Characterization of 1-Tough Graphs using Factors

Hongliang Lu *
School of Mathematics and Statistics
Xi'an Jiaotong University
Xi'an, Shaanxi 710049, China

Mikio Kano †
Ibaraki University, Hitachi, Ibaraki, Japan

Abstract

For a graph G, let $odd(G)$ and $\omega(G)$ denote the number of odd components and the number of components of G, respectively. Then it is well-known that G has a 1-factor if and only if $odd(G-S) \le |S|$ for all $S \subset V(G)$. Also it is clear that $odd(G-S) \le \omega(G-S)$. In this paper we characterize a 1-tough graph G, which satisfies $\omega(G-S) \le |S|$ for all $\emptyset \ne S \subset V(G)$, using an H-factor of a set-valued function $H : V(G) \to \{\{1\}, \{0, 2\}\}$. Moreover, we generalize this characterization to a graph that satisfies $\omega(G-S) \le f(S)$ for all $\emptyset \ne S \subset V(G)$, where $f : V(G) \to \{1, 3, 5, \ldots\}$.

1 Introduction

We consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $iso(G)$ and $odd(G)$ the number of isolated vertices and the number of odd components of G, respectively. For a set $\mathcal{S}$ of connected graphs, a spanning

*luhongliang@mail.xjtu.edu.cn; Supported by the National Natural Science Foundation of China under grant No.11471257 and Fundamental Research Funds for the Central Universities

†mikio.kano.math@vc.ibaraki.ac.jp; Supported by JSPS KAKENHI Grant Number 16K05248

subgraph F of G is called an $\mathcal{S}$*-factor* if each component of F is isomorphic to an element of $\mathcal{S}$. For example, let C_n denote the cycle of order $n \geq 3$, and let K_2 denote the complete graph of order 2. Thus each component of a $\{K_2, C_n : n \geq 3\}$-factor is K_2 or a cycle, and a $\{K_2\}$-factor is simply a *1-factor*. A graph G is said to be *factor-critical* if for every vertex x of G, $G - x$ has a 1-factor. We begin with the 1-factor theorem.

Theorem 1 (The 1-factor theorem, [11]) *A connected graph G either has a 1-factor or is factor-critical if and only if*

$$odd(G - S) \leq |S| \qquad \text{for all} \quad \emptyset \neq S \subset V(G). \tag{1}$$

Assume that a connected graph G satisfies (1). If G has even order, then G has a 1-factor, otherwise, G is factor-critical. Moreover, the 1-factor theorem is usually stated as follows: a graph G has a 1-factor if and only if $odd(G-S) \leq |S|$ for all $S \subset V(G)$. By letting $S = \emptyset$ in this form, we obtain that every component of G is of even order. However as mentioned in the above theorem, if we use $\emptyset \neq S \subset V(G)$ instead of $S \subset V(G)$, then the order of G is not necessarily even, and if G has odd order and satisfies (1), then G is factor-critical. This fact is shown as follows.

It is known that a graph H of even order satisfies $odd(H - X) \equiv |X| \pmod 2$ for every $X \subset V(H)$. Assume that a connected graph G has odd order and satisfies (1), and let x be any vertex of G. Then $G - x$ has even order, and for every $S \subset V(G-x)$, it follows from (1) and the property given above that

$$\begin{aligned} &odd(G - x - S) = odd(G - (S \cup \{x\})) \leq |S \cup \{x\}| = |S| + 1 \quad \text{and} \\ &odd(G - x - S) \equiv |S| \pmod 2. \end{aligned}$$

Thus $odd(G - x - S) \leq |S|$. So $G - x$ has a 1-factor by the usual 1-factor theorem, and hence G is factor-critical. Conversely, if G is factor-critical, then for $\emptyset \neq S \subset V(G)$ and $y \in S$, we have $odd(G - S) = odd(G - y - (S - y)) \leq |S - y| \leq |S|$ since $G - y$ has a 1-factor. Hence (1) holds.

The next theorem is also well-known.

