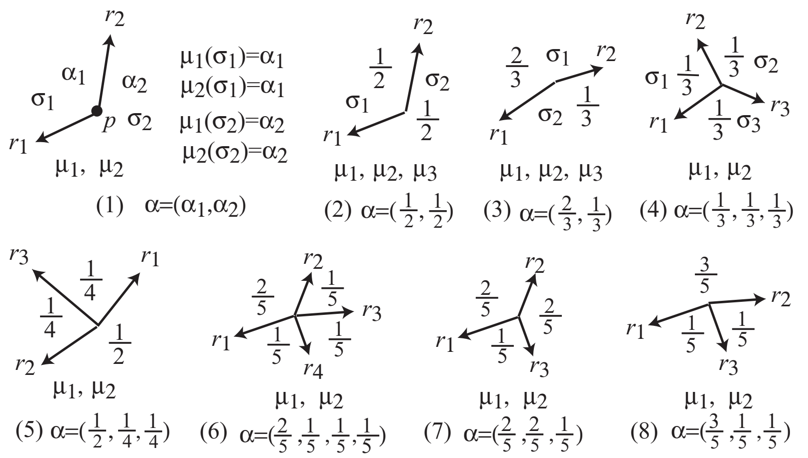

Figure 27: (1) A simultaneous (α_1, α_2)-partition of μ_1 and μ_2 by a 2-fan with prescribed apex p (2) A simultaneous $(\frac{1}{2}, \frac{1}{2})$-partition of μ_1, μ_2, μ_3 by a 2-fan; (3) A simultaneous $(\frac{2}{3}, \frac{1}{3})$-partition of μ_1, μ_2, μ_3 by a 2-fan; (6) A simultaneous $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ partition of μ_1 and μ_2 by a 4-fan.

The following theorem seems to follow from the statement 1 of Theorem 2.41, but is independent of it since the sector σ_1 with $\alpha_1 < \frac{1}{2}$ in (1) of Fig. 27 might be non-convex.

Theorem 2.42 (Aichholzer et al. [5]) *Assume that R and B are in the plane, and let $0 < \alpha \leq \frac{1}{2}$ be a real number. Then there exists a convex set that contains exactly $\lceil \alpha |R| \rceil$ red points and $\lceil \alpha |B| \rceil$ blue points. Moreover, there exists a convex set that contains exactly $\left\lceil \frac{|R|+1}{2} \right\rceil$ red points and $\left\lceil \frac{|B|+1}{2} \right\rceil$ blue points (see Fig. 28).*

Theorem 2.43 (Bárány and Matoušek [22]) *Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a vector such that $\alpha_i > 0$ for every i and $\alpha_1 + \cdots + \alpha_k = 1$. Then*

1. *For any $k \geq 2$ and any α, there are 4 probability measures that cannot be simultaneously α-partitioned by a k-fan (see (1) of Fig. 29) .*

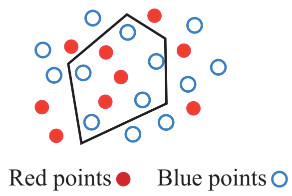

Figure 28: A convex set containing exactly $\lceil\alpha|R|\rceil = 3$ red points and $\lceil\alpha|B|\rceil = 5$ blue points, where $\alpha = \frac{1}{3}$, $|R| = 9$ and $|B| = 13$.

2. *For any $k \geq 3$ and any α, there are 3 probability measures that cannot be simultaneously α-partitioned by a k-fan (see (2) of Fig. 29).*

3. *For any $k \geq 5$ and any α, there are 2 probability measures that cannot be simultaneously α-partitioned by a convex k-fan.*

4. *For $k = 4$ and any α, there are 2 probability measures that cannot be simultaneously α-partitioned by a 4-fan.*

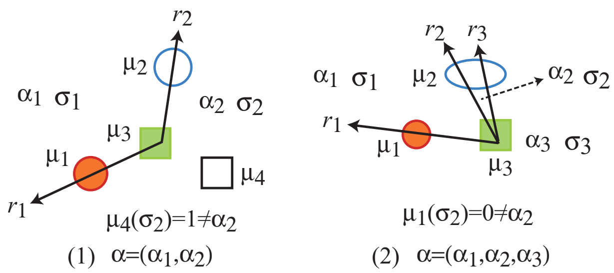

Figure 29: (1) A configuration of 4 measures which has no simultaneous (α_1, α_2)-partition by a 2-fan; (2) A configuration of 3 measures which has no simultaneous $(\alpha_1, \alpha_2, \alpha_3)$-partition by a 3-fan.

In order to show the statement (1) of Theorem 2.43, it suffices to show the non-existence for $k = 2$ since an α-partition by a k-fan implies an α'-partition by a $(k-1)$-fan by removing one ray.

Theorem 2.44 (Bárány and Matoušek [23]) *Assume that two probability measures μ_1 and μ_2 are give on the plane. Then there exists a 4-fan that simultaneously $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$-partitions both μ_1 and μ_2 (see (1) of Fig. 30).*

The discrete version of Statement 2 of Theorem 2.41 is strengthened as follows in the case where $\alpha = (\frac{1}{2}, \frac{1}{2})$.

Theorem 2.45 (Živaljević [119]) *Assume that 3 probability measures μ_1, μ_2 and μ_3 are given on the plane. Then for every positive real number α_1 and α_2 such that $\alpha_1 + \alpha_2 = 1$, there exists a 2-fan that simultaneously (α_1, α_2)-partition μ_1, μ_2 and μ_3 (see (2) of Fig. 30).*

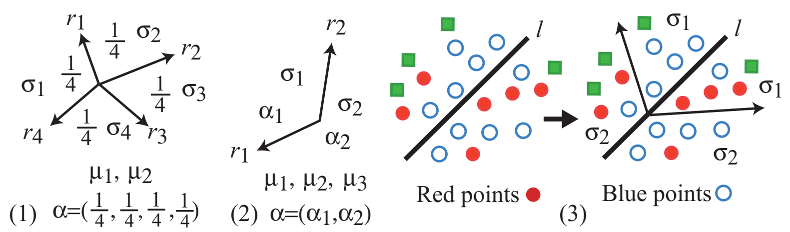

Figure 30: (1) A simultaneous $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$-partition of μ_1 and μ_2 by a 4-fan; (2) A simultaneous (α_1, α_2)-partition of μ_1, μ_2 and μ_3 by a 2-fan (3) A simultaneous $(\frac{1}{2}, \frac{1}{2})$-partition of R, B and G by a 2-fan with apex lying on a given line l.

The discrete version of Statement 2 of Theorem 2.41 is strengthened as follows. Also similar results on two measures defined on the plane, and on two measures defined on the unit sphere $\mathbb{S}^2$ are obtained [25].

Theorem 2.46 (Bereg [25]) *Assume that R, B and G are in the plane and that $|R|$, $|B|$ and $|G|$ are all even. Moreover a line l is given in the plane. Then there exists a 2-fan with an apex lying on l that simultaneously $(\frac{1}{2}, \frac{1}{2})$-partitions all R, B and G (see (3) of Fig. 30).*

We now consider a set S of points on the unit sphere $\mathbb{S}^2$ in the space $\mathbb{R}^3$. A point set S is said to be *in general position* if (i) no three points of S lie on the same great circle, and (ii) no two points of S are antipodal (i.e. their midpoint is the sphere's center). A k-fan on $\mathbb{S}^2$ is formed by a point $\boldsymbol{x} \in \mathbb{S}^2$ and k great semicircles $\ell_1, \ell_2, \ldots, \ell_k$ starting from $\boldsymbol{x}$ and ending at $-\boldsymbol{x}$, listed in clockwise order. The spherical lune σ_i is determined by ℓ_i and ℓ_{i+1} for all $1 \leq i \leq k$, where σ_k is determined by ℓ_k and $\ell_{k+1} = \ell_1$ (see (1) of Fig. 31). The k-fan is called *convex* if every dihedral angle is at most π. Given a probability measure μ on $\mathbb{S}^2$, the k-fan *equipartitions* μ if $\mu(\sigma_i) = \frac{1}{k}$ for all $1 \leq i \leq k$.

Theorem 2.47 (Bereg [25]) *Assume that R, B and G are on the sphere $\mathbb{S}^2$ in general position and $|R|$, $|B|$ and $|G|$ are all even. Then there exists a 2-fan with apex on a given great circle that simultaneously $(\frac{1}{2}, \frac{1}{2})$-partitions all R, B and G.*

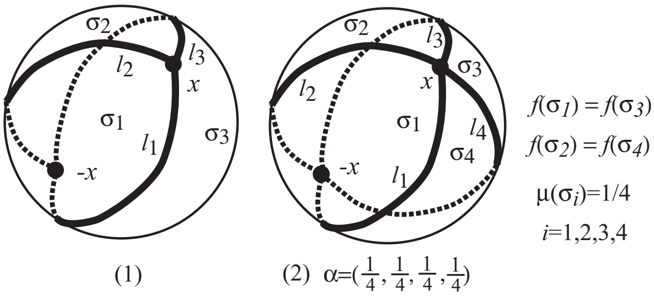

Figure 31: (1) A 3-fan on the unit sphere $\mathbb{S}^2$ with three great semicircles ℓ_1, ℓ_2, ℓ_3 with an apex x and 3 lunes $\sigma_1, \sigma_2, \sigma_3$; (2) A $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$-partition of μ that satisfies $f(\sigma_1) = f(\sigma_3)$ and $f(\sigma_2) = f(\sigma_4)$.

Theorem 2.48 (Blagojević and Blagojević [35]) *For any two probability measures μ_1 and μ_2 on the unit sphere $\mathbb{S}^2$, there exists a 3-fan that simultaneously $(\alpha, \alpha, 1-2\alpha)$-partitions both μ_1 and μ_2 for every $0 < \alpha < \frac{1}{2}$.*

Theorem 2.49 (Bárány, Blagojević and Blagojević [21]) *Assume μ is a probability measure on the unit sphere $\mathbb{S}^2$, and f is a continuous function defined on the lunes in $\mathbb{S}^2$. Then there is a convex 3-fan with lunes σ_1, σ_2 and σ_3 that satisfy*

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_3) \quad \textit{and} \quad f(\sigma_1) = f(\sigma_2) = f(\sigma_3).$$

Theorem 2.50 (Bárány, Blagojević and Blagojević [21]) *Let μ be a probability measure on the unit sphere $\mathbb{S}^2$, and f be a continuous function defined on the lunes in $\mathbb{S}^2$. Then*
(i) there exists a convex 4-fan that equipartitions μ and satisfies $f(\sigma_1) = f(\sigma_3)$ and $f(\sigma_2) = f(\sigma_4)$ (see (2) of Fig. 31).
(ii) there exists a convex 4-fan that equipartitions μ and satisfies $f(\sigma_1) = f(\sigma_2)$ and $f(\sigma_3) = f(\sigma_4)$.
(iii) there exists a convex 5-fan that equipartitions μ and satisfies $f(\sigma_1) = f(\sigma_2) = f(\sigma_3)$.
(iv) there exists a convex 5-fan that equipartitions μ and satisfies $f(\sigma_1) = f(\sigma_2) = f(\sigma_4)$.

