Existence of all generalized fractional (g, f)-factors of graphs

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Abstract

Let G be a graph and $\mathbb{R}^+$ denote the set of non-negative real numbers. For a vertex v of G, $E_G(v)$ denotes the set of edges incident with v. Let $\varphi : E(G) \to \mathbb{R}^+$ and $f : V(G) \to \mathbb{R}^+$. Then a generalized fractional f-factor of G is a real-valued function $\omega : E(G) \to \mathbb{R}^+$ that satisfies $0 \le \omega(e) \le \varphi(e)$ for every $e \in E(G)$ and $f(v) = \sum_{e \in E_G(v)} \omega(e)$ for every $v \in V(G)$. For two functions $g, f : V(G) \to \mathbb{R}^+$ with $g \le f$, we say that G has all generalized fractional (g, f)-factors if G has a generalized fractional h-factor for every $h : V(G) \to \mathbb{R}^+$ satisfying $g(x) \le h(x) \le f(x)$ for all $x \in V(G)$. In this paper, we present a necessary and sufficient condition for a graph G to have all generalized fractional (g, f)-factors, and moreover, our proof is self-contained and does not use the (g, f)-factor theorem or the fractional (g, f)-factor theorem.

1 Introduction

First we consider usual *fractional factors* of graphs. Let G be a multigraph with vertex set $V(G)$ and edge set $E(G)$, which may have multiple edges but has no loops. We denote the order of G by $|G|$, that is, $|G| = |V(G)|$.

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For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G, and by $E_G(v)$ the set of edges of G incident with v. Let $\mathbb{Z}^+$ denote the set of non-negative integers, and let $f : V(G) \to \mathbb{Z}^+$ be an integer-valued function. Under this notation, an *f-factor* of G is a spanning subgraph F of G that satisfies $\deg_F(x) = f(x)$ for all $x \in V(G)$. On the other hand, a *fractional f-factor* of G is a real-valued function $\omega : E(G) \to [0, 1]$ that satisfies

$$\sum_{e \in E_G(v)} \omega(e) = f(v) \quad \text{for every} \quad v \in V(G).$$

Obviously, an f-factor is a fractional f-factor ω satisfying $\omega(e) \in \{0, 1\}$ for every $e \in E(G)$, and vice versa.

For a function ψ defined on $V(G)$ and a subset $X \subseteq V(G)$, we write

$$\psi(X) := \sum_{x \in X} \psi(x).$$

Let $g, f : V(G) \to \mathbb{Z}^+$ with $g \le f$, that is, $g(x) \le f(x)$ for all $x \in V(G)$. In this situation, we say that G has *all (g, f)-factors* if G has an h-factor for every $h : V(G) \to \mathbb{Z}^+$ satisfying the condition that

$$g(x) \le h(x) \le f(x) \quad \text{for all } x \in V(G), \text{ and} \quad h(V(G)) \ \text{ is even.}$$

Similarly, we say that G has *all fractional (g, f)-factors* if G has a fractional h-factor for every $h : V(G) \to \mathbb{Z}^+$ satisfying

$$g(x) \le h(x) \le f(x) \quad \text{for all } x \in V(G).$$

Necessary and sufficient conditions for a graph to have all (g, f)-factors or all fractional (g, f)-factors were obtained as follows.

Theorem 1 (Niessen [4]) *Let G be a multigraph and $g, f : V(G) \to \mathbb{Z}^+$ with $g \le f$ and in the case where $g = f$, assume that $f(V(G))$ is even. Then G has all (g, f)-factors if and only if for all $S, T \subseteq V(G)$ with $T \cap S = \emptyset$,*

$$g(S) - f(T) + \sum_{x \in T} \deg_{G-S}(x) - q^*_G(S, T, g, f) \ge \begin{cases} 0 & \text{if } g = f \\ -1 & \text{otherwise,} \end{cases}$$

where $q^*_G(S, T, g, f)$ denotes the number of components C of $G - (S \cup T)$ such that either there exists $x \in V(C)$ with $g(x) < f(x)$ or $f(V(C)) + e_G(V(C), T)$ is odd, and $e_G(V(C), T)$ denotes the number of edges of G that joins a vertex in $V(C)$ to a vertex in T.