Theorem 2 ([12], Theorem 7.2 in [1]) *A connected graph G of order at least 2 has a $\{K_2, C_n : n \geq 3\}$-factor if and only if*

$$iso(G - S) \leq |S| \qquad \text{for all} \quad \emptyset \neq S \subset V(G). \tag{2}$$

Since $iso(G-S) \leq odd(G-S)$, if a connected graph G of order at least 2 satisfies (1), then G satisfies (2), and so G has a $\{K_2, C_n : n \geq 3\}$-factor. We can construct such a factor as follows. Assume that G satisfies (1). If G has even order, then G has a 1-factor, which is clearly a $\{K_2, C_n : n \geq 3\}$-factor. Assume that G has odd order, and let u and v be two adjacent vertices of G. Since G is factor-critical, $G - u$ has a 1-factor M_u and $G - v$ has a 1-factor M_v. Then $M_u \cup M_v$ is a union of two matchings of G, and each component of $M_u \cup M_v$ is a K_2, an even cycle, or a path connecting u and v. Hence $(M_u \cup M_v) + uv$ is a $\{K_2, C_n : n \geq 3\}$-factor of G, which contains at most one odd cycle.

We denote by $\omega(G)$ the number of components of G. A connected graph G is said to be *t-tough* if $|S| \geq t\omega(G-S)$ for every $S \subset V(G)$ with $\omega(G-S) > 1$. It is obvious that

$$iso(G-S) \leq odd(G-S) \leq \omega(G-S) \qquad \text{for all} \quad \emptyset \neq S \subset V(G).$$

In this paper, we first characterize a connected graph G that satisfies $\omega(G - S) \leq |S|$ for all $\emptyset \neq S \subset V(G)$. Such a graph is called *1-tough*. Bauer, Hakimi and Schmeichel [3] showed that for any positive rational number t, the t-tough problem, which is a problem of checking a graph to be t-tough or not, is NP-Hard.

In this paper, we give a characterization of a 1-tough graph in terms of graph factors. Later we generalize this characterization by using a function $f : V(G) \to \{1, 3, 5, \ldots\}$. Some results related to our theorems are found in [2, 4, 5, 6, 7, 9, 10].

2 Characterization of 1-tough graphs

In this section, we give a characterization of a graph G that satisfies $\omega(G - S) \leq |S|$ for all $\emptyset \neq S \subset V(G)$. In order to state our theorem, we need some notions and definitions. Let $\mathbf{Z}$ denote the set of integers. For two vertices x and y of a graph, an edge joining x to y is denoted by xy or yx. The degree of a vertex v in a subgraph H is denoted by $\deg_H(v)$. For two vertex sets X and Y of G, not necessary to be disjoint, we denote by $e_G(X, Y)$ the number of edges of G joining a vertex of X to a vertex of Y. If C is a component of $G - S$, then we briefly write $e_G(C, S)$ for $e_G(V(C), S)$. For a vertex set X of G, the subgraph of G induced by X is denoted by $\langle X \rangle_G$. For a function $h : V(G) \to \mathbf{Z}$, a subset $X \subseteq V(G)$ and a component C of $G - S$ for some $S \subset V(G)$, we write

$$h(X) := \sum_{x \in X} h(x) \quad \text{and} \quad h(C) := \sum_{x \in V(C)} h(x).$$

For any vertex x of G, let G^x denote the graph obtained from G by adding a new vertex x' together with a new edge xx', that is, $G^x = G + xx'$. Let $H : V(G) \to \{\{1\}, \{0, 2\}\}$ be a set-valued function. So $H(v)$ is equal to $\{1\}$ or $\{0, 2\}$ for each vertex v. We write

$$H^{-1}(1) := \{v \in V(G) : H(v) = \{1\}\}.$$

A spanning subgraph F of G is called an *H-factor* if $\deg_F(v) \in H(v)$ for all $v \in V(G)$. This H-factor is also called a *{1,{0,2}}-factor*. It is clear that if G has an H-factor, then $|H^{-1}(1)|$ must be even by the Handshaking Lemma. So if $|H^{-1}(1)|$ is odd, then G has no H-factor. For a function $H : V(G) \to \{\{1\}, \{0, 2\}\}$ and a vertex x of G, we define $H^x : V(G^x) \to \{\{1\}, \{0, 2\}\}$ as follows.

$$H^x(v) = \begin{cases} \{1\} & \text{if } v = x', \\ H(v) & \text{otherwise.} \end{cases} \tag{3}$$

A graph G is said to be *H-critical* or *{1,{0,2}}-critical* if G^x has an H^x-factor for every vertex x of G.