The following negative result is concerning two measures defined on $\mathbb{R}^2$, but the same result holds for two measures defined on $\mathbb{S}^2$.

Theorem 2.51 (Bárány, Blagojević and Blagojević [21]) *There exist two mass distributions μ and τ on $\mathbb{R}^2$ such that there is no convex 5-fan that equipartitions μ and satisfies $\tau(\sigma_i) = \tau(\sigma_{i+1}) = \tau(\sigma_{i+2}) = \tau(\sigma_{i+3})$ for some $1 \leq i \leq 5$, where subscripts are taken modulo 5.*

3 Geometric Graphs

A graph drawn in the plane is called a *geometric graph* if all its edges are straight line segments. If no two edges of a geometric graph except possibly having a common endpoint cross, we call it a *non-crossing geometric graph* or a *planar straight line graph*.

A graph G with colored vertices is called *properly colored* if every edge joins two vertices of distinct colors. If the vertices of a properly colored graph G are colored by two colors, then G is called an *alternating graph* or a *bichromatic graph*.

Recall that R, B and G denote a set of red points, blue points, and green points in the plane, respectively. We always assume that R and B (or R, B and G) are in general position. Moreover, for k-colored point sets $Q_1, Q_2, \ldots, Q_k$ in the plane, we also assume that Q_1, $Q_2, \cdots, Q_k$ are in general position.

Theorem 2.23 on $R \cup B$ is generalized to multicolored point sets as follows.

Theorem 3.1 ([8], [86]) *Let $k \geq 2$ and $n \geq 1$ be integers. Assume that k-colored point sets $Q_1, Q_2, \ldots, Q_k$ are in the plane. Let $S = Q_1 \cup Q_2 \cup \cdots \cup Q_k$. If $|S| = 2n$ and $|Q_i| \leq n$ for every $1 \leq i \leq k$, then (i) there exists a non-crossing geometric properly colored perfect matching on S; and (ii) the plane can be partitioned into n convex polygons so that each open polygon contains exactly two points of S with distinct colors (see Fig. 32).*

A matching is said to be *monochromatic* if every edge joins two points of the same color. There does not always exist a non-crossing geometric monochromatic perfect matching on $R \cup B$ in the plane (see Fig. 33). Thus it is an interesting problem to determine the maximum number of points that can be covered by this type of matching. The following theorem gives one such answer.

Theorem 3.2 (Dumitrescu and Steiger [52]) *Assume that R and B are in the plane. Then there exists a non-crossing geometric monochromatic*

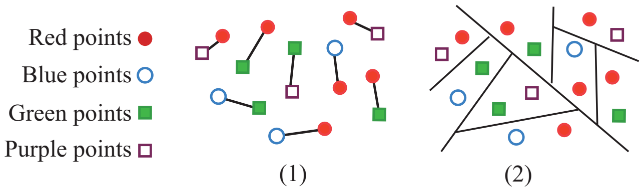

Figure 32: (1) A non-crossing geometric properly colored perfect matching on 4-colored point sets; (2) A partition of the plane into 8 convex polygons each of which contains 2 points with distinct colors.

matching that covers at least $\frac{5}{6} = 0.833\cdots$ *of* $R \cup B$*. Moreover, there exists* $R \cup B$ *for which no non-crossing geometric monochromatic matching covers more than* $\frac{155}{156} = 0.9935\cdots$ *of* $R \cup B$*.*

Improving the result by Dumitrescu and Steiger [52], Dumitrescu and Kaye obtained the following theorem.

Theorem 3.3 (Dumitrescu and Kaye [50]) *Assume that* R *and* B *are in the plane. Then there exists a non-crossing geometric monochromatic matching that covers at least* $\frac{6}{7} = 0.857\cdots$ *of* $R \cup B$*. Furthermore, there exists* $R \cup B$ *for which no non-crossing geometric monochromatic matching covers more than* $\frac{94}{95} = 0.9894\cdots$ *of* $R \cup B$*.*

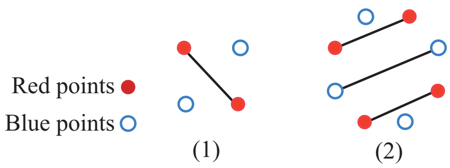

Figure 33: Two configurations of $R \cup B$ which cannot be covered by non-crossing geometric monochromatic matchings.

Garijo et. al [56, 57] studied problems of finding non-crossing geometric monochromatic k-factors of $R \cup B$, where a monochromatic k-factor is a k-regular graph whose edges join two vertices with the same color. In their papers, they allowed the use of *Steiner points*, and what they called *white points* whose positions on the plane are given in advance. Steiner and white points have no color assigned to them, until they are matched with a red or a blue point, inhering its color. Namely, white points have given positions and free colors, and Steiner points have free positions and free colors. Moreover, every non-crossing geometric monochromatic k-factor of $R \cup B$ must cover all points of $R \cup B$ but can use some white points and adding Steiner points. Note that $k \leq 5$ since the maximum degree of a regular planar graph is at most 5. Among other things, they prove the following results:

Theorem 3.4 (Garijo et. al [56]) *Assume that $S = R \cup B$ is in the plane and $|S| = n$. Then*
(i) n white points is always sufficient and sometimes necessary to obtain a non-crossing geometric monochromatic perfect matching of S.
(ii) for any positive integer m, $n-m$ white points and $\lfloor \frac{m}{3} \rfloor$ Steiner points are sufficient to obtain a non-crossing geometric monochromatic perfect matching of S (see (1) and (2) of Fig. 34).

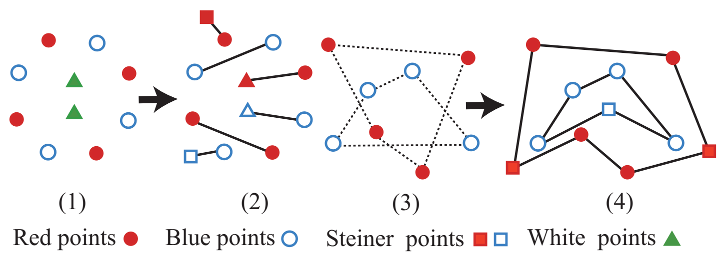

Figure 34: (1) $R \cup B$ and 2 white points; (2) A non-crossing geometric monochromatic perfect matching of $R \cup B$; (3) $R \cup B$; (4) Two non-crossing geometric monochromatic Hamiltonian cycles of R and B.

Theorem 3.5 (Garijo et. al [57]) *Assume that $S = R \cup B$ is in the plane and $|S| = n$. Then*
(i) $\lfloor \frac{n}{2} \rfloor$ Steiner points are sufficient and sometimes necessary to obtain two non-crossing geometric monochromatic Hamiltonian cycles of R and B (see (3) and (4) of Fig. 34).
(ii) $\frac{2n}{5}+4$ Steiner points suffice to obtain two non-crossing geometric monochromatic 2-factors of R and B.
(iii) $n+4$, $2n$ and $5n$ Steiner points suffice to construct non-crossing geometric monochromatic 3-, 4-, and 5-factors of S.

3.1 Alternating Paths

A topic that has received a lot of attention is that of finding long non-crossing geometric alternating paths whose vertices are in $R \cup B$. A path passing

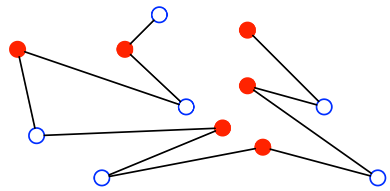

Figure 35: A non-crossing geometric alternating Hamiltonian path on $R \cup B$.

through all the points of $R \cup B$ is called a *Hamiltonian path* on $R \cup B$, see Figure 35

For $R \cup B$ given in the plane in convex position, in 1990, Akiyama and Urrutia [15] gave an $O(n^2)$-time algorithm to evaluate whether a non-crossing geometric alternating Hamiltonian path on $R \cup B$ exists. At around the same time, Erdős [97] conjectured that any set $R \cup B$ with $|R| = |B| = n$ in convex position always has a non-crossing geometric alternating path that covers $\frac{3n}{2}$ points of $R \cup B$. This conjecture was later proven to be false by Abellanas, García, Hurtado and Tejel [2] and Kynčl, Pach and Tóth [97].

Let us define $\ell(n)$ as the maximum number m such that for every $R \cup B$ with $|R| = |B| = n$ in convex position in the plane, there exists a non-crossing geometric alternating path that covers m points of $R \cup B$. It is easy to see that $\ell(n) \geq n$ since for a bisector l_1 of $R \cup B$, we may assume that $\mathit{left}(l_1)$ contains k red points ($k \geq n/2$), and so $\mathit{right}(l_1)$ contains k blue points, and hence there exists a non-crossing geometric path that covers these k red points and k blue points [97] (see (1) of Fig. 36).

On the other hand, consider the configuration $R \cup B$ in (2) of Fig. 36 for which $|R| = |B| = n = 4k$ and a longest non-crossing geometric alternating path covers $6k + 2 = \frac{3n}{2} + 2$ points [97]. Thus $\ell(n) \leq \frac{3n}{2} + 2$. Actually they obtained the following upper and lower bounds of $\ell(n)$.

Theorem 3.6 (Kynčl, Pach and Tóth [97]) *There exist constants $c, c' > 0$ such that*

$$n + c\sqrt{\frac{n}{\log n}} < \ell(n) < \frac{4}{3}n + c'\sqrt{n}.$$

They made the following conjecture.

Conjecture 3.7 ([97]) *The upper bound of Theorem 3.6 is asymptotically tight, namely,*

$$\left|\ell(n) - \frac{4}{3}n\right| = o(n)$$

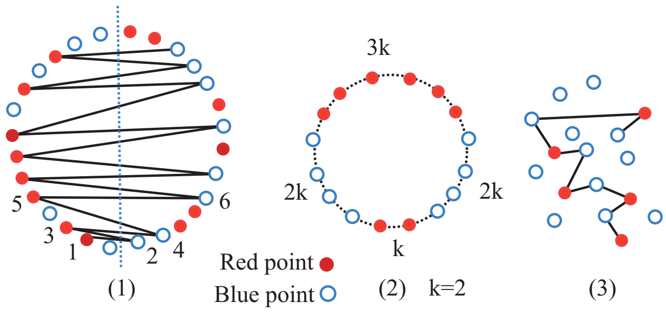

Figure 36: (1) A non-crossing geometric alternating path that covers at least n points of $R \cup B$ with $|R| = |B| = n$; (2) A configuration of $R \cup B$ with $|R| = |B| = n = 4k$ in which a longest non-crossing geometric alternating path covers $6k + 2 = \frac{3n}{2} + 2$ points [97]; (3) A non-crossing geometric alternating path that covers $B' \cup R$, where $B' \subset B$.