Theorem 2 (Lu [2]) *Let G be a multigraph and $g, f : V(G) \to \mathbb{Z}^+$ with $g \le f$. Then G has all fractional (g, f)-factors if and only if for all $S, T \subseteq V(G)$ with $T \cap S = \emptyset$,*

$$g(S) - f(T) + \sum_{x \in T} \deg_{G-S}(x) \ge 0.$$

Let $k \ge 1$ be an integer. If G has all fractional (g, f)-factors for the functions g and f defined by letting $g(x) = 1$ and $f(x) = k$ for all $x \in V(G)$, then we say that G has *all fractional* $[1, k]$*-factors*. Let kG denote the graph obtained from G by replacing each edge of G with k parallel edges. Then the following theorem holds.

Theorem 3 (Hu, Kano and Yu [3]) *Let $k \ge 1$ be an integer and G be a multigraph. Then kG has all fractional $[1, k]$-factors if and only if*

$$k \cdot iso(G - S) \le |S| \quad \textit{for all } S \subset V(G), \tag{1}$$

where $iso(G - S)$ denotes the number of isolated vertices of $G - S$.

Many other results on fractional factors of graphs can be found in the book [5].

Now we turn our attention to new factors, which we will call *generalized fractional factors*. Let G be a *general graph*, which may have multiple edges and loops. For a vertex v of G, $E_G(v)$ denotes the set of edges incident with v including loops incident with v. Let $\mathbb{R}^+$ denote the set of non-negative real numbers.

Let $\varphi : E(G) \to \mathbb{R}^+$ and $f : V(G) \to \mathbb{R}^+$. Then a *generalized fractional f-factor* of G with respect to φ is a real-valued function $\omega : E(G) \to \mathbb{R}^+$ that satisfies $0 \le \omega(e) \le \varphi(e)$ for every $e \in E(G)$ and

$$f(v) = \sum_{e \in E_G(v)} \omega(e) \quad \text{for every} \quad v \in V(G) \tag{2}$$

(see Figure 1). For an edge e, $\omega(e)$ is called the *weight of e by ω*, and $\varphi(e)$ is called the *capacity* of e. Clearly, for a function $f' : V(G) \to \mathbb{Z}^+$, a fractional f'-factor ω is a generalized fractional f'-factor ω satisfying $0 \le \omega(e) \le 1$ for every $e \in E(G)$ (i.e., $\varphi(e) = 1$ for all $e \in E(G)$).

For two functions $g, f : V(G) \to \mathbb{R}^+$ with $g \le f$, we say that G has *all generalized fractional (g, f)-factors* if G has a generalized fractional h-factor for every $h : V(G) \to \mathbb{R}^+$ satisfying $g(x) \le h(x) \le f(x)$ for all $x \in V(G)$. On

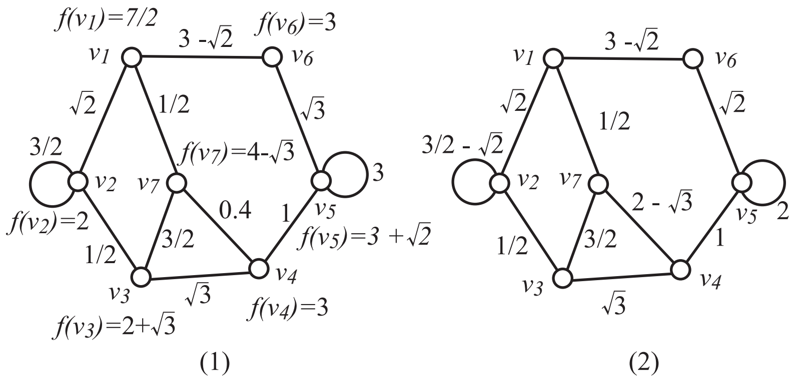

Figure 1: (1) A general graph G, where numbers denote the capacities $\varphi(e), e \in E(G)$. (2) A generalized fractional f-factor ω, where numbers denote the weights $\omega(e), e \in E(G)$.

the other hand, we say that G has *all generalized fractional* $\{g, f\}$*-factors* if G has a generalized fractional h'-factor for every $h' : V(G) \to \mathbb{R}^+$ satisfying

$$h'(x) = g(x) \quad \text{or} \quad f(x) \qquad \text{for all} \quad x \in V(G).$$

Note that if $g < f$, then there are $2^{|G|}$ such functions h'. In this paper, we prove the following theorem.

