Properly Colored Spanning Trees in Edge-Colored Graphs

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Abstract

A subgraph H of an edge-colored graph G is called a properly colored subgraph if no two adjacent edges of H have the same color, and is called a rainbow subgraph if no two edges of H have the same color. We prove the following two theorems and show that the conditions on the minimum color degree are sharp. Let G be an edge-colored graph with minimum color degree $\delta^c(G)$. If $\delta^c(G) \geq |G|/2$, then G has a properly colored spanning tree. Moreover, if $\delta^c(G) \geq |G|/2$ and the set of edges colored with any fixed color forms a subgraph of order at most $(|G|/2) + 1$, then G has a rainbow spanning tree. We also give a new proof of a necessary and sufficient condition for the existence of properly colored spanning trees in edge-colored compete graphs which appeared in (Abouelaoualim et al, Proceedings of CTW 09, Paris, 115-119). Also we generalize it to edge-colored balanced compete bipartite graphs.

Keywords: spanning tree, rainbow, properly colored, edge colored graph

1 Introduction

In this paper we consider finite simple graphs, which have neither loops nor multiple edges. For a graph $G = (V(G), E(G))$, let $V(G)$, $E(G)$ and $|G|$

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denote the vertex set, the edge set and the order of G, respectively. Thus $|G| = |V(G)|$. For a vertex v, the degree of v in G is denoted by $d_G(v)$, and the minimum degree of G is denoted by $\delta(G)$.

If every edge of a graph G is colored, then G is called an *edge-colored graph* or briefly a *colored graph*. Let G be a colored graph. For a vertex v of G, the *color degree* of v, denoted by $d^c_G(v)$, is the number of distinct colors appeared in the edges incident with v, and the *minimum color degree* of G, denoted by $\delta^c(G)$, is the minimum value among the color degrees of all vertices of G.

Let H be a subgraph of a colored graph G. Then H is called a *properly colored subgraph* if no two adjacent edges of H have the same color. On the other hand, if no two edges of H have the same color, then H is called a *rainbow subgraph* or a *heterochromatic subgraph*. The complete graph of order n is denoted by K_n, and the complete bipartite graph with partite sets of order m and n is denoted by $K_{m,n}$, and $K_{1,n}$ is called a *star*, where $m, n \geq 1$ are integers. For a star $K_{1,n}$ with $n \geq 2$, the vertex of degree n is called its *center*, and the *center* of $K_{1,1}$ is any chosen vertex.

The classical Dirac's theorem in [6] states that every graph G with order at least 3 and minimum degree $\delta(G) \geq |G|/2$ contains a Hamiltonian cycle. A natural question is the following: Does an edge-colored graph G with $\delta^c(G) \geq |G|/2$ have a properly colored Hamiltonian cycle? However, Fujita and Magnant [9] showed that there exists a coloring of K_{2m} with $\delta^c(K_{2m}) = m$ which has no properly colored Hamiltonian cycle. Furthermore, in [12], Lo showed that the lower bound cannot be better than $(2/3)|G|$. Besides, he proved the following theorem.

Theorem 1 (Lo [11])**.** *For any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored graph G with $\delta^c(G) \geq (\frac{2}{3} + \varepsilon)|G|$ and $|G| \geq n_0$ contains a properly colored cycle of length l for all $3 \leq l \leq |G|$.*

Therefore, we tend to consider a properly colored spanning tree in an edge-colored graph. However the following is known, and so it is difficult to find a properly colored spanning tree.

Theorem 2 ([1])**.** *Finding a properly colored spanning tree in an edge-colored graph is NP-complete.*

In this paper, we show that an edge-colored graph G with $\delta^c(G) \geq |G|/2$ has a properly colored spanning tree. In fact, we prove the following three theorems, and show that the condition on the minimum color degree is sharp, which is shown in Section 3.

Theorem 3. *Let G be an edge-colored connected graph. If*

$$\delta^c(G) \geq \frac{|G|}{2},$$

then G has a properly colored spanning tree.

Notice that the above theorem follows immediately from the following Theorem 5, and it is shown in the beginning of Section 3.

Theorem 4. *Let G be an edge-colored connected graph having the property that for every color c, the set of edges colored with c forms a subgraph of G with order at most $\frac{|G|}{2} + 1$. If*

$$\delta^c(G) \geq \frac{|G|}{2},$$

then G has a rainbow spanning tree.