Theorem 3.8 (Kano and Kaneko [76]) *Assume that $R \cup B$ in the plane and $|R| = n \geq 3$.*
(i) If $|B| \geq (n+1)(2n-4)+1$, then we can find a subset $B' \subset B$ with n points such that there exists a non-crossing geometric alternating path on $R \cup B'$ (see (3) of Fig. 36).
(ii) There exists a configuration of $R \cup B$ with $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ for which there is no non-crossing geometric alternating path on $R \cup B'$ for any $B' \subset B$ with $|B'| = n$.

Cibulka et al. [42] showed that if $R \cup B$ forms a *double-chain*, then there exists a non-crossing geometric alternating Hamiltonian path on $R \cup B$ (see Fig. 37).

Theorem 3.9 (Cibulka et al. [42]) *Assume that $R \cup B$ lies on a double-chain $\mathcal{C}_1 \cup \mathcal{C}_2$, $|\,|R| - |B|\,| \leq 1$ and $|\mathcal{C}_i \cap (R \cup B)| \geq \frac{1}{5}|R \cup B|$ for $i = 1, 2$. Then there exists a non-crossing geometric alternating Hamiltonian path on $R \cup B$ (see Fig. 37).*

The alternating path problem has also been studied for the case when each edge can have one bend, that is, when an edge can be a polygonal line consisting of two line segments with a common endpoint.

Theorem 3.10 (Di Giacomo el al. [49]) *Assume that $R \cup B$ is in the plane and $|R| = |B|$. Then there exists a non-crossing Hamiltonian alternating path on $R \cup B$ each of whose edges has at most one bend.*

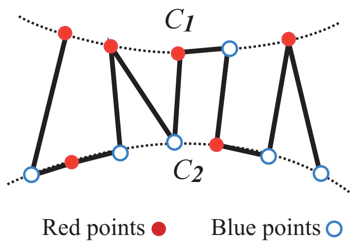

Figure 37: $R \cup B$ that forms a double-chain $\mathcal{C}_1 \cup \mathcal{C}_2$, and a non-crossing geometric alternating path that covers $R \cup B$.

3.2 Hamiltonian Cycles

A cycle passing through all points of a set S is called a *Hamiltonian cycle* on S. Allowing for crossings among the edges of a geometric graph, we have the following result:

Theorem 3.11 (Kaneko, Kano and Yoshimoto [83]) *Assume that $R \cup B$ is in the plane and $|R| = |B| = n \geq 2$. Then there exists a geometric alternating Hamiltonian cycle on $R \cup B$ that has at most $n - 1$ crossings (see (1) and (2) of Fig.38). This upper bound on the number of crossings is sharp.*

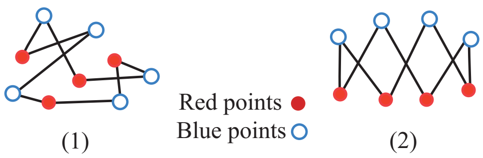

Figure 38: (1) A geometric alternating Hamiltonian cycle on $R \cup B$ with 3 crossings; (2) A geometric alternating Hamiltonian cycle on $R \cup B$ with $|R| - 1$ crossings.

A geometric graph is called a 1-*plane* if every its edge is crossed by at most one other edge. The problem of finding 1-plane alternating geometric paths and cycles on $R \cup B$ was first studied by Claverol et.al [44], and their results were improved in the recent paper by Claverol et.al [43]. Thus we mention only the recent result in [43].

Let $\tau(R, B)$ denote the number of unordered pairs $\{x, y\}$ of vertices of $conv(R \cup B)$ such that one of $\{x, y\}$ is red and the other is blue, and xy is an edge of $conv(R \cup B)$ (see Fig. 39). A *run* of $R \cup B$ is a maximal set of consecutive points of the same color on the boundary of $conv(R \cup B)$. Then $\tau(R, B)$ is equal to the number of runs of $R \cup B$ if the vertices of $conv(R \cup B)$ are not monochromatic, otherwise $\tau(R, B) = 0$ and the number of runs is one.

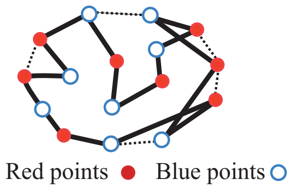

Figure 39: $R \cup B$ with $|R| = |B| = 8$ and $\tau(R, B) = 6$, and a 1-plane Hamiltonian alternating cycle on $R \cup B$ with 2 crossings.

Theorem 3.12 (Claverol et al. [43]) *Assume that $R \cup B$ is in the plane and that $|B| = |R| = n \geq 2$. Then there exists a 1-plane Hamiltonian alternating cycle on $R \cup B$ with at most*

$$n - \max\left\{\frac{\tau(R, B)}{2}, 1\right\} \quad \textit{crossings (see Fig. 39).}$$

3.3 Alternating Paths in k-Colored Point Sets

While obtaining sharp bounds on the length of non-crossing geometric alternating paths for 2-colored point sets in convex position has been elusive so far, the problem for 3-colored point sets (and in general k-colored, k odd) has been solved in full;

Theorem 3.13 (Merino, Salazar and Urrutia [103]) *Assume that $R \cup B \cup G$ is in the plane in convex position and that $|R| = |B| = |G| = n$. Then there exists a non-crossing geometric alternating path of length at least $2n+1$. This bound is sharp (see Fig. 40).*

For k-colored point sets in convex position, they proved the following:

Theorem 3.14 (Merino, Salazar and Urrutia [103]) *Let $k \geq 3$ be an integer. Assume that a set $Q_1 \cup Q_2 \cup \cdots \cup Q_k$ of k-colored point sets is in the plane in convex position, and that $|Q_1| \geq \cdots \geq |Q_k|$ and $|Q_1| + \cdots + |Q_k| = n$. Then there exists a non-crossing geometric alternating path of length at least $n - |Q_1|$.*

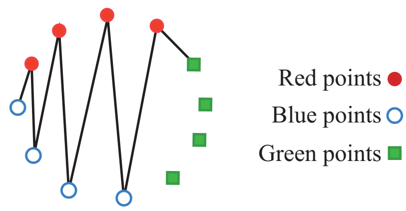

Figure 40: A point configuration achieving the bounds in Theorem 3.13.

3.4 Spanning Trees

One of the first results on bichromatic spanning trees for bicolored point sets is the following.:

Theorem 3.15 (Abellanas et al.[1]) *Assume that $R \cup B$ is in the plane and $|R| = |B|$. Then there exists a non-crossing geometric bichromatic spanning tree on $R \cup B$ having maximum degree at most $O(\log |R|)$.*

This result was improved by Kaneko.

Theorem 3.16 (Kaneko [72]) *Assume that $R \cup B$ is in the plane and $|R| = |B|$. Then there exists a non-crossing geometric bichromatic spanning tree on $R \cup B$ whose maximum degree is at most 3. The upper bound 3 is sharp (see (1) of Fig. 41).*

The following theorem is a generalization of the above theorem, and solves the outstanding problem on this topic.

Theorem 3.17 (Biniaz, Bose, Maheshwari and Smid [34]) *Assume that $R \cup B$ is in the plane and $|R| \leq |B|$. Then there exists a non-crossing geometric bichromatic spanning tree on $R \cup B$ whose maximum degree is at most*

$$\max\left\{3, \left\lceil \frac{|B|-1}{|R|} \right\rceil + 1\right\}.$$

This upper bound is sharp (see (2) of Fig.41).

Theorem 3.18 (Hoffmann and Tóth [65]) *Let S be a set of colored points in the plane in general position, and let H be a non-crossing geometric properly colored disconnected graph on S having no isolated vertices. Then there exists a non-crossing properly colored geometric connected graph G on S that is obtained from H by adding some new edges joining two vertices with distinct colors and satisfies $\deg_G(v) \leq \deg_H(v)+2$ for every vertex v. Moreover, this degree condition is tight (see Fig. 42).*

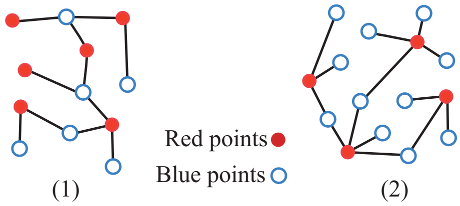

Figure 41: (1) A non-crossing geometric bichromatic spanning tree on $R \cup B$ with maximum degree 3; (2) A non-crossing geometric bichromatic spanning tree on $R \cup B$ with maximum degree $4 = \left\lceil \frac{|B|-1}{|R|} \right\rceil + 1$.

A non-crossing geometric properly colored connected graph G given in Theorem 3.18 is called an *encompassing graph of* H.

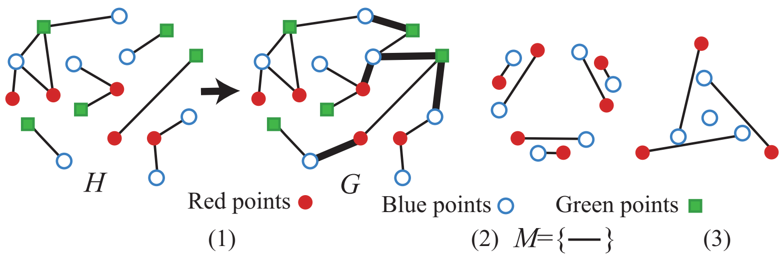

Figure 42: (1) A non-crossing geometric properly colored disconnected graph H without isolated vertices, and an encompassing graph G of H; (2) A non-crossing geometric bichromatic matching M for which there exists no non-crossing geometric bichromatic Hamiltonian path containing M, which implies the degree condition $\deg_G(v) \leq \deg_H(v) + 2$ in Theorem 3.18 is necessary; (3) A configuration which shows the necessity of the condition that H has no isolated vertex.

If $R \cup B$ with $|R| = |B|$ is given in the plane, then by Theorem 2.23, there is a non-crossing geometric bichromatic perfect matching M on $R \cup B$. Applying Theorem 3.18 to M, we can obtain a non-crossing geometric bichromatic spanning tree on $R \cup B$ with maximum degree at most 3. Namely, another proof of Theorem 3.16 is obtained. Moreover, Theorem 3.17 is proved by using Theorem 3.18.

For a given $R \cup B$ in the plane, we say that a graph G *covers* $R \cup B$ with bichromatic edges if there exists a non-crossing geometric graph which is isomorphic to G, whose vertex set is $R \cup B$ and each of whose edges joins a red point to a blue point. Analogously, for a fixed graph G of k vertices, we say that $R \cup B$ has a *bichromatic G-covering* or *can be G-covered with bichromatic edges* if $|R \cup B| = t \cdot k$ for some integer $t \geq 1$, and the graph G_t resulting from the union of t copies of G covers $R \cup B$.