Theorem 4 *Let G be a general graph. Let $\varphi : E(G) \to \mathbb{R}^+$ and $g, f : V(G) \to \mathbb{R}^+$ be functions satisfying $g(x) \le f(x)$ for all $x \in V(G)$. Then the following three statements are equivalent.*

(i) *G has all generalized fractional (g, f)-factors with respect to φ.*

(ii) *G has all generalized fractional $\{g, f\}$-factors with respect to φ.*

(iii) *For all $S, T \subset V(G)$ with $S \cap T = \emptyset$, it follows that*

$$g(S) - f(T) + \sum_{y \in T} \deg_{(G-S,\varphi)}(y) \ge 0, \tag{3}$$

where

$$\deg_{(G-S,\varphi)}(y) = \sum_{e \in E_{G-S}(y)} \varphi(e). \tag{4}$$

Note that the notation $\deg_{(G-S,\varphi)}(y)$ is defined by (4).

2 Proof of Theorem 4

In this section, we prove Theorem 4. Our proof is self-contained and does not use the (g, f)-factor theorem (Theorem 4.1 by Lovász in [1]) or the fractional (g, f)-factor theorem (Theorem 8.2.1 by Anstee in [5]).

Let G be a general graph as in Theorem 4, and let $\varphi : E(G) \to \mathbb{R}^+$. Then we briefly call the pair (G, φ) a *weighted general graph*, and a generalized fractional factor of G with respect to φ is referred to as a *generalized fractional factor* of (G, φ). Recall that for a vertex v of G, $E_G(v)$ denotes the set of edges incident with v including loops incident with v. We first prove the following theorem.

Theorem 5 *Let (B, φ) be a weighted bipartite multigraph with bipartition (S_0, T_0), which has no loops. Let $h : V(B) \to \mathbb{R}^+$ such that $h(S_0) = h(T_0)$. Then (B, φ) has a generalized fractional h-factor if and only if*

$$h(X) - h(Y) + \sum_{y \in Y} \deg_{(B-X,\varphi)}(y) \geq 0 \qquad (5)$$

for all $X \subseteq S_0$ and $Y \subseteq T_0$.

Proof. We first prove the necessity. Assume that B has a generalized fractional h-factor ω. Let $X \subseteq S_0$ and $Y \subseteq T_0$. Then

$$h(X) + \sum_{y \in Y} \deg_{(B-X,\varphi)}(y) \geq \sum_{x \in X} \deg_{(B,\omega)}(x) + \sum_{y \in Y} \deg_{(B-X,\omega)}(y)$$
$$\geq \sum_{y \in Y} \deg_{(B,\omega)}(y) = \sum_{y \in Y} h(y) = h(Y).$$

Hence (5) holds, and the necessity is proved.

We next prove the sufficiency. For two vertices u and v in a network, an arc with tail u and head v is denoted by (u, v). We construct a network N with vertex set $\{s\} \cup S_0 \cup T_0 \cup \{t\}$ as follows: (i) for every vertex $x \in S_0$, add an arc (s, x) with capacity $h(x)$, (ii) for every edge $e = xy \in E(G)$ with $x \in S_0$ and $y \in T_0$, add an arc $arc(e) = (x, y)$ with capacity $\varphi(e)$, and (iii) for every vertex $y \in T_0$, add an arc (y, t) with capacity $h(y)$. For two disjoint vertex sets U and W of N, let $A(U, W)$ denote the set of arcs (u, w) with $u \in U$ and $w \in W$.

Let C be a cut-set separating s from t in N. We show that the capacity of C is at least $h(S_0)$. Set $X = \{x \in S_0 : (s, x) \in C\}$ and $Y = \{y \in$

$T_0 : (y,t) \in C\}$. Then $A(S_0 - X, T_0 - Y) \subseteq C$. Thus we may assume that $C = A(s,X) \cup A(S_0 - X, T_0 - Y) \cup A(Y,t)$. Then its capacity is

$$\begin{aligned} & h(X) + \sum_{arc(e) \in A(S_0 - X, T_0 - Y)} \varphi(e) \ + h(Y) \\ = \ & h(X) + \sum_{y \in T_0 - Y} \deg_{(B-X,\varphi)}(y) \ + h(Y). \end{aligned} \tag{6}$$