Theorem 5. *Let G be an edge-colored connected graph having the property that for every color c, the set of edges colored with c forms a star. If*

$$\delta^c(G) \geq \frac{|G|}{2},$$

then G has a rainbow spanning tree.

We first explain that Theorem 5 is an easy consequence of Theorem 4. Assume that G satisfies the conditions of Theorem 5. Let S be a star with center u induced by the set of edges colored with any fixed color. Then it follows that

$$d_S(u) \leq d_G(u) - (d_G^c(u) - 1) \leq (|G| - 1) - \left(\frac{|G|}{2} - 1\right) = \frac{|G|}{2},$$

and so $d_S(u) \leq |G|/2$. Thus $|S| \leq (|G|/2) + 1$. Hence G satisfies the conditions of Theorem 4, and thus Theorem 5 follows from Theorem 4.

We now mention a few known results on rainbow spanning trees. Some other results related to our theorems can be found in [10]. The maximum color degree of an edge-colored graph G is denoted by $\Delta^c(G)$, which is the maximum value among the color degrees of all vertices of G.

Theorem 6 (Brualdi and Hollingsworht [4]). *The edge-colored complete graph K_{2n} ($n \geq 3$) has two edge disjoint rainbow spanning trees if the set of edges colored with any color forms a perfect matching of K_{2n}.*

Theorem 7 (Akbari and Alipour [2]). *Assume that the edges of the complete graph K_n are colored with t colors. If $t \geq n-1$ and $\Delta^c(K_n) \leq (n+3)/2$, then K_n has a rainbow spanning tree.*

Theorem 8 (Suzuki [14]). *The edge-colored complete graph K_n has a rainbow spanning tree if the number of edges colored with any color is at most $\frac{n}{2}$.*

There are some other results on rainbow spanning trees, and most of them give sufficient conditions for a colored complete graph or a colored complete bipartite graph to have a rainbow spanning tree (see [10]). On the other hand, Theorem 4 gives a sufficient condition for a colored general graph to have a rainbow spanning tree. The following theorem gives a criterion for a colored graph to have a rainbow spanning tree, and we use Theorem 9 in the proof of Theorem 4.

Theorem 9 (Akbari and Alipour [2], and Suzuki [14]). *An edge-colored connected graph G has a rainbow spanning tree if and only if for any r colors ($1 \leq r \leq |G|-2$), the removal of all the edges colored with these r colors from G results in a graph having at most $r+1$ components.*

2 Proof of Theorem 4

In this section, we prove Theorem 4.

Proof of Theorem 4. It is easy to see that the theorem holds if $2 \leq |G| \leq 3$. Hence we may assume that $|G| \geq 4$.

Assume that G has no rainbow spanning tree. By Theorem 9, there exist r colors such that $1 \leq r \leq |G|-2$ and the removal of all the edges colored with these r colors from G results in a graph that has at least $r+2$ components.

Let $n = |G|$, and let $X_1, X_2, ..., X_r$ be the subgraphs of G induced by the set of edges colored with each of these r colors, respectively. We call X_i a *monochromatic subgraph*. Then every X_i has order at most $(n/2)+1$ by the assumption. Let $W_1, W_2, ..., W_\ell$, $\ell \geq r+2$, be the components of $G - \bigcup_{i=1}^{r} E(X_i)$. Let H be the spanning subgraph of G defined as

$$H = G - \bigcup_{i=1}^{r} E(X_i) = (V(G), \bigcup_{j=1}^{\ell} E(W_j)).$$

Claim 2.1. $r \geq \lceil n/2 \rceil$.

Proof. Without loss of generality, we may assume that W_1 is a smallest component of H, namely, $|W_1| = \min\{|W_i| : 1 \le i \le \ell\}$. It is obvious that $|W_1| \le n/(r+2)$. Let v be a vertex of W_1. Then $d^c_G(v) \ge \delta^c(G) \ge n/2$, and so

$$\frac{n}{2} \le d^c_G(v) \le |W_1| - 1 + r \le \frac{n}{r+2} - 1 + r. \tag{1}$$