Theorem 3.19 (Abrego et al. [3]) *Let $g \geq 0$ and $h \geq 0$ be integers such that $g + h \geq 1$. Assume that $R \cup B$ is in the plane in convex position, $|R| = 3g + h$ and $|B| = g + 3h$. Then at least $|R \cup B| - 4$ points of $R \cup B$ can be $K_{1,3}$-covered with bichromatic edges. This bound is best possible (see (2) of Fig. 43).*

bichromaitc

Theorem 3.20 (Abrego et al. [3]) *Assume that R and B are in the plane and that $|B| \leq |R| \leq 3|B|$. Then at least $\frac{8}{9}(|R \cup B| - 8)$ points can be $K_{1,3}$-covered with bichromatic edges (see (1) of Fig. 43).*

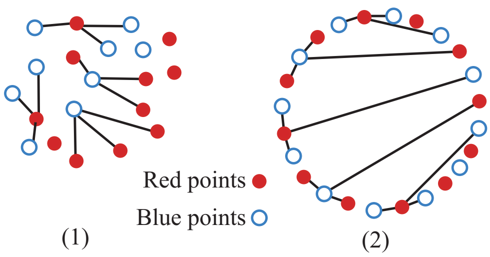

Figure 43: (1) For $R \cup B$ in the plane, at least $\frac{8}{9}(|R \cup B| - 8)$ points are $K_{1,3}$-covered with bichromatic edges; (2) For $R \cup B$ in convex position with $|R| = |B| = 3 \cdot 3 + 3$, at least $|R \cup B| - 4$ points are $K_{1,3}$-covered with bichromatic edges.

3.5 Monochromatic Spanning Trees on Colored Point Sets

Given a point set S in the plane, a non-crossing geometric spanning tree on S is briefly called an *S-spanning tree*. Tokunaga [117] studied the following problem: Given a set $R \cup B$ in the plane, find an R-spanning tree and a B-spanning tree such that the number of intersections of their edges is as small as possible. Recall that $\tau(R, B)$ denotes the number of unordered pairs $\{x, y\}$ of vertices of $conv(R \cup B)$ such that one of $\{x, y\}$ is red and the other is blue, and xy is an edge of $conv(R \cup B)$.

Theorem 3.21 (Tokunaga [117]) *Assume that R and B are in the plane. Then there exist an R-spanning tree and a B-spanning tree whose edges intersect exactly*

$$\max\left\{\frac{\tau(R,B)-2}{2}, 0\right\}$$

times. This is optimal (see Fig. 44).

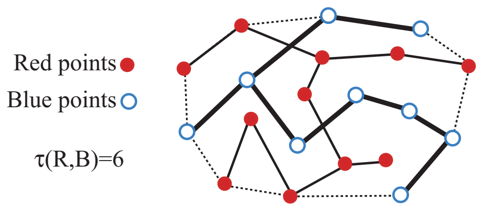

Figure 44: $R \cup B$ with $\tau(R,B) = 6$, and an R-spanning tree and a B-spanning tree, whose edges intersect 2 times.

The following result concerns the problem of finding non-crossing geometric monochromatic spanning trees on multicolored point sets.

Theorem 3.22 (Kano, Merino and Urrutia [85]) *Let $k \geq 3$ be an integer. Assume that k-colored point sets $Q_1, Q_2, \ldots, Q_k$ are in the plane. Then for each $Q_i, 1 \leq i \leq k$, we can find a Q_i-spanning tree T_i, $i = 1, \ldots, k$, such that the total number of intersections among the edges of $T_1, T_2, \ldots, T_k$ is at most*

$$(k-1)\left(n - \frac{k}{2}\right), \qquad \textit{where} \quad |Q_1 \cup \ldots \cup Q_k| = n.$$

This bound is tight within a factor $\frac{3}{2}$ from the optimal solution (see [98]).

In addition, they also proved that if $|R \cup B| = n$, then the minimum weight spanning trees of R and B intersect at most $8n$ times, where the *weight of an edge* is its length, and the *weight of a tree* is the sum of the weights of its edges. They posed the following problem that remains open:

Problem 3.23 (Kano, Merino and Urrutia [85]) *Is it true that the edges of the minimum weight spanning trees of any two point sets R and B with $|R \cup B| = n$ intersect at most $2n - c$ times for some constant c.*

Given a set S of even number of points in the plane, let M_S denote a geometric perfect matching on S. Then the *weight of a matching* of M_S

is defined to be the sum of the lengths of its edges. Given R and B with $|R| = 2r$, and $|B| = 2b$, let M_R and M_B be the minimum weight perfect matchings on R and B, respectively. Then M_R and M_B are non-crossing, and the following holds:

Theorem 3.24 (Merino, Salazar and Urrutia [104]) *Assume that R and B are in the plane, $|R| = 2r$ and $|B| = 2b$. Then the edges of M_R and M_B intersect at most $r + b - 1$ times. The bound is tight (see Fig. 45).*

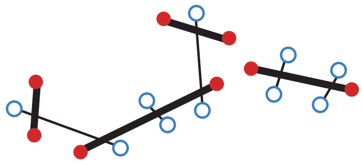

Figure 45: Two minimum weight matching M_R and M_B on R and B, respectively, which intersect 7 times.

The intersection graph of M_R and M_B is the graph whose vertices are the edges of M_R and M_B, two of which are adjacent if they are incident to a common vertex. The proof of Theorem 3.24 follows from the next Lemma:

Lemma 3.25 (Merino, Salazar, and Urrutia [104]) *The intersection graph of M_R and M_B is a forest (i.e., every component is a tree).*

For multicolored point sets in the plane, we have the following result.

Theorem 3.26 (Merino, Salazar and Urrutia [104]) *Let $k \geq 2$ be an integer. Assume that k-colored point sets $Q_1, Q_2, \dots, Q_k$ are in the plane, and that $|Q_i|$ is even for every $1 \leq i \leq k$ and $|Q_1| + |Q_2| + \cdots + |Q_k| = 2n$. Let M_i be a minimum weight perfect matching on Q_i for every $1 \leq i \leq k$. Then the edges of $M_1 \cup M_2 \cup \cdots \cup M_k$ intersect at most $(k-1)n - \frac{k(k-1)}{2}$ times. The bound is sharp.*

4 Empty Polygons and Balanced Lines

In this section, we first consider problems on monochromatic empty polygons (i.e., monochromatic holes) on colored point sets in the plane. Recall that R, B and G always denote *a set of red points, a set of blue points* and *a set* *of green points*, respectively. Moreover, we always assume that R and B (or R, B and G) are in the plane in general position.

Let S be a set of points in the plane in general position, and P be a polygon whose vertices are contained in S. If P has k-vertices, then P is called a *k-gon*. Of course, a 3-gon and 4-gon are also called a triangle and quadrilateral, and so on. A polygon P is said to be *empty* if P contains no point of S in its interior. An empty k-gon is often called a *k-hole*. Note that in some papers a k-hole needs to be convex, but in this paper, a k-hole might be non-convex.

Let X be a subset of S. Then X is called *empty* if $conv(X)$ is empty (i.e., no point of S lies in the interior of $conv(X)$), and X is said to be *convex* if the set of vertices of $conv(X)$ is X (see Fig. 46). An empty convex subset X of S with $|X| = k$ is often called a *convex k-hole*. A family of empty subsets $\{Y_1, Y_2, \ldots, Y_k\}$ of S is called *compatible* if any two $conv(Y_i)$ and $conv(Y_j)$ have disjoint relative interiors, that is, if $conv(Y_i)$ and $conv(Y_j)$ have no common interior point (see (2) of Fig. 46).

It is not hard to show that every $R \cup B$ of 10 points has an empty monochromatic triangle from the fact that every set of 10 points contains an empty convex pentagon. Grima et al. [61] showed that 9 points are necessary and sufficient for every $R \cup B$ to have a monochromatic empty triangle. The following theorems can be considered as variations of Erdős-Szekeres Theorem for colored point sets in the plane.

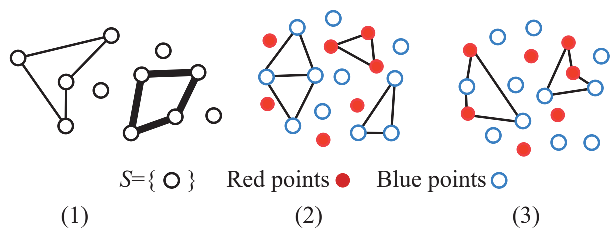

Figure 46: (1) A point set S, a non-convex empty quadrilateral (i.e., a non-convex 4-hole) drawn with lines, and a convex empty quadrilateral (i.e., a convex 4-hole) drawn with bold lines; (2) Four compatible empty monochromatic triangles in $R \cup B$; (3) Two balanced 4-holes in $R \cup B$.

Theorem 4.1 (Aichholzer et al. [9]) *Assume that R and B are in the plane and $|R \cup B| = n$. Then $R \cup B$ has $\Omega(n^{5/4})$ empty monochromatic triangles, namely, $R \cup B$ has at least $cn^{5/4}$ empty monochromatic triangles for sufficiently large n and for a constant positive real number c.*

They conjectured the following:

Conjecture 4.2 (Aichholzer et al. [9]) *Assume that R and B are in the plane and $|R \cup B| = n$. Then $R \cup B$ has $\Omega(n^2)$ empty monochromatic triangles.*

Theorem 4.3 (Pach and Tóth [110]) *Assume that R and B are in the plane and that $|R \cup B| = n$. Then $R \cup B$ has $\Omega(n^{4/3})$ empty monochromatic triangles.*

Theorem 4.4 (Devillers, Hurtado, Károlyi and Seara [47]) *Assume that R and B are in the plane and $|R \cup B| = n \geq 5$. Then $R \cup B$ has at least $\lceil \frac{n}{4} \rceil - 2$ compatible monochromatic empty triangles (see (2) of Fig. 46). Moreover, this bound is tight up to a constant factor.*

For 3-colored point sets, the following negative result holds.

Theorem 4.5 (Devillers, Hurtado, Károlyi and Seara [47]) *There exists an arbitrarily large set $R \cup B \cup G$ in the plane that has no monochromatic empty triangle.*

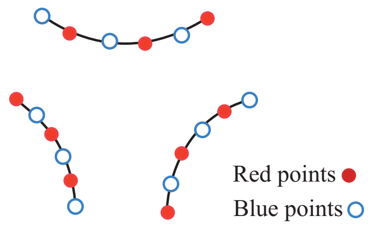

Figure 47: A set $R \cup B$ with 18 points having no monochromatic 4-hole.