Let $g, f : V(G) \to \mathbf{Z}$ be functions such that $g(v) \le f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$, where we allow that $g(x) < 0$ and $\deg_G(y) < f(y)$ for some vertices x and y (see Theorem 6.1 in [1]). Then a spanning subgraph F of G is called a *parity (g, f)-factor* if

$$g(v) \le \deg_F(v) \le f(v) \quad \text{and} \quad \deg_F(v) \equiv f(v) \pmod 2$$

for all $v \in V(G)$. The following theorem gives a criterion for a graph to have a parity (g, f)-factor.

Theorem 3 (Lovász, [8], Theorem 6.1 in [1]) *Let G be a connected graph and $g, f : V(G) \to \mathbf{Z}$ such that $g(v) \le f(v)$ and $g(v) \equiv f(v) \pmod 2$ for all $v \in V(G)$. Then G has a parity (g, f)-factor if and only if for any two disjoint subsets S, T of $V(G)$,*

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} \deg_G(x) \; - e_G(S, T) - q(S, T) \ge 0, \tag{4}$$

where $q(S, T)$ denotes the number of components C of $G - S - T$, called g-odd components, such that $f(C) + e_G(C, T) \equiv 1 \pmod 2$. If necessary, we write $\eta(G; S, T)$ and $q(G; S, T)$ for $\eta(S, T)$ and $q(S, T)$ to express the graph G.

Note that if (4) holds, then $\eta(\emptyset, \emptyset) = -q(\emptyset, \emptyset) \ge 0$, which implies that $|f(V(G))| \equiv 0 \pmod 2$. The following lemma will prove useful.

Lemma 4 *Let G, g, f, S, T and $\eta(S,T)$ be the same as Theorem 3. Then*

$$\eta(S,T) \equiv f(V(G)) \equiv \sum_{x \in V(G)} f(x) \pmod 2.$$

Proof. Let $C_1, C_2, \ldots, C_m$ be the g-odd components of $G-(S\cup T)$, and let $D_1, D_2, \ldots, D_r$ be the other components of $G-(S\cup T)$. Then $m = q(S,T)$, $f(C_i)+e_G(C_i,T) \equiv 1 \pmod 2$ for $1 \le i \le m$, and $f(D_j)+e_G(D_j,T) \equiv 0 \pmod 2$ for $1 \le j \le r$. Hence

$$\begin{aligned} m &\equiv \sum_{i=1}^{m}(f(C_i)+e_G(C_i,T)) + \sum_{j=1}^{r}(f(D_j)+e_G(D_j,T)) \\ &\equiv \sum_{x\in V(G)-(S\cup T)} f(x) + e_G(V(G)-(S\cup T),T) \pmod 2. \end{aligned}$$

Since $g(x) \equiv f(x) \pmod 2$ and $-k \equiv k \pmod 2$ for every integer k, we have the following.

$$\begin{aligned} \eta(S,T) &\equiv f(S)+f(T)+\sum_{x\in T}\deg_G(x)+e_G(S,T)+m \\ &\equiv f(S)+f(T)+e_G(V(G),T)+e_G(S,T) \\ &\qquad + \sum_{x\in V(G)-(S\cup T)} f(x) + e_G(V(G)-(S\cup T),T) \\ &= \sum_{x\in V(G)} f(x) + e_G(V(G),T) + e_G(V(G)-T,T) \\ &= f(V(G)) + 2|E(\langle T\rangle_G)| \quad (\text{by } e_G(T,T) = 2|E(\langle T\rangle_G)|) \\ &\equiv \sum_{x\in V(G)} f(x) \pmod 2. \end{aligned}$$

Therefore the lemma holds. □

The next theorem is our first result, which gives a characterization of a 1-tough graph.