Hence

$$n \le \frac{2(r+2)(r-1)}{r} = 2(r+1) - \frac{4}{r}.$$

If n is even, then this inequality implies $n \le 2r$, and hence $\lceil n/2 \rceil \le r$. If n is odd, then (1) implies $(n+1)/2 \le d^c_G(v) \le n/(r+2) - 1 + r$, and so $n \le 2r + 1 - 6/r$. This implies $n \le 2r - 1$. Therefore $\lceil n/2 \rceil = (n+1)/2 \le r$. □

Claim 2.2. *Assume $n \ge 4$. Then for every r, $\lceil n/2 \rceil \le r \le n-2$, it follows that*

$$\frac{n^2}{4} - \frac{r}{2}\Big(\Big\lfloor \frac{n}{2} \Big\rfloor + 1\Big) \ge \binom{n-r-1}{2} + 1. \tag{2}$$

Proof. Consider a function $f(r)$ with $\lceil n/2 \rceil \le r \le n-2$ defined by

$$\begin{aligned} f(r) &= \frac{n^2}{4} - \frac{r}{2}\Big(\Big\lfloor \frac{n}{2} \Big\rfloor + 1\Big) - \binom{n-r-1}{2} - 1 \\ &= -\frac{r^2}{2} + \Big(n - 2 - \frac{1}{2} \cdot \Big\lfloor \frac{n}{2} \Big\rfloor\Big) r - \frac{n^2}{4} + \frac{3n}{2} - 2. \end{aligned}$$

In order to show that $f(r) \ge 0$ for all $\lceil n/2 \rceil \le r \le n-2$, it suffices to show that $f(\lceil n/2 \rceil) \ge 0$ and $f(n-2) \ge 0$. If n is even, then we have

$$\begin{aligned} f(n/2) &= -\frac{n^2}{8} + \Big(\frac{3n-8}{4}\Big)\frac{n}{2} - \frac{n^2}{4} + \frac{3n}{2} - 2 \\ &= \frac{n}{2} - 2 \ge 0, \end{aligned}$$

and

$$\begin{aligned} f(n-2) &= -\frac{(n-2)^2}{2} + \Big(\frac{3n-8}{4}\Big)(n-2) - \frac{n^2}{4} + \frac{3n}{2} - 2 \\ &= 0. \end{aligned}$$

If n is odd, then $\lceil n/2 \rceil = (n+1)/2$ and $\lfloor n/2 \rfloor = (n-1)/2$, and we have

$$\begin{aligned} f((n+1)/2) &= -\frac{(n+1)^2}{8} + \left(\frac{3n-7}{4}\right)\left(\frac{n+1}{2}\right) - \frac{n^2}{4} + \frac{3n}{2} - 2 \\ &= \frac{3n}{4} - 3 \geq 0, \end{aligned}$$

and

$$\begin{aligned} f(n-2) &= -\frac{(n-2)^2}{2} + \left(\frac{3n-7}{4}\right)(n-2) - \frac{n^2}{4} + \frac{3n}{2} - 2 \\ &= \frac{n}{4} - \frac{1}{2} \geq 0. \end{aligned}$$

Therefore the claim is proved. □

It is easy to see that a graph of order n with ℓ components has a maximum number of edges when it consists of one complete graph of order $n - \ell + 1$ and $\ell - 1$ isolated vertices. Thus it follows from $\ell \geq r + 2$ that

$$|E(H)| \leq \binom{n-\ell+1}{2} \leq \binom{n-r-1}{2}. \tag{3}$$

On the other hand, the removal of a monochromatic subgraph X_i from G decreases at most $\lfloor n/2 \rfloor + 1$ of $\sum_{v \in V(G)} d^c_G(v)$. Hence it follows from $\delta^c(G) \geq n/2$ that

$$\sum_{v \in V(G)} d^c_H(v) \geq \sum_{v \in V(G)} d^c_G(v) - r(\left\lfloor \frac{n}{2} \right\rfloor + 1) \geq \frac{n^2}{2} - r(\left\lfloor \frac{n}{2} \right\rfloor + 1).$$

By Claim 2.2, we obtain

$$|E(H)| \geq \frac{1}{2} \sum_{v \in V(G)} d^c_H(v) \geq \binom{n-r-1}{2} + 1.$$

The above inequality contradicts (3). Consequently the proof is complete. □

3 Proof of Theorem 3

In this section, we prove Theorem 3, and show that the condition on $\delta^c(G)$ of Theorems 3 and 4 is sharp. In an edge-colored G, a *monochromatic component* is a maximal connected monochromatic subgraph of G, and for every edge e of G, let *color*(e) denote the color of e.