We next consider empty monochromatic quadrilaterals (i.e., monochromatic 4-holes). Devillers et al. [47] obtained $R \cup B$ with 18 points that has no monochromatic 4-hole (see Fig. 47). Huemer and Seara [67] found $R \cup B$ with 36 points that has no monochromatic convex 4-hole. Moreover, Koshelev [95] obtained another $R \cup B$ of 46 points having no monochromatic convex 4-hole. Devillers et al. [47] proposed the following conjecture on monochromatic quadrilaterals.

Conjecture 4.6 (Devillers et al. [47]) *Every $R \cup B$ in the plane with sufficiently large number of points contains a monochromatic convex 4-hole.*

The above conjecture remains open, but the following theorem was obtained.

Theorem 4.7 (Aichholzer et al. [11]) *Every $R \cup B$ in the plane with $|R \cup B| \geq 2760$ contains a monochromatic 4-hole.*

For given R and B in the plane, if a 4-hole has 2 red vertices and 2 blue vertices, then it is called a *balanced 4-hole* (see (3) of Fig. 46). The following theorem shows the existence of balanced 4-holes.

Theorem 4.8 (Bereg et. al [29]) *Assume that R and B are in the plane and $|R| = |B| = n$. Then $R \cup B$ has at least $\frac{n^2-4n}{12}$ balanced 4-holes. This bound is tight up to a constant factor.*

If we consider 5-holes, the situation changes as the following theorem shows.

Theorem 4.9 (Devillers, Hurtado, Karolyi, Seara [47]) *For every integer n, there exists $R \cup B$ in the plane with $|R \cup B| = n$ that contains no convex monochromatic 5-hole.*

We now turn our attention from holes to balanced lines. Assume that R and B are in the plane and $|R| = |B|$. In this case, a line l is called a *balanced line* if it passes through one red point and one blue point and *left*(l) contains the same number of red points as blue points, that is, *left*(l) contains k red points and k blue points for some integer $1 \leq k \leq |R| - 1$ (see (1) of Fig. 48).

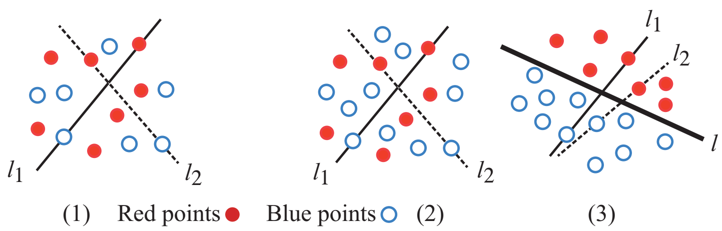

Figure 48: (1) $R \cup B$ with $|R| = |B|$ and two balanced lines l_1 and l_2; (2) $R \cup B$ with $|B| = |R| + 4$ and two generalized balanced lines l_1 and l_2; (3) $R \cup B$ with $|B| = |R| + 4$ which is separated by a line l, and two generalized balanced lines l_1 and l_2.

The following theorem by Pach and Pinchasi [109] settled a conjecture of George Baloglou [20].

Theorem 4.10 (Pach and Pinchasi [109]) *Assume that R and B are in the plane and $|R| = |B| = n$. Then $R \cup B$ has at least n balanced lines. This bound is best possible.*

Assume that R and B are in the plane and $|B| = |R| + 2\delta$ for some positive integer δ. In this case, a line is called a *generalized balanced line* if it passes through one red point and one blue point and *left*(l) contains k red points and $k + \delta$ blue points for some integer $1 \leq k \leq |R| - 1$ (see (2) and (3) of Fig. 48). For generalized balanced lines, the following theorem was obtained by Sharir and Welz [113] and by Orden, Ramos and Salazar [105] with different proof techniques.

Theorem 4.11 ([113], [105]) *Assume that R and B are in the plane and $|R| = n$ and $|B| = n + 2\delta$, where $\delta \geq 0$ is an integer. Then $R \cup B$ has at least n generalized balanced lines. Equality holds if R and B are separated by a line (see (3) of Fig. 48).*

5 Miscellaneous Topics

In this section, we collect some research topics on colored point sets in the plane, which are not dealt with in previous sections. Recall that R, B and G always denote *a set of red points, a set of blue points* and *a set of green points*, respectively. Moreover, we assume that R and B (or R, B and G) are in the plane in general position unless explicitly stated otherwise.

5.1 Compatible Graphs

For two non-crossing geometric graphs H_1 and H_2 on the same point set in the plane, if $H_1 \cup H_2$ has no crossing, then we say that *H_1 is compatible with H_2* or *H_1 and H_2 are compatible.* Assume that $R \cup B$ is in the plane and $|R| = |B|$. Then there are non-crossing geometric alternating perfect matchings on $R \cup B$, or simply referred to as *RB-matchings* (see Fig. 49). The *transformation graph* of RB-matchings contains one vertex for each RB-matching and an edge joining two such vertices if and only if the corresponding two RB-matchings are compatible.

Theorem 5.1 (Aloupis, Barba, Langerman and Souvaine [19]) *Assume that R and B are in the plane and $|R| = |B| = n$. Then the transformation graph of RB-matchings is connected.*

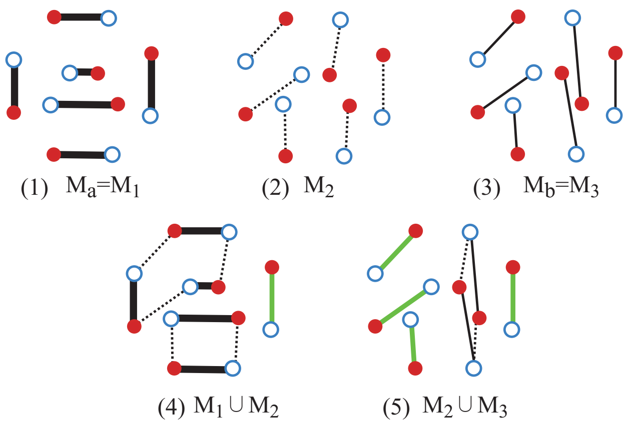

Figure 49: (1) A RB-matching $M_a = M_1$; (3) A RB-matching $M_b = M_3$; (2) A RB-matching M_2 such that M_1 and M_2 are compatible (see (4)) and M_2 and M_3 are compatible (see (5)).

In other words, Theorem 5.1 tells us that given any two RB-matchings M_a and M_b on $R \cup B$, there exists a sequence of RB-matchings $M_a = M_1, M_2, \ldots,$ $M_k = M_b$ such that $M_i \cup M_{i+1}$ contains no crossing (see Fig. 49). In [19], the authors introduced the problem of determining the *diameter* of the transformation graph of $R \cup B$. This was solved as follows:

Theorem 5.2 (Aichholzer el al. [7]) *The diameter of the transformation graph of RB-matchings is at most $2n$, where $|R| = |B| = n$. The bound is asymptotically tight.*

5.2 Polygon Enclosing or Excluding Monochromatic Points

We say that a polygon P *encloses* a point set S if all the points of S belong to the interior of P. Given R and B in the plane, we consider a problem of finding a polygon with vertex set R that encloses as many blue points as possible. If there is a polygon with vertex set R that encloses B, then B is contained in the interior of $conv(R)$. However this condition is not sufficient (see (1) and (3) of Fig. 50). For this problem, the following results are obtained.

Theorem 5.3 (Czyzowciz, Hurtado, Urrutia and Zaguia [46]) *Assume that R and B are in the plane, and that B is contained in the interior of $conv(R)$. Then there exists a polygon with vertex set R that encloses at least $\frac{|B|}{2}$ blue points.*

The following theorem and the above theorem are independent of each other.

Theorem 5.4 (Hurtado et al. [69]) *Assume that R and B are in the plane, and that B is contained in the interior of conv$(R \cup B)$. If the number of vertices of conv$(R \cup B)$ is greater than $|B|$, then there is a polygon with vertex reset R that encloses B (see (2) of Fig. 50).*

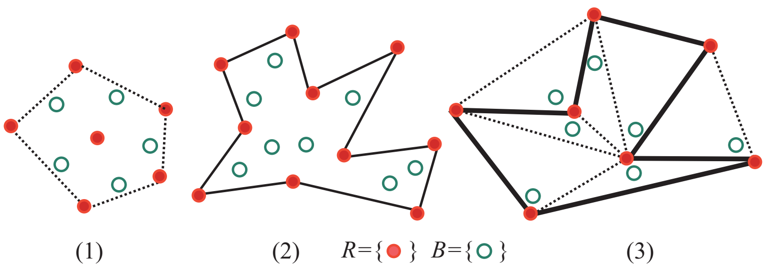

Figure 50: (1) No polygon with vertex set R encloses B; (2) A polygon with vertex set R encloses B; (3) Every polygon with vertex set R encloses exactly 5 blue points.

Theorem 5.5 (Hurtado et al. [69]) *There are configurations of $R \cup B$ that satisfy the following two conditions: (i) B belongs to the interior of conv(R); and (ii) any polygon with vertex set R encloses exactly $\frac{|B|+1}{2}$ points of B.*

In the above theorems, we consider a problem of finding a red polygon that encloses many blue points. We next consider the dual problem, namely, we want to find a red polygon that contains a small number of blue points. In particular, we want to find a red polygon whose interior contains no blue point, that is, it excludes all blue points (and possibly some red points in the interior of *conv*(R), see Fig. 51). The following theorem says that finding such red polygon is always possible as long as the number of red points is large.

Theorem 5.6 (Fulek, Keszegh, Moríc and Uljarevíc [54]) *Let $b \geq 1$ and $n \geq 1$ be integers. Then there exists a number $K(b) = O(b^4)$ that possesses the following property. Assume that R and B are in the plane, $|B| = b$, $|R| = n + k$, $k \geq K(b)$, and that conv(R) contains B and k red points in its interior. Then there exists a subset $R' \subset R$ such that some polygon with vertex set R' excludes B and R' contains all the vertices of conv(R) (see Fig. 51).*

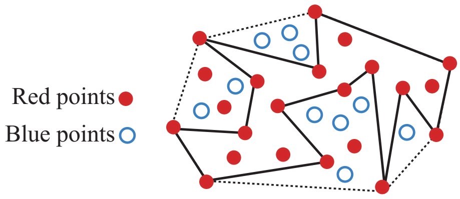

Figure 51: A subset $R' \subset R$ such that a polygon with vertex set R' excludes B and R' includes all the vertices of $conv(R)$.

Given $R \cup B$ in the plane, if the plane is partitioned into some convex polygons so that for each polygon P, the data points in P are either mostly red points or mostly blue points, then we say that $R \cup B$ is *well-separated*. Otherwise, we have an *uniform distribution*. For this problem, some formal definition and results in relation to this problem are listed by by Berge et al. [30] and Díaz-Báñez et al. [48].

5.3 Bichromatic Lines

Hereafter we deal with $R \cup B$ in the plane, *which is not in general position*. If a line l passes through at least two points of $R \cup B$, then we say that $R \cup B$ *determines* l. A line is said to be *bichromatic* if it passes through at least one red point and at least one blue point (see Fig. 52). On the other hand, a line that passes through at least two points of $R \cup B$ and passes through only points of the same color is called *monochromatic*.