By (5), we have $h(X) - h(T_0 - Y) + \sum_{y \in T_0 - Y} \deg_{(B-X,\varphi)}(y) \geq 0$, and substituting this inequality into (6), we have

$$\begin{aligned} & h(X) + \sum_{y \in T_0 - Y} \deg_{(B-X,\varphi)}(v) + h(Y) \\ \geq \ & h(T_0 - Y) + h(Y) = h(T_0) = h(S_0), \end{aligned}$$

as desired. Of course, $A(s,S_0)$ and $A(T_0,t)$ are cut-sets with capacity $h(S_0) = h(T_0)$. Hence by the *max-flow min-cut Theorem* (Theorem 4.3.9 of [6]), B has a flow $f : Arc(N) \to \mathbb{R}^+$ such that (i) $f((s,u)) = h(u)$ for all $u \in S_0$, (ii) $f(arc(e)) \leq \varphi(e)$ for all $e \in E(B)$, and (iii) $f((y,t)) = h(v)$ for all $y \in T_0$. Namely, the function ω defined by $\omega(e) = f(arc(e))$, $e \in E(B)$, is the desired generalized fractional h-factor. □

We need the following notation. For two disjoint vertex sets X and Y of a weighted graph (G,φ), let

$$\varphi(X,Y) = \sum_{e \in E_G(X,Y)} \varphi(e).$$

Theorem 6 *Let (G,φ) be a weighted general graph with $\varphi : E(G) \to \mathbb{R}^+$, and let $h : V(G) \to \mathbb{R}^+$. Then (G,φ) has a generalized fractional h-factor if and only if*

$$\theta_{(G,\varphi)}(S,T;h) := h(S) - h(T) + \sum_{y \in T} \deg_{(G-S,\varphi)}(y) \geq 0 \tag{7}$$

for all $S,T \subset V(G)$ with $S \cap T = \emptyset$, where $\deg_{(G-S,\varphi)}(y) = \sum_{e \in E_{G-S}(y)} \varphi(e)$ (see (4)) and $\theta_{(G,\varphi)}(S,T;h)$ is defined by (7) and is written $\theta(S,T;h)$ when no confusion can arise.

Proof. In this proof, we briefly call a generalized fractional h-factor a *fractional h-factor*. Assume first that G has a fractional h-factor ω. Let $S,T \subset$

$V(G)$ with $S \cap T = \emptyset$. Since $\theta(\emptyset, \emptyset; h) = 0$, we may assume that $S \cup T \neq \emptyset$. Then

$$h(S) + \sum_{y \in T} \deg_{(G-S,\varphi)}(y) \geq \sum_{x \in S} \deg_{(G,\omega)}(x) + \sum_{y \in T} \deg_{(G-S,\omega)}(y)$$
$$\geq \sum_{y \in T} \deg_{(G,\omega)}(y) = \sum_{y \in T} h(y) = h(Y).$$

Hence (7) holds, and the necessity is proved.

We next prove the sufficiency by induction on $|G|$. Assume that (7) holds. First assume $|G| = 1$. Let $V(G) = \{v\}$ and $E(G) = \{e_1, \ldots, e_k\}$, where e_i is a loop incident with v. By (7), $\theta(\emptyset, \{v\}; h) = -h(v) + \sum_{1 \leq i \leq k} \varphi(e_i) \geq 0$. Hence define $\omega : E(G) \to \mathbb{R}^+$ so that $\omega(e_i) \leq \varphi(e_i)$ for all $1 \leq i \leq k$ and $\sum_{1 \leq i \leq k} \omega(e_i) = h(v)$. Then ω is the desired fractional h-factor.

Hence we may assume that $|G| \geq 2$. We may also assume that G is connected; for otherwise, we can apply the induction hypothesis to each component of G.

Let

$$\mathcal{F} = \{\varphi' : E(G) \to \mathbb{R}^+ \mid \varphi'(e) \leq \varphi(e) \text{ for all } e \in E(G) \text{ and } (7) \text{ holds with } \varphi \text{ replaced by } \varphi'.\},$$

and let

$$m = \inf\{\sum_{e \in E(G)} \varphi'(e) : \varphi' \in \mathcal{F}\}.$$

It is easy to see that there exists $\varphi_0 \in \mathcal{F}$ such that $\sum_{e \in E(G)} \varphi_0(e) = m$. Also if (G,φ_0) has a fractional h-factor ω, then ω is clearly a fractional h-factor of (G, φ) as well. Thus replacing φ by φ_0, we may assume that

$$\sum_{e \in E(G)} \varphi(e) = m.$$

Let

$$\beta = \min\{\theta(S, T; h) \mid S, T \subseteq V(G), S \cap T = \emptyset, S \cup T \neq \emptyset\}.$$

Suppose that $\beta > 0$. Take one $e_0 \in E(G)$ with $\varphi(e_0) > 0$, and define φ' by

$$\varphi'(e) = \begin{cases} \varphi(e_0) - \min\{\beta, \varphi(e_0)\} & \text{if } e = e_0, \\ \varphi(e) & \text{otherwise.} \end{cases}$$