Proof of Theorem 3. First, we take a spanning subgraph G_1 of G with minimum number of edges that satisfies $\delta^c(G_1) = \delta^c(G)$. Then the deletion of any edge of G_1 reduces $\delta^c(G_1)$. Thus every monochromatic component of G_1 is a star since otherwise there exists a monochromatic path $P = v_1v_2v_3v_4$ of length 3, but the deletion of the edge v_2v_3 does not reduce $\delta^c(G_1)$, a contradiction.

We construct a new edge-colored graph G^* from G_1 as follows: For any monochromatic component S of G_1, we recolor the edges of S by a new color c_S depending only on S, namely, all monochromatic components of G_1 are colored with distinct colors and all edges of the same monochromatic component of G_1 are colored with the same new color.

It is obvious that for each color c, the set of edges of G^* colored with c forms a star and $\delta^c(G^*) = \delta^c(G_1) = \delta^c(G) \geq |G|/2 = |G^*|/2$. By Theorem 4, G^* contains a rainbow spanning tree T^*. By recoloring all the edges of T^* with their original colors in G_1, we obtain a properly colored spanning tree of G_1, which is the desired spanning tree of G. Consequently, Theorem 3 is proved. □

We next show that the lower bound $\delta^c(G) \geq |G|/2$ is tight. We first consider the case where $|G| = 2m + 1$. Let K_m be a rainbow complete graph with $V(K_m) = \{u_1, u_2, \ldots, u_m\}$, in which all the edges have distinct colors. Let $v_1, v_2, \ldots, v_{m+1}$ be $m + 1$ new vertices, and let $c_1, c_2, \ldots, c_m$ be new m colors not appearing in K_m. We construct an edge-colored graph G_1 from K_m and $\{v_1, v_2, \ldots, v_{m+1}\}$ by adding a new edge u_iv_j colored with c_i for all $1 \leq i \leq m$ and $1 \leq j \leq m + 1$. Then G_1 satisfies $\delta^c(G_1) = m \geq (|G_1| - 1)/2$ and every monochromatic component of G_1 is a star. Note that in G_1, a properly colored spanning graph is also a rainbow spanning tree and vice versa. If we delete the m monochromatic stars induced by colors $c_1, c_2, \ldots, c_m$, then we obtain a graph with $m + 2$ components. Hence by Theorem 9, G_1 does not have a rainbow spanning tree.

Next consider the case where $|G| = 2m + 2$. We construct a graph G_2 from K_m and $m + 2$ new vertices $v_1, v_2, \ldots, v_{m+2}$ in the same way as given above. Hence $|G_2| = 2(m + 1)$, $\delta^c(G_2) = m \geq |G_2|/2 - 1$, and G_2 has no rainbow spanning tree. Therefore the lower bound on $\delta^c(G)$ is tight. □

In [5], Čada et al. proposed the following conjecture:

Conjecture 10. *Let G be an edge-colored graph of order n, and let k be a positive integer. If $\delta^c(G) \geq \frac{n+k}{2}$, then G contains a rainbow cycle of length at least k.*

As a corollary of Theorem 3, we prove a properly colored version of this conjecture.

Corollary 11. *Let k and n be two positive integers such that $1 \le k \le n-1$, and let G be an edge-colored graph of order n. If $\delta^c(G) \ge \frac{n+k+3}{2}$, then every properly colored path with length at most k is contained in a properly colored cycle with length at least $k+1$. Especially, every edge of G is contained in a properly colored cycle with length at least $k+1$.*

Proof. First, by $\delta^c(G) \ge k$, every properly colored path with length less than k can be extended to a properly colored path with length k. Take such a path $P = uw_1w_2...w_{k-1}v$ of length k and with endpoints u and v. Let $W = \{w_1, ..., w_{k-1}\}$ and $H = G - W$. Let H_1 be the subgraph of H obtained from H by deleting the edges adjacent to u with color $color(uw_1)$ and the edges adjacent to v with color $color(w_{k-1}v)$. Then, for every $x \in V(H_1)$, we have

$$d^c_{H_1}(x) \ge \delta^c(G) - |W| \ge \frac{n+k+3}{2} - (k+1) = \frac{n-k+1}{2}.$$