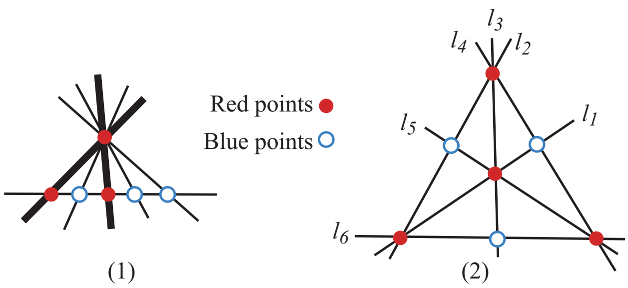

Figure 52: (1) Two monochromatic lines (bold lines) and 4 bichromatic lines; (2) A set $R \cup B$ determines 6 bichromatic lines. No line passes through precisely one red point and one blue point.

The first result on this topic is on monochromatic lines, and it can be considered as color type of Gallai-Sylvester Theorem, which says that a set

S of points in the plane which is not collinear determines a line which passes through exactly two points of S. Also it is easy to see that Gallai-Sylvester Theorem implies that S determines at least $|S|$ lines (Erdős-de Bruijn Theorem).

Theorem 5.7 (Motzkin [101]) *Assume that $R \cup B$ is in the plane and not collinear. Then $R \cup B$ determines a monochromatic line (see (1) of Fig. 52).*

We next consider the number of bichromatic lines.

Theorem 5.8 (Pach and Pinchasi [108]) *Assume that $R \cup B$ is in the plane, $|R| = |B| = n \geq 2$ and $R \cup B$ is not collinear. Then there exist at least $\frac{n}{2}$ bichromatic lines that passes through at most two red points and at most two blue points. Moreover, there exist at least $n+1$ bichromatic lines (see (1) of Fig. 52).*

Theorem 5.9 (Pach and Pinchasi [108]) *Assume that $R \cup B$ is in the plane, $|R| = n$, $|B| = cn$ and $R \cup B$ is not collinear, where $c \geq 1$. Then the number of bichromatic lines passing through at most $8c$ points is at least*

$$\frac{1}{25c^2} \times \text{ (the total number of lines determined by } R \cup B).$$

In particular, there exists at least one such line.

The next theorem generalizes Theorem 5.7.

Theorem 5.10 (Kleitman and Pinchasi [94]) *Assume that $R \cup B$ is in the plane and that neither R nor B is collinear. If $|R| = n$ and $n - 1 \leq |B| \leq n$, then $R \cup B$ determines at least $|R \cup B| - 3$ bichromatic lines (see (2) of Fig. 52).*

Conjecture 5.11 (Kleitman and Pinchasi [94]) *Assume that $R \cup B$ is in the plane and that neither R nor B is collinear. If $|R| = n$ and $n - 1 \leq |B| \leq n$, then $R \cup B$ determines at least $|R \cup B| - 1$ bichromatic lines.*

A line passing through i red points and j blue points such that $i + j \geq 2$ and $|i - j| \leq 1$ is called an *equichromatic line.* The following theorem proved Conjecture 5.11 for sufficiently large $|R|$.

Theorem 5.12 (Purdy and Smith [111]) *Assume that $R \cup B$ is in the plane, $|R| = n$, $|B| = n - k$ for $k \in \{0, 1\}$, and that neither R nor B is collinear. If $n \geq 78 + k$, then the number of equichromatic lines is at least $2n - k - 1 = |R \cup B| - 1$. In particular, Conjecture 5.11 is true for all $n \geq 79$.*

Theorem 5.13 (Purdy and Smith [111]) *Assume that $R \cup B$ is in the plane, $|R| = n$, $|B| = n - k$ for $k \geq 0$, and that $R \cup B$ is not on a line. Let t be the total number of lines determined by $R \cup B$. Then the number of equichromatic lines is at least $\frac{1}{4}(t + 2n + 3 - k(k+1))$.*

By the Erdős-de Bruijn Theorem, which says $t \geq 2n - k$, where t is given in the above theorem, we have the following corollary from the above theorem.

Corollary 5.14 (Purdy and Smith [111]) *Let R and B be the same as Theorem 5.13. Then the number of equichromatic lines is at least $n + \frac{1}{4}(3 - k(k+2))$. If $k \in \{0, 1\}$, then the number of equichromatic lines is at least $n + 1 - k$.*

6 Colored Point Sets in the Plane Lattice

In this section we consider colored point sets in the *integer plane lattice* $\mathbb{Z}^2$. For simplicity, we will refer to $\mathbb{R}^2$ and $\mathbb{Z}^2$ as *the plane* and *the plane lattice*, respectively. A set S of points in the plane lattice is said to be *in general position* if every vertical line or horizontal line contains at most one point of S (see Fig. 53). An *L-line* with corner $q \in \mathbb{R}^2$ consists of a horizontal half-line and a vertical half-line emanating from the common apex q (see Fig. 53). It will be shown that L-lines and L-line segments in the plane lattice play similar roles as lines and line segments, respectively, in the plane.

Recall that R, B and G always denote *a set of red points, a set of blue points* and *a set of green points*, respectively. Moreover, we always assume that R and B (or R, B and G) are *in the plane lattice in general position.* For R and B in the plane lattice, a *bichromatic matching* on $R \cup B$ consists of L-line segments joining red points to blue points.

Theorem 6.1 (Kano and Suzuki [87]) *Assume that R and B are in the plane lattice and $|R| = |B|$. Then there exists a non-crossing bichromatic perfect matching on $R \cup B$ with L-line segments (see (2) of Fig. 53).*

An L-line partitions the plane into two regions. When we look for an L-line having some property, we often consider an L-line whose corner is not in the plane lattice so that the L-line does not pass through any point of the plane lattice.

Theorem 6.2 (Uno, Kawano and Kano [118]) *Assume that R and B are in the plane lattice, and that $|R| = 2m$ and $|B| = 2n$ for some integers $m, n \geq 1$. Then there exists an L-line in the plane that bisects both R*

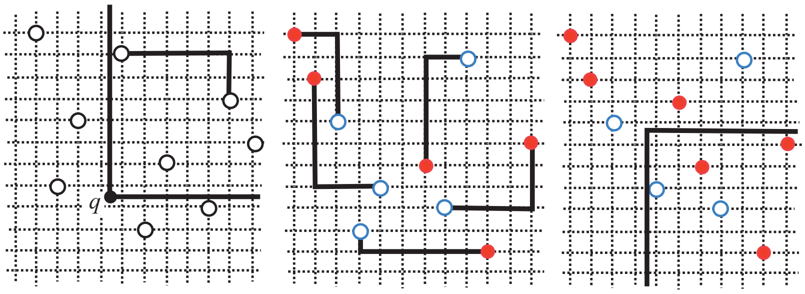

Figure 53: (1) A point set S in the plane lattice $\mathbb{Z}^2$ in general position, an L-line with corner $q \in \mathbb{R}^2$, and an L-line segment joining two points of S; (2) A non-crossing bichromatic perfect matching on $R \cup B$ with L-line segments; (3) An L-line in the plane that bisects R and B in the plane lattice.

and B (see (3) of Fig. 53). Namely, each region determined by the L-line contains exactly m red points and n blue points.

If the plane is partitioned into some regions by horizontal half-lines and vertical half-lines, then we call such a partition *an orthogonal partition* (see Fig. 54). Note that a half-line might be a line or a line segment. A set X in the plane is said to be *orthogonally convex* if the intersection of X with every horizontal or vertical line is connected (see Fig. 56). Theorem 6.2 is generalized as follows.

Theorem 6.3 (Orthogonal Balanced Partition (discrete), Bereg [26]) *Let $m, n \geq 1$ and $k \geq 2$ be integers. Assume that R and B are in the plane lattice, and $|R| = km$ and $|B| = kn$. Then there exists a partition of the plane into k orthogonally convex regions by at most $k-1$ horizontal half-lines and at most $k-1$ vertical half-lines such that every region contains exactly n red points and m blue points (see Fig. 54).*

Theorem 6.4 (Orthogonal Balanced Partition (continuous), Bereg [26]) *Let $k \geq 2$ be an integer and μ_1 and μ_2 be two mass distributions on the plane. Then there exists a partition of the plane into k orthogonally convex regions $X_1, X_2, \ldots, X_k$ by at most $k-1$ horizontal half-lines and at most $k-1$ vertical half-lines such that*

$$\mu_1(X_i) = \frac{\mu_1(\mathbb{R}^2)}{k} \quad \textit{and} \quad \mu_2(X_i) = \frac{\mu_2(\mathbb{R}^2)}{k} \qquad \textit{for all } 1 \leq i \leq k.$$

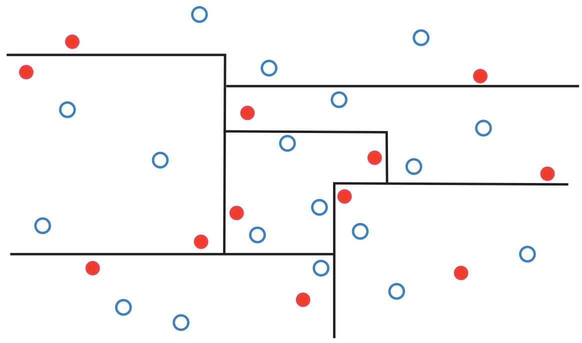

Figure 54: An orthogonal balanced partition of the plane lattice into 6 convex regions by 5 horizontal half-lines and 3 vertical half-lines.

It is a corollary of Theorem 2.41 that two measures defined on the plane can be simultaneously bisected by a 2-fan whose apex is a given point in the plane. The next theorem shows a similar result holds for the plane lattice using L-rays, where an *L-ray* is an L-line emanating from a point called an *apex* (see Fig. 55).

Theorem 6.5 (Kano and Suzuki [87]) *Assume that R and B are in the plane lattice, and that $|R| = 2m$ and $|B| = 2n$ for integers $m \geq 1$ and $n \geq 1$. Let p be a point in the plane but not in the plane lattice. Then there exist two L-rays emanating from p that bisect both R and B (see Fig. 55).*

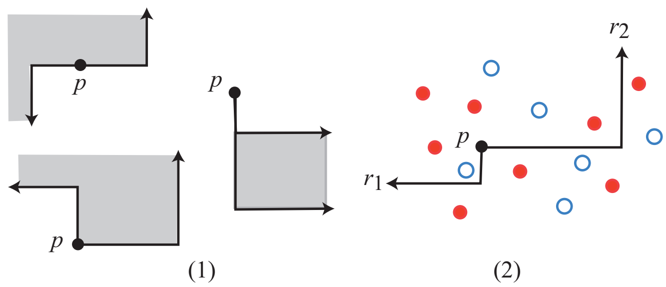

Figure 55: (1) Partitions of the plane by two L-rays emanating from the apex p. (2) Two L-rays r_1 and r_2 emanating from the given apex p that bisect both R and B.