Theorem 5 *Let G be a connected graph. Then the following two statements hold.*

(i) *G has an H-factor for every $H : V(G) \to \{\{1\},\{0,2\}\}$ with $|H^{-1}(1)|$ even if and only if*

$$\omega(G-S) \le |S|+1 \quad \textit{for all} \quad S \subset V(G). \tag{5}$$

(ii) *G is H-critical for every $H : V(G) \to \{\{1\}, \{0,2\}\}$ with $|H^{-1}(1)|$ odd if and only if*

$$\omega(G-S) \le |S| \quad \textit{for all} \quad \emptyset \ne S \subset V(G). \tag{6}$$

Proof. We first prove the statement (i), starting with sufficiency. Let $H : V(G) \to \{\{1\}, \{0,2\}\}$ be any set-valued function such that $|H^{-1}(1)|$ is even. Let M be a sufficiently large odd integer. Define $f : V(G) \to \mathbf{Z}$ as

$$f(v) = \begin{cases} 1 & \text{if } H(v) = \{1\}, \\ 2 & \text{otherwise.} \end{cases}$$

Next define $g : V(G) \to \mathbf{Z}$ as

$$g(v) = \begin{cases} -M & \text{if } H(v) = \{1\}, \\ -M-1 & \text{otherwise.} \end{cases}$$

Then it is easy to see that G has an H-factor if and only if G has a parity (g,f)-factor. We use Theorem 3. Let S and T be two disjoint subsets of $V(G)$. If $T \ne \emptyset$, then $-g(T)$ is sufficiently large, and so

$$\eta(S,T) = f(S) - g(T) + \sum_{x \in T} \deg_G(x) - e_G(S,T) - q(S,T) \ge 0.$$

Thus we may assume that $T = \emptyset$. It follows that $\eta(\emptyset,\emptyset) = -q(\emptyset,\emptyset) = 0$ since $f(V(G)) \equiv |H^{-1}(1)| \equiv 0 \pmod 2$ and G is connected. Hence we may assume $S \ne \emptyset$. By $q(S,\emptyset) \le \omega(G-S)$ and (5), we have

$$\eta(S,\emptyset) = f(S) - q(S,\emptyset) \ge |S| - \omega(G-S) \ge -1.$$

By $f(V(G)) \equiv 0 \pmod 2$ and Lemma 4, the above inequality implies $\eta(S,\emptyset) \ge 0$. Therefore G has the desired H-factor.

We now prove the necessity. Suppose that there exists a subset $\emptyset \ne S' \subset V(G)$ such that

$$\omega(G-S') \ge |S'| + 2. \tag{7}$$

Let $C_1, C_2, \dots, C_a$ be the odd components of $G - S'$, and let $D_1, D_2, \dots, D_b$ be the even components of $G - S'$, where $|V(C_i)|$ is odd and $|V(D_j)|$ is even. If $b \ge 1$, then take a vertex $w_i \in D_i$ for every $1 \le i \le b$, and let $W \subseteq \{w_i : 1 \le i \le b\}$ such that $|W| \in \{b-1, b\}$ and $|V(G)| - |W|$ is even. If $b = 0$, then take $W \subseteq V(C_1)$ such that $|W| \in \{0,1\}$ and $|V(G)| - |W|$ is even.

We define $H : V(G) \to \{\{1\}, \{0, 2\}\}$ as

$$H(v) = \begin{cases} \{0, 2\} & \text{if } v \in W, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then $|H^{-1}(1)|$ is even by $H^{-1}(1) = V(G) - W$ and by the choice of W. Let M be a sufficiently large odd integer, and define $f, g : V(G) \to \mathbf{Z}$ as

$$f(v) = \begin{cases} 2 & \text{if } v \in W \\ 1 & \text{otherwise,} \end{cases}$$

and

$$g(v) = \begin{cases} -M - 1 & \text{if } v \in W \\ -M & \text{otherwise.} \end{cases}$$

Then it is easy to see that G has an H-factor if and only if G has a parity (g, f)-factor. By the definitions of g, f, all but at most one component of $G - S'$ are g-odd components. Thus we have $q(S', \emptyset) \geq \omega(G - S') - 1$. We use Theorem 3. Since $f(S') = |S'|$, we obtain by (7) that

$$\eta(S', \emptyset) = f(S') - q(S', \emptyset) \leq |S'| - \omega(G - S') + 1 \leq -1.$$

Therefore G has no parity (g, f)-factor, which implies G has no H-factor.