Then

$$\theta_{(G,\varphi')}(S, T; h) \geq \theta_{(G,\varphi)}(S, T; h) - \beta \geq 0$$

for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$. Moreover, we have $\sum_{e \in E(G)} \varphi'(e) = m - \min\{\beta, \varphi(e_0)\}$, which contradicts the minimality of m. Thus $\beta = 0$, namely, there exist $S_0, T_0 \subseteq V(G)$ with $S_0 \cap T_0 = \emptyset$ and $S_0 \cup T_0 \neq \emptyset$ such that $\theta_{(G,\varphi)}(S_0, T_0; h) = 0$.

Note that if $V(G) = S_0 \cup T_0$ (i.e., if $G' = G - (S_0 \cup T_0)$ is an empty graph), then we can skip the following two claims, Claims 1 and 2, and proceed to Claim 3. Here we consider the case where $V(G) \neq S_0 \cup T_0$.

Claim 1. *Let $v \in V(G) - (S_0 \cup T_0)$. Then $h(v) - \varphi(v, T_0) \geq 0$.*

Proof. It follows from (7) that

$$\theta(S_0 \cup \{v\}, T_0; h) = \theta(S_0, T_0; h) + h(v) - \varphi(v, T_0) \geq 0.$$

Hence the claim follows from the fact that $\theta(S_0, T_0; h) = 0$. □

Let

$$G' = G - (S_0 \cup T_0), \quad \text{and} \quad h'(v) = h(v) - \varphi(v, T_0) \quad \text{for all} \quad v \in V(G').$$

We now consider these G' and h'.

Claim 2. *Let $S', T' \subseteq V(G')$ with $S' \cap T' = \emptyset$. Then $\theta_{(G',\varphi)}(S', T'; h') \geq 0$.*

Proof. Let $S = S_0 \cup S'$ and $T = T_0 \cup T'$. Then

$$\sum_{y \in T_0} \deg_{(G-S_0,\varphi)}(y) = \sum_{y \in T_0} \deg_{(G-S,\varphi)}(y) + \varphi(S', T_0), \tag{8}$$

$$\sum_{y \in T'} \deg_{(G'-S',\varphi)}(y) = \sum_{y \in T'} \deg_{(G-S,\varphi)}(y) - \varphi(T', T_0). \tag{9}$$

Hence

$$\begin{aligned}
& \theta_{(G',\varphi)}(S', T'; h') \\
= \ & h'(S') - h'(T') + \sum_{y \in T'} \deg_{(G'-S',\varphi)}(y) \\
= \ & h(S) - h(S_0) - \varphi(S', T_0) - \big(h(T) - h(T_0) - \varphi(T', T_0)\big) \\
& \qquad + \sum_{y \in T'} \deg_{(G-S,\varphi)}(y) - \varphi(T', T_0) \qquad \text{(by (9))} \\
= \ & \theta_{(G,\varphi)}(S, T; h) - h(S_0) - \varphi(S', T_0) + h(T_0) - \sum_{y \in T_0} \deg_{(G-S,\varphi)}(y) \\
= \ & \theta_{(G,\varphi)}(S, T; h) - h(S_0) + h(T_0) - \sum_{y \in T_0} \deg_{(G-S_0,\varphi)}(y) \qquad \text{(by (8))} \\
= \ & \theta_{(G,\varphi)}(S, T; h) - \theta_{(G,\varphi)}(S_0, T_0; h) \\
= \ & \theta_{(G,\varphi)}(S, T; h) \geq 0. \qquad \text{(because } \theta_{(G,\varphi)}(S_0, T_0; h) = 0\text{)}
\end{aligned}$$

Hence Claim 2 holds. $\square$

By Calim 2, it follows from the induction hypothesis that (G', φ) has a fractional h'-factor ω'.

Claim 3. *Let $y \in T_0$. Then $h(y) - \deg_{(G-S_0,\varphi)}(y) \geq 0$.*

Proof. The claim follows from $\theta(S_0, T_0; h) = 0$ and the following.

$$\theta(S_0, T_0 - y; h) = \theta(S_0, T_0; h) + h(y) - \deg_{(G-S_0,\varphi)}(y) \geq 0. \quad \square$$

Next we consider the weighted bipartite multigraph B with bipartition (S_0, T_0) and edge set $E_G(S_0, T_0)$. Define $h'' : S_0 \cup T_0 \to \mathbb{R}^+$ by

$$h''(v) = \begin{cases} h(v) & \text{if } v \in S_0, \\ h(v) - \deg_{(G-S_0,\varphi)}(v) & \text{if } v \in T_0. \end{cases}$$

Claim 4. $h''(S_0) = h''(T_0)$.