Recall that a set X in the plane is said to be *orthogonally convex* if the intersection of X with every horizontal or vertical line is connected. The *orthogonal convex hull* of a point set S is the intersection of all connected orthogonally convex sets including S (see (1) and (2) of Fig. 56).

Theorem 6.6 (Bereg et al. [31]) *Assume that R, B and G are in the plane lattice and $|R| = |B| = |G| = n$. If all the vertices of the orthogonal convex hull of $R \cup B \cup G$ have the same color, then there exists a nontrivial balanced L-line in the plane, namely, there exists an integer $1 \leq k \leq n-1$ and an L-line L^* such that a region determined by L^* contains exactly k points of each color (see Fig. 56).*

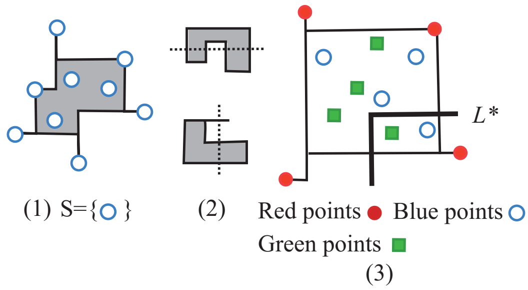

Figure 56: (1) The orthogonal convex hull of S; (2) Two non-orthogonal convex sets ;(3) $R \cup B \cup G$ satisfying the conditions of Theorem 6.6 and its balanced L-line L^*.

The next theorem shows that a similar result on an R-spanning tree and a B-spanning tree given in Theorem 3.21 holds for the plane lattice. Here we define $\tau^*(R, B)$ by using the minimum rectangle containing $R \cup B$ but not the orthogonal convex hull of $R \cup B$ in the plane lattice. For R and B in the plane lattice, let $\tau^*(R, B)$ denote the number of unordered pairs $\{x, y\}$ of points of $R \cup B$ such that one of $\{x, y\}$ is red and the other is blue, and x and y are on the consecutive edges of the minimum rectangular containing $R \cup B$ (see Fig. 57). Then $\tau^*(R, B)$ is even and $0 \leq \tau^*(R, B) \leq 4$.

Theorem 6.7 (Kano and Suzuki [87]) *Assume that R and B are in the plane lattice. If $\tau^*(R, B) \leq 2$, then there exist two non-crossing spanning trees on R and B, respectively, whose edges are L-line segments and whose maximum degrees are at most 3. If $\tau^*(R, B) = 4$, then there are two such spanning trees having one crossing (see Fig. 57).*

In the following theorem, each edge of a path is not necessary an L-line segment, but is a shortest path in the plane that connects two of its end-points and consists of vertical and horizontal line segments in the plane, which is called an *orthogeodesic path*.

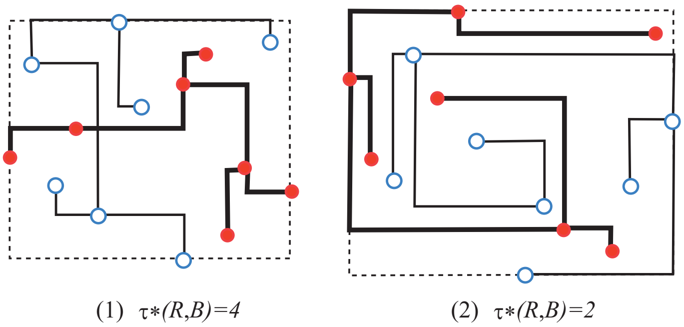

Figure 57: (1) R and B with $\tau^*(R, B) = 4$, and two spanning trees on R and B, respectively, which have one crossing; (2) R and B with $\tau^*(R, B) = 2$, and two spanning trees on R and B, respectively, which have no crossing.

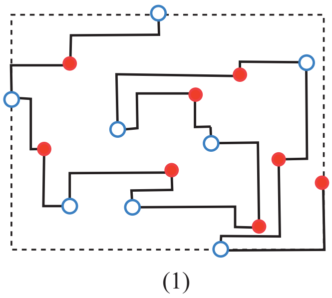

Figure 58: A non-crossing alternating Hamiltonian path on $R \cup B$ in the plane each of whose edges is orthogeodesic and has at most two bends

Theorem 6.8 (Di Giacomo et al. [58]) *Assume that R and B are in the plane lattice and that $||R| - |B|| \leq 1$. Then there exists a non-crossing alternating Hamiltonian path on $R \cup B$ each of whose edges is an orthogeodesic path in the plane and has at most two bends (see Fig.58).*

7 Measures on Higher Dimensional Spaces

In this section, we collect some results on measures defined on a higher dimensional space $\mathbb{R}^d$, $d \geq 3$, which are directly related to some results given in the previous sections. Thus we do not deal with results on the following topics: colorful version of Helly-type theorem and of Tverberg's theorem and others. Many theorems in this section are proved using topological methods, and for these methods, the reader is referred to the paper [120] by Živaljević, the book *"Using the Borsuk-Ulam Theorem"* [100] by Matoušek, and others.

For simplicity and for our discrete geometry, we assume that every measure μ defined on $\mathbb{R}^d$, $d \geq 3$, satisfies the following conditions: (i) μ is absolutely continuous with respect to the Lebesgue measures; and (ii) there is a bounded domain $D \subset \mathbb{R}^n$ such that $0 < \mu(D) = \mu(\mathbb{R}^n) < \infty$. Thus every open set X of $\mathbb{R}^n$ is measurable and $\mu(h) = 0$ for every hyperplane h. Such a measure is called *a mass distribution*. Moreover, if μ also satisfies $\mu(\mathbb{R}^d) = 1$, then μ is called *a probability measure*. In many results, however, there is no essential difference between mass distributions and probability measures.

The following theorem is the Ham-sandwich Theorem in higher dimensional space.

Theorem 7.1 (Ham-sandwich Theorem (continuous), Stone and Tukey [116], Steinhaus [115]) *Let $d \geq 3$ be an integer. Assume that d mass distributions $\mu_1, \mu_2, \ldots, \mu_d$ are defined on $\mathbb{R}^d$. Then there exists a hyperplane h such that each half-space H defined by h satisfies*

$$\mu_i(H) = \frac{\mu_i(\mathbb{R}^d)}{2} \qquad \textit{for all} \quad 1 \leq i \leq d.$$

The following theorem is well-known and is a starting point for understanding partition problems in higher dimensional space.

Theorem 7.2 (Akiyama and Alon [13]) *Let $d \geq 2$ and $n \geq 1$ be integers. Assume that a d-colored point set S is in $\mathbb{R}^d$ in general position, $|S| = dn$ and S contains exactly n points of each color. Then S can be partitioned into n disjoint subsets $X_1, X_2, \ldots, X_n$ so that every X_i contains exactly one point of each color and all $conv(X_i)$ are pairwise disjoint.*

Notice that in the above theorem, S is a set of points in $\mathbb{R}^d$ in general position and X_i contains d points of S, and so $conv(X_i)$ is a $(d-1)$-dimensional simplex. The Ham-sandwich Theorem (Theorem 7.1) was generalized to balanced partitions in higher dimensional space, and it was done by Karasev [91] and Soberón [114] independently.

Theorem 7.3 (Balanced Partition Theorem (continuous), Karasev [91], Soberón [114]) *Let $d \geq 2$ and $n \geq 2$ be integers. Assume that d mass distributions $\mu_1, \mu_2, \ldots, \mu_d$ are defined on $\mathbb{R}^d$. Then there exists a partition of $\mathbb{R}^d$ into n convex regions $C_1, C_2, \ldots, C_n$ that satisfy*

$$\mu_i(C_j) = \frac{\mu_i(\mathbb{R}^d)}{n} \qquad \textit{for all} \;\; 1 \leq j \leq n \quad \textit{and} \;\; 1 \leq i \leq d.$$

It is not easy to obtain a discrete version of balanced partition theorem from a continuous version of it by case analysis. The following discrete version of the Balanced Partition Theorem was proved by Blagojević, Rote, Steinmeyer and Ziegler [36]. They obtained this theorem from Theorem 7.3 by making use of the so-called *Integer Flow Theorem*, which states that every fractional flow in a network with integer capacities can be realized by an integer flow.

Theorem 7.4 (Balanced Partition Theorem (discrete), Blagojević, Rote, Steinmeyer and Ziegler [36]) *Let $d \geq 2$ and $n \geq 2$ be integers. Assume that a d-colored point set S is in $\mathbb{R}^d$ in general position. Let $c_1, c_2, \ldots, c_d$ be the d colors, and let S_i denote the set of points in S colored with c_i. Then there exists a partition of S into n disjoint subsets $X_1, X_2, \ldots, X_n$ that satisfy $conv(X_i) \cap conv(X_j) = \emptyset$ for all $i \neq j$ and*

$$\#\{\text{points in } X_j \text{ colored with } c_i\} = \left\lfloor \frac{|S_i|}{n} \right\rfloor \text{ or } \left\lceil \frac{|S_i|}{n} \right\rceil$$

for all $1 \leq i \leq d$ and $1 \leq j \leq n$.

The following theorem is a generalization of Theorem 2.15.

Theorem 7.5 (Hamburger Theorem (continuous), Kano and Kynčl [84]) *Let $d \geq 2$ be an integer. Assume that $d+1$ mass distributions $\mu_1, \mu_2, \ldots, \mu_{d+1}$ are defined on $\mathbb{R}^d$. Let $\omega_i = \mu_i(\mathbb{R}^d)$ for all i, and $\omega = \min\{\omega_i : 1 \leq i \leq d+1\}$. Assume that $\omega_1 + \omega_2 + \cdots + \omega_{d+1} = 1$ and $\omega_i \leq \frac{1}{d}$ for all i. Then there exists a hyperplane h such that each open half-space H defined by h satisfies*

$$\mu_i(H) \leq \frac{1}{d}\big(\mu_1(H) + \mu_2(H) + \cdots + \mu_{d+1}(H)\big) \quad \text{for } 1 \leq i \leq d+1, \text{ and}$$

$$\mu_1(H) + \mu_2(H) + \cdots + \mu_{d+1}(H) \geq \min\left\{\frac{1}{2}, 1 - d\omega\right\} \geq \frac{1}{d+1}.$$

A discrete version of The Hamburger Theorem is the following.