We next prove the statement (ii). Let $H : V(G) \to \{\{1\}, \{0, 2\}\}$ be any set-valued function such that $|H^{-1}(1)|$ is odd. Let x be any chosen vertex of G, and define H^x as in (3). Then $(H^x)^{-1}(1) = H^{-1}(1) \cup \{x'\}$ contains an even number of vertices. We shall show that G^x and g, f satisfy the condition of Theorem 3, where g and f are defined as in the previous proof of the statement (i) and $H^x(x') = \{1\}$, $f(x') = 1$ and $g(x') = -M$. Let S and T be two disjoint subsets of $V(G^x) = V(G) \cup \{x'\}$. By the same argument given above, we may assume that $T = \emptyset$. It follows that $\eta(\emptyset, \emptyset) = -q(\emptyset, \emptyset) = 0$ since $f(V(G^x)) \equiv |(H^x)^{-1}(1)| \equiv 0 \pmod 2$ and G^x is connected. Hence we may assume that $S \neq \emptyset$. If S contains x', then $\omega(G^x - S) = \omega(G - (S - \{x'\}))$, and so it follows from (6) that

$$\eta(G^x; S, \emptyset) = f(S) - q(G^x; S, \emptyset) \geq |S| - \omega(G - (S - \{x'\})) \geq 1.$$

Hence we may assume that S does not contain x'. If S does not contain x, then $\omega(G^x - S) = \omega(G - S)$, and so

$$\eta(G^x; S, \emptyset) = f(S) - q(G^x; S, \emptyset) \geq |S| - \omega(G - S) \geq 0.$$

If S contains x, then $\omega(G^x - S) = \omega(G - S) + 1$. Thus

$$\eta(G^x; S, \emptyset) = f(S) - q(G^x; S, \emptyset) \geq |S| - \omega(G - S) - 1 \geq -1. \tag{8}$$

On the other hand, since

$$\sum_{v\in V(G^x)} f(v) \equiv |(H^x)^{-1}(1)| \equiv 0 \pmod 2,$$

it follows from Lemma 4 and (8) that $\eta(G^x; S, \emptyset) \geq 0$. Consequently G^x has an H^x-factor, and therefore G is H-critical.

Next we prove the necessity of (ii). Suppose that there exists a subset $\emptyset \neq S' \subset V(G)$ such that

$$\omega(G - S') \geq |S'| + 1. \tag{9}$$

Let $C_1, C_2, \ldots, C_a$ be the odd components of $G - S'$, and $D_1, D_2, \ldots, D_b$ be the even components of $G - S'$, where $|V(C_i)|$ is odd and $|V(D_j)|$ is even. If $b \geq 1$, then take a vertex $w_i \in D_i$ for every $1 \leq i \leq b$, and let $W \subseteq \{w_i : 1 \leq i \leq b\}$ such that $|W| \in \{b-1, b\}$ and $|V(G)| - |W|$ is odd. If $b = 0$, then let $W \subseteq V(C_1)$ such that $|W| \in \{0, 1\}$ and $|V(G)| - |W|$ is odd. Moreover, choose one vertex x from S', and let $G^x = G + xx'$.