Proof. Since $\theta_{(G,\varphi)}(S_0, T_0; h) = 0$, we get

$$\begin{aligned} 0 = \theta_{(G,\varphi)}(S_0, T_0; h) &= h(S_0) - h(T_0) + \sum_{y \in T_0} \deg_{(G-S_0,\varphi)}(y) \\ &= h''(S_0) - h''(T_0). \end{aligned}$$

Hence Claim 4 holds. $\square$

Claim 5. *Let $X \subseteq S_0$ and $Y \subseteq T_0$. Then $\theta_{(B,\varphi)}(X, Y; h'') \geq 0$.*

Proof. Let $y \in Y$. Then $h''(y) = h(y) - \deg_{(G-S_0,\varphi)}(y)$, and

$$\deg_{(B-X,\varphi)}(y) = \deg_{(G-X,\varphi)}(y) - \deg_{(G-S_0,\varphi)}(y).$$

Hence

$$\begin{aligned} \theta_{(B,\varphi)}(X, Y; h'') &= h''(X) - h''(Y) + \sum_{y \in Y} \deg_{(B-X,\varphi)}(y) \\ &= h(X) - h(Y) + \sum_{y \in Y} \deg_{(G-S_0,\varphi)}(y) \\ &\quad + \sum_{y \in Y} \deg_{(G-X,\varphi)}(y) - \sum_{y \in Y} \deg_{(G-S_0,\varphi)}(y) \\ &= \theta_{(G,\varphi)}(X, Y; h) \geq 0. \end{aligned}$$

Therefore Claim 5 holds. $\square$

By Claims 4 and 5, it follows from Theorem 5 that (B, φ) has a fractional h''-factor ω''. Define $\omega : E(G) \to \mathbb{R}^+$ by

$$\omega(e) = \begin{cases} \omega'(e) & \text{if } e \in E(G'), \\ \omega''(e) & \text{if } e \in E(B), \\ \varphi(e) & \text{if } e \in E(G - S_0) - E(G'), \\ 0 & \text{otherwise.} \end{cases}$$

Then for vertices $x \in S_0$, $y \in T_0$ and $v \in V(G')$, we have

$$\sum_{e \in E_G(x)} \omega(e) = h''(x) = h(x),$$
$$\sum_{e \in E_G(y)} \omega(e) = h''(y) + \deg_{(G-S_0,\varphi)}(y) = h(y),$$
$$\sum_{e \in E_G(v)} \omega(e) = h'(v) + \varphi(v, T_0) = h(v).$$

Hence ω is the desired fractional h-factor of (G, φ). Consequently Theorem 6 is proved. □

Now we are ready to prove Theorem 4.

Proof of Theorem 4. It is obvious that (i) implies (ii). We now prove that (ii) implies (iii).

Let $S, T \subset V(G)$ with $S \cap T = \emptyset$. Since (3) holds for $S = T = \emptyset$, we may assume that $S \cup T \neq \emptyset$. Consider the function $h : V(G) \to \mathbb{R}^+$ defined by

$$h(v) = \begin{cases} g(v) & \text{if } v \in S, \\ f(v) & \text{otherwise.} \end{cases}$$

Then by (ii), G has a generalized fraction h-factor. By Theorem 6, we have $\theta(S, T; h) = h(S) - h(T) + \sum_{y \in T} \deg_{(G,\varphi)}(y) \geq 0$, which implies that $g(S) - f(T) + \sum_{y \in T} \deg_{(G,\varphi)}(y) \geq 0$. Hence (iii) holds.

Finally we assume that (iii) holds. Let $h : V(G) \to \mathbb{R}^+$ be a function satisfying $g(x) \leq h(x) \leq f(x)$ for all $x \in V(G)$. Then for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$, we have

$$\begin{aligned} & h(S) - h(T) + \sum_{y \in T} \deg_{(G,\varphi)}(y) \\ \geq\ & g(S) - f(T) + \sum_{y \in T} \deg_{(G,\varphi)}(y) \\ \geq\ & 0. \qquad \text{(by (iii))} \end{aligned}$$

By Theorem 6, G has a generalized fractional h-factor. Therefore (i) holds. Consequently Theorem 4 is proved. □

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