Theorem 7.6 (Hamburger Theorem (discrete), Kano and Kynčl [84]) *Let $d \geq 2$ and $n \geq 2$ be integers. Assume that a $(d+1)$-colored point set $X_1 \cup X_2 \cup \cdots \cup X_{d+1}$ is in $\mathbb{R}^d$ in general position and that*

$$|X_1 \cup X_2 \cup \cdots \cup X_{d+1}| = dn \quad \text{and} \quad |X_i| \leq n \quad \text{for all } 1 \leq i \leq d+1.$$

Then there exists a hyperplane h such that each open half-space H defined by h satisfies

$$|X_i \cap H| \le \frac{1}{d}|(X_1 \cup X_2 \cup \cdots \cup X_{d+1}) \cap H| \quad \text{for } 1 \le i \le d+1,$$

and $|(X_1 \cup X_2 \cup \cdots \cup X_{d+1}) \cap H|$ is a positive integer multiple of d.

By using the above theorem, we can easily prove the following conjecture in the case of $r = d+1$ by induction on n.

Conjecture 7.7 (Kano and Suzuki [88]) *Let $d \ge 3$, $r \ge d+1$ and $n \ge 2$ be integers. Let $X_1 \cup X_2 \cup \cdots \cup X_r$ be an r-colored point set in $\mathbb{R}^d$ in general position such that $|X_1| + |X_2| + \cdots + |X_r| = dn$ and $|X_i| \le n$ for every $1 \le i \le r$. Then $X_1 \cup X_2 \cup \cdots \cup X_r$ can be partitioned into n disjoint sets $Y_1, Y_2, \ldots, Y_n$ so that every Y_i has d points with distinct colors and all $conv(Y_i)$ are pairwise disjoint.*

Recall that for a point set X of $\mathbb{R}^d$, a subset $Y \subset X$ is called an *island spanned by* X if $X \cap conv(Y) = Y$. Equivalently, we say that X *spans* Y. A colored point set is called *j-colorful* if it contains at least j points with distinct colors.

Theorem 7.8 (Holmsen, Kynčl and Valculescu [63]) *Let $d \ge 2$ and $n \ge 2$ be integers, and let S be a d-colored point set in $\mathbb{R}^d$ in general position. Suppose that $|S| = (d+1)n$ and that there are at least n points in each color class. Then S can be partitioned into n sets $X_1, X_2, \ldots, X_n$ so that every X_i contains exactly $d+1$ points, every X_i is d-colorful and all $conv(X_i)$ are pairwise disjoint.*

Note that the conclusion in the above theorem can be reworded as follows: "Then S spans n pairwise disjoint d-colorful $(d+1)$-islands". Moreover, the authors of Theorem 7.8 made the following conjecture.

Conjecture 7.9 (Holmsen, Kynčl and Valculescu [63]) *Let k, m and d be integers such that $k, m \ge d \ge 2$. Let S be an m-colored point set of kn points in $\mathbb{R}^d$ in general position. Suppose that S admits a partition into n disjoint d-colorful sets of size k. Then S spans n pairwise disjoint d-colorful k-islands.*

Let us give some remarks on Conjecture 7.9. First, this conjecture says that if S has a partition $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ such that each Y_i contains k points and is d-colorful, then S can be partitioned into $Z_1 \cup Z_2 \cup \cdots \cup Z_n$ so that each Z_i contains k points, is d-colorful, and all $conv(Z_j)$ are pairwise disjoint. Namely, the partition $Z_1 \cup Z_2 \cup \cdots \cup Z_n$ has the same combinatorial properties as $Y_1 \cup Y_2 \cup \cdots \cup Y_n$ together with a geometric property $conv(Z_i) \cap conv(Z_j) = \emptyset$ for all $i \neq j$.

Next, Theorem 7.2 proves Conjecture 7.9 in the case where $k = m = d$. The case where $m \geq k = d = 2$ is settled by Theorem 3.1. The case where $k \geq m = d = 2$ and the size of the color classes are divisible by n follows from the Balanced Partition Theorem (Theorem 2.24). Theorem 2.33 solves the case where $m = d = 2$, and Theorem 7.8 solves the case where $k = d + 1$ and $m = d$, and the case where $m = d + 1$, $k = d$ was solved in [84].

In order to explain the next theorem, we need some definitions and remarks originally presented in [16]. Let $\mu_1, \mu_2, \ldots, \mu_{d+1}$ be $d + 1$ probability measures on $\mathbb{R}^d$, and let $\varepsilon \in (0, \frac{1}{2})$ be a real number. Then the set of measures is called *ε-not-permuted* if for any half-space H with $\mu_i(H) < \varepsilon$ for all $1 \leq i \leq d + 1$, we have

$$\mu_i(H) \geq \mu_j(H) \qquad \text{for some } i = i(\varepsilon) \ < \ j = j(\varepsilon).$$

For $\varepsilon > 0$, consider all half-spaces H in $\mathbb{R}^d$ such that $\mu_i(H) < \varepsilon$ for all i and values $\mu_i(H)$ are pairwise distinct. If we arrange the values $\mu_i(H)$ in the ascending order, then we get some permutation of $\{1, 2, \ldots, d + 1\}$. So there is an order of μ_i such that the measures μ_i are ε-not-permuted if and only if in such a way we cannot get all possible permutations of the $d + 1$ element set. In Theorem 3.2, we are actually interested in measures that are ε-not-permuted for at least one order of them ([16]).

A natural example of ε-not-permuted measures appears when the support of one measure lies in the interior of the convex hull of the union of supports of the other d measures. In this case the measures are ε-not-permuted for sufficiently small ε.

Theorem 7.10 (Akopyan and Karasev [16]) *Let $\mu_1, \mu_2, \ldots, \mu_{d+1}$ be $d+1$ probability ε-not-permuted measures on $\mathbb{R}^d$ for some $\varepsilon \in (0, \frac{1}{2})$. Then there exists a half-space H such that*

$$\mu_1(H) = \mu_2(H) = \cdots = \mu_{d+1}(H) \in \left[\varepsilon, \frac{1}{2}\right]$$

For every directed line ℓ in $\mathbb{R}^d$, we write $y \leq_\ell x$ if the projection of a point y to ℓ has no larger coordinate than the projection of a point x to ℓ. A hyperplane h in $\mathbb{R}^d$ is called *balanced* if each half-space defined by h contains precisely the same number of points of each color. The following theorem is a discrete version of the above theorem and a generalization of Theorem 2.14.

Theorem 7.11 (Akopyan and Karasev [16]) *Let $X_1, X_2, \cdots, X_{d+1}$ be $d+1$ sets of points in $\mathbb{R}^d$ in general position such that $|X_i| = n$ for every $1 \le i \le d+1$ (i.e., X_i is a set of n points colored with i). Assume that for every directed line ℓ there exist two colors $a = a(\ell)$ and $b = b(\ell)$, $a < b$, and a point $x \in X_a$ such that for every $y \in X_b$, it follows that $y \le_\ell x$ holds. Then there exists a balanced hyperplane h, namely, there exists a hyperplane h such that each half-space defined by h contains precisely the same number of points of each X_i.*

The following theorem is a generalization of Theorem 2.42.

Theorem 7.12 (Akopyan and Karasev [16]) *Suppose that $d+1$ probability measures $\mu_1, \mu_2, \ldots, \mu_{d+1}$ are defined on $\mathbb{R}^d$, and let $0 < \alpha < \frac{1}{2}$ be a real number. Then there always exists a convex set C in $\mathbb{R}^d$ such that*

$$\mu_1(C) = \mu_2(C) = \cdots = \mu_{d+1}(C) = \alpha$$

if and only if $\alpha = \frac{1}{m}$ for a positive integer m.

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_m\}$ be a finite set of hyperplanes in $\mathbb{R}^d$, $\{A_1, A_2, \ldots, A_m\}$ be affine functions such that the zero set of A_i is H_i, and let $P^{\mathcal{H}} = A_1 A_2 \cdots A_m$ be the product of these affine functions (see Fig. 14). If μ is a mass distribution in $\mathbb{R}^d$, we say that $\mathcal{H}$ bisects μ if

$$\mu\big(\{\boldsymbol{v} \in \mathbb{R}^d : P^{\mathcal{H}}(\boldsymbol{v}) > 0\}\big) = \frac{\mu(\mathbb{R}^d)}{2}. \tag{1}$$

Theorem 7.13 (Hubard and Karasev [66]) *Let $d \ge 2$ be a power of two, $n \ge 1$ be an integer, and let $\mu_1, \mu_2, \ldots, \mu_{dn}$ be dn mass distributions on $\mathbb{R}^d$. Then there exists an arrangement of at most n hyperplanes $H_1, H_2, \ldots, H_m$, $m \le n$, that bisect every measure $\mu_i, 1 \le i \le dn$, namely, equation (1) holds for all μ_i, $1 \le i \le dn$.*

The following conjecture is open.

Conjecture 7.14 (Barba, Pilz and Schnider [24]) *Let $d \ge 3$ and $n \ge 1$ be integers and let $\mu_1, \mu_2, \ldots, \mu_{dn}$ be dn mass distributions on $\mathbb{R}^d$. Then there exist n hyperplanes $H_1, H_2, \ldots, H_n$ that bisect every measure $\mu_i, 1 \le i \le dn$.*

For a convex body C in $\mathbb{R}^d$, let $\partial(C)$ denote the $(d-1)$-dimensional surface area of C. In particular, if $d = 2$ and C is a convex set in the plane, then $\partial(C)$ denotes the perimeter of C.

Theorem 7.15 (Karasev [91]) *Let $1 \leq k < d$ be integers. Let C be a convex body in $\mathbb{R}^d$, $\mu_1, \mu_2, \ldots, \mu_k$ be probability measures on C, and $\sigma_1, \sigma_2, \ldots, \sigma_{n-k}$ be probability measures on $\partial(C)$. Then for any integer $q \geq 1$, the body C can be partitioned into q convex parts $X_1, X_2, \ldots, X_q$ so that*

$$\mu_i(X_1) = \mu_2(X_i) = \cdots = \mu_i(X_q) \qquad \text{for all } 1 \leq i \leq k,$$

and

$$\sigma_j(X_1 \cap \partial(C)) = \sigma_j(X_2 \cap \partial(C)) = \cdots = \sigma_j(X_q \cap \partial(C))$$

for all $1 \leq j \leq d - k$.

Consider a problem of the number of empty monochromatic simplices for a given colored point set in a space $\mathbb{R}^d$. Recall that for a set S of points in $\mathbb{R}^d$, a subset $X \subset S$ is said to be *empty* if $conv(X) \cap S = X$.

Theorem 7.16 (Aichholzer et al. [10]) *Let $d \geq 2$ be an integer. Any 2-colored set S of n points in $\mathbb{R}^d$ in general position determines $\Omega(n^{d-2/3})$ empty monochromatic d-simplices.*

Theorem 7.17 (Aichholzer et al. [10]) *Let $d \geq k \geq 3$ be integers. Any k-colored set S of n points in $\mathbb{R}^d$ in general position determines $\Omega(n^{d-k+1+2^{-d}})$ empty monochromatic d-simplices.*

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