We define $H^x : V(G^x) \to \{\{1\}, \{0, 2\}\}$ as

$$H^x(v) = \begin{cases} \{0, 2\} & \text{if } v \in W, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then $(H^x)^{-1}(1) = (V(G) - W) \cup \{x'\}$ and so $|(H^x)^{-1}(1)|$ is even. Let M be a sufficiently large odd integer, and define $f, g : V(G^x) \to \mathbf{Z}$ as

$$f(v) = \begin{cases} 2 & \text{if } v \in W \\ 1 & \text{otherwise,} \end{cases}$$

and

$$g(v) = \begin{cases} -M-1 & \text{if } v \in W \\ -M & \text{otherwise.} \end{cases}$$

Then it is easy to see that G^x has an H^x-factor if and only if G^x has a parity (g, f)-factor. We use Theorem 3. Since $f(S') = |S'|$ and $q(G^x; S', \emptyset) \geq \omega(G - S') - 1 + |\{x'\}| = \omega(G - S')$, we obtain by (9) that

$$\eta(G^x; S', \emptyset) = f(S') - q(G^x; S', \emptyset) \leq |S'| - \omega(G - S') \leq -1.$$

Therefore G^x has no parity (g, f)-factor, which implies G is not H-critical. Consequently, the proof of Theorem 5 is complete. □

Remark: Tutte's 1-Factor Theorem builds a relation between 1-factors and odd components. For an even component C, by picking a vertex $v \in V(C)$

and assigning $\{0,2\}$ to v, even component C becomes an H-odd component. Theorem 5 builds a relation between factors and 1-toughness. Consider Petersen Graph P, which is 1-tough. Let x_1, x_2 be two adjacent vertices of P. Note that $P-x-y$ is a cycle of length eight C_8. We denote $C_8 = v_1v_2\dots, v_8v_1$. Define

$$H(v) = \begin{cases} \{0,2\} & \text{if } v \in \{x_1, v_1, v_2\}, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then $\{x_1'x_1x_2, v_8v_1v_2v_3, v_4v_5, v_6v_7\}$ is a H^{x_1}-factor of P^{x_1}.

3 {(1,f)-odd, even}-factors

In this section, we generalize Theorem 5 by using an odd integer valued function f. Let G be a graph, let $f : V(G) \to \{1,3,5,\dots\}$ be a function, and let

$$2N = \max\{f(x) : x \in V(G)\} + 1$$

be an even integer. Define a set-valued function H_f on $V(G)$ by

$$H_f(v) = \{1,3,\dots,f(v)\} \quad \text{or} \quad \{0,2,\dots,2N\} \quad \text{for each } v \in V(G). \tag{10}$$

Thus for a given function f, there are $2^{|V(G)|}$ set-valued functions H_f. For a set-valued function H_f on $V(G)$, define

$$H_f^{-1}(f) := \{v \in V(G) : H_f(v) = \{1,3,\dots,f(v)\}\ \}.$$

A spanning subgraph F of G is called an *H_f-factor* if $\deg_F(v) \in H_f(v)$ for all $v \in V(G)$. This H_f-factor is also called an *{(1,f)-odd, even}-factor*. For a vertex x of G, we define a graph $G^x = G + xx'$. Moreover, for a function H_f on $V(G)$, define the function H_f^x on $V(G^x)$ as follows.

$$H_f^x(v) = \begin{cases} \{1\} & \text{if } v = x', \\ H_f(v) & \text{otherwise.} \end{cases} \tag{11}$$

A graph is said to be *H_f-critical* or *{(1,f)-odd, even}-critical* if G^x has an H_f^x-factor for every vertex x of G.

In this section, we prove the following theorem.

Theorem 6 *Let G be a connected graph, and let $f : V(G) \to \{1,3,5,\dots\}$ be a function. Then the following two statements hold.*

(i) *G has an H_f-factor for every function H_f with $|H_f^{-1}(f)|$ even if and only if*

$$\omega(G-S) \le f(S) + 1 \quad \textit{for all} \quad S \subset V(G). \tag{12}$$

(ii) *G is H_f-critical for every function H_f with $|H_f^{-1}(f)|$ odd if and only if*

$$\omega(G-S) \le f(S) \quad \textit{for all} \quad \emptyset \neq S \subset V(G). \tag{13}$$

Proof. Since this theorem can be proved in a similar way as Theorem 5, we omit some details of the proof. We first prove the sufficiency for each of (i) and (ii). Assume that G satisfies (12). Let H_f be any set-valued function defined by (10) such that $|H_f^{-1}(f)|$ is even. Let M be a sufficiently large odd integer. Define $f_1, g_1 : V(G) \to \mathbf{Z}$ as

$$f_1(v) = \begin{cases} f(v) & \text{if } H_f(v) = \{1, 3, \ldots, f(v)\}, \\ 2N & \text{otherwise,} \end{cases}$$

and

$$g_1(v) = \begin{cases} -M & \text{if } H_f(v) = \{1, 3, \ldots, f(v)\}, \\ -M-1 & \text{otherwise.} \end{cases}$$

It is easy to see that G has an H_f-factor if and only if G has a parity (g_1, f_1)-factor. We use Theorem 3. Let S and T be two disjoint subsets of $V(G)$. If $T \neq \emptyset$, then $-g_1(T)$ is sufficiently large, and so $\eta(S,T) \ge 0$. Thus we may assume that $T = \emptyset$. It follows that $\eta(\emptyset, \emptyset) = -q(\emptyset, \emptyset) = 0$ since $|H_f^{-1}(f)|$ is even and G is connected. Hence we may assume that $S \neq \emptyset$. By $f_1(S) \ge f(S)$, $q(S, \emptyset) \le \omega(G-S)$ and by (12), we have

$$\eta(S, \emptyset) = f_1(S) - q(S, \emptyset) \ge f(S) - \omega(G-S) \ge -1.$$

Since $f_1(V(G)) \equiv |H_f^{-1}(f)| \equiv 0 \pmod 2$, the above inequality implies $\eta(S, \emptyset) \ge 0$ by Lemma 4. Therefore G has the desired H_f-factor.

We next assume that G satisfies (13). In this case, it is also assumed that $|H_f^{-1}(f)|$ is odd. Let x be any chosen vertex of G. We shall show that G^x and f_1, g_1 satisfy the conditions of Theorem 3, where $f_1(x') = 1$ and $g_1(x') = -M$. Let S and T be two disjoint subsets of $V(G^x) = V(G) \cup \{x'\}$. By the same argument given above, we may assume $T = \emptyset$. It follows that $\eta(G^x; \emptyset, \emptyset) = -q(G^x; \emptyset, \emptyset) = 0$ since $\{v \in V(G^x) : f_1(v) \equiv 1 \pmod 2\} = \{x'\} \cup H_f^{-1}(f)$ contains an even number of vertices and G^x is connected. Hence we may assume that $S \neq \emptyset$. If S contains x', then $\omega(G^x - S) \le \omega(G - (S - x'))$, and so $\eta(G^x; S, \emptyset) \ge f(S) - \omega(G - (S - x')) \ge 1$. Thus we may assume that S does not contain x'. If S does not contain x, then $\omega(G^x - S) = \omega(G - S)$, and so $\eta(G^x; S, \emptyset) \ge f(S) - \omega(G-S) \ge 0$. If S contains x, then $\omega(G^x - S) = \omega(G-S) + 1$, and thus $\eta(G^x; S, \emptyset) \ge f(S) - \omega(G-S) - 1 \ge -1$, which implies $\eta(G^x; S, \emptyset) \ge 0$ by Lemma 4 and $f_1(V(G^x)) \equiv |H_f^{-1}(f) \cup \{x'\}| \equiv 0 \pmod 2$. Therefore G^x has a H_f^x-factor. Consequently G is H_f-critical.

We now prove the necessity for each of (i) and (ii). First consider (i). Assume that there exists a subset $\emptyset \neq S' \subset V(G)$ such that

$$\omega(G - S') \geq f(S') + 2. \tag{14}$$

Let $C_1, C_2, \ldots, C_a$ be the odd components of $G - S'$, and let $D_1, D_2, \ldots, D_b$ be the even components of $G - S'$. If $b \geq 1$, then take a vertex $w_i \in D_i$ for every $1 \leq i \leq b$, and let $W \subseteq \{w_i : 1 \leq i \leq b\}$ such that $|W| \in \{b-1, b\}$ and $|V(G)| - |W|$ is even. If $b = 0$, then take $W \subseteq V(C_1)$ such that $|W| \in \{0, 1\}$ and $|V(G)| - |W|$ is even.

We define $H_f : V(G) \to \{\{1, 3, \ldots, f(v)\}, \{0, 2, \ldots, 2N\}\}$ as

$$H_f(v) = \begin{cases} \{0, 2, \ldots, 2N\} & \text{if } v \in W, \\ \{1, 3, \ldots, f(v)\} & \text{otherwise.} \end{cases}$$

Then $|H_f^{-1}(f)|$ is even by $H_f^{-1}(f) = V(G) - W$ and by the choice of W. Let M be a sufficiently large odd integer, and define $f_2, g_2 : V(G) \to \mathbf{Z}$ as

$$f_2(v) = \begin{cases} 2N & \text{if } v \in W \\ f(v) & \text{otherwise,} \end{cases}$$

and

$$g_2(v) = \begin{cases} -M - 1 & \text{if } v \in W \\ -M & \text{otherwise.} \end{cases}$$

Then G has an H_f-factor if and only if G has a parity (g_2, f_2)-factor. We use Theorem 3. Since $f_2(S') = f(S')$ and $q(S', \emptyset) \geq \omega(G - S') - 1$, it follows from (14) that

$$\eta(S', \emptyset) = f_2(S') - q(S', \emptyset) \leq f(S') - \omega(G - S') + 1 \leq -1.$$

Therefore G has no parity (g_2, f_2)-factor, and thus G has no H_f-factor.

Next consider (ii). Suppose that there exists a subset $\emptyset \neq S' \subset V(G)$ such that

$$\omega(G - S') \geq f(S') + 1. \tag{15}$$

Let $C_1, C_2, \ldots, C_a$ be the odd components of $G - S'$, and $D_1, D_2, \ldots, D_b$ be the even components of $G - S'$. If $b \geq 1$, then take a vertex $w_i \in D_i$ for every $1 \leq i \leq b$, and let $W \subseteq \{w_i : 1 \leq i \leq b\}$ such that $|W| \in \{b-1, b\}$ and $|V(G)| - |W|$ is odd. If $b = 0$, then let $W \subseteq V(C_1)$ such that $|W| \in \{0, 1\}$ and $|V(G)| - |W|$ is odd. Define a set-valued function H_f on $V(G)$ as

$$H_f(v) = \begin{cases} \{0, 2, \ldots, 2N\} & \text{if } v \in W, \\ \{1, 3, \ldots, f(v)\} & \text{otherwise.} \end{cases}$$

Then $|(H_f)^{-1}(f)| = |V(G) - W|$ is odd.

Choose one vertex x from S', and let $G^x = G + xx'$. Then define a function H_f^x on $V(G^x)$ as in (11). Let M be a sufficiently large odd integer, and define $f_2, g_2 : V(G^x) \to \mathbf{Z}$ as

$$f_2(v) = \begin{cases} 2N & \text{if } v \in W, \\ f(v) & \text{if } v \in V(G) - W, \\ 1 & \text{if } v = x'. \end{cases}$$

and

$$g_2(v) = \begin{cases} -M-1 & \text{if } v \in W \\ -M & \text{if otherwise.} \end{cases}$$

Then it is easy to see that G^x has an H_f^x-factor if and only if G^x has a parity (g_2, f_2)-factor. We use Theorem 3. Since all but at most one component of $G - S'$ are g_2-odd components, we have $q(G; S', \emptyset) \geq \omega(G - S') - 1$. Note that x' is an isolated vertices of $G^x - S'$ and $G - S' = G^x - S' - x'$. Thus we have $q(G^x; S', \emptyset) \geq \omega(G - S') - 1 + |\{x'\}| = \omega(G - S')$. Since $f_2(S') = f(S')$, we obtain by (15) that

$$\eta(G^x; S', \emptyset) = f(S') - q(G^x; S', \emptyset) \leq f(S') - \omega(G - S') \leq -1.$$

Therefore G^x has no parity (g_2, f_2)-factor, which implies G is not H_f-critical. Consequently, the proof of Theorem 6 is complete. $\square$

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