Since $|H_1| = n-k+1$, it follows that $\delta^c(H_1) \ge |H_1|/2$. By Theorem 3, H_1 contains a properly colored spanning tree T_1. Hence, T_1 contains a path $P_1 = ux_1x_2...x_av$ connecting u and v. Since $color(ux_1) \ne color(uw_1), color(x_av) \ne color(w_{k-1}v)$ and $x_i \notin W$ for every $1 \le i \le a$, $P_1 \cup P$ is the desired properly colored cycle with the length at least $k+1$. □

4 Other Results

A *1-tree-cycle system* of G is a set of vertex disjoint subgraphs consisting of one tree T and some cycles $C_1, \dots, C_d$. If this 1-tree-cycle system satisfies $V(T) \cup V(C_1) \cup \cdots \cup V(C_k) = V(G)$, then it is called a *spanning 1-tree-cycle system*. Moreover, if the tree T and every cycle C_i are properly colored, then it is called a *properly colored 1-tree-cycle system*. The following theorem has already appeared in [1], but they didn't give a complete proof in that paper. We now give a new short proof here.

Theorem 12 ([1])**.** *An edge-colored complete graph K_n contains a properly colored spanning tree if and only if it has a properly colored spanning 1-tree-cycle system.*

Proof. Since a properly colored spanning tree is itself a properly colored spanning 1-tree-cycle, it suffices to prove the sufficiency. Suppose that K_n contains a properly colored spanning 1-tree-cycle system $\{T, C_1, \dots, C_k\}$, where T is a tree and every C_i is a cycle. We prove the theorem by induction on k. Suppose that the theorem holds for $k = 1$. Then by applying the theorem with $k = 1$ to the complete subgraph induced by $V(T) \cup V(C_1)$, we obtain a properly colored spanning tree T_1 in it. This implies that there is a properly colored spanning 1-tree-cycle system $T_1 \cup C_2 \cup \cdots \cup C_k$ in K_n. Hence, by the induction hypotheses, there exists a properly colored spanning tree T_k in K_n. Therefore it suffice to prove that the theorem holds for $k = 1$.

Now we assume that K_n contains vertex disjoint properly colored a tree T and a cycle C_1 such that $V(T) \cup V(C_1) = V(G)$. We claim that K_n contains a properly colored spanning tree. Suppose to the contrary that G has no properly colored spanning tree. Let $C_1 = v_1 v_2 ... v_t$ be the properly colored cycle of order t, where v_i denotes a vertex, and for convenience, v_0 and v_{t+1} denote v_t and v_1, respectively. For every edge e of K_n, let $color(e)$ denote the color of e. For any edge $e \in E(T)$, $k(e)$ denotes the number of edges which join an end-vertex of e to C_1 and has the same color as e.

Claim 4.1. *For every edge xv_i of K_n joining $x \in V(T)$ to $v_i \in V(C_1)$, there exists a unique edge xz in T such that $color(xz) = color(xv_i)$.*

Proof. Suppose that there exists an edge xv_i joining T to C_1 such that no edge of T incident with x has the color $color(xv_i)$. Then $T + xv_i + (C_1 - v_i v_{i+1})$ or $T + xv_i + (C_1 - v_i v_{i-1})$ is a properly colored spanning tree of K_n, a contradiction. The uniqueness is obvious since T is properly colored. □

Claim 4.2. *Let $e = xy$ be an edge of T. If an edge xv_i joining T to C_1 has the same color as e, and if $color(v_i v_{i-1}) \neq color(e)$, $color(v_i v_{i+1}) \neq color(e)$, then $k(e) \leq t$.*

Proof. In fact, if $color(xv_j) \neq color(e)$ for every $1 \leq j \leq t$, then it is obvious that $k(e) \leq t$. Assume that there is an edge $xv_a, 1 \leq a \leq t$, that has the same color as e. Then at least one of edges of $v_a v_{a-1}$ and $v_a v_{a+1}\}$ has a distinct color from e, and let $v_a v_b$ the other edge of $v_a v_{a-1}$ and $v_a v_{a+1}$. Then the $T - e + xv_a + (C - v_a v_b)$ is a properly colored spanning tree, a contradiction. □

Claim 4.3. *For every edge $e = xy$ of T, we have $k(e) \leq t$.*

Proof. Let $e = xy$ be an edge of T. Assume that $k(e) > t$. By the same argument given in the proof of Claim 4.2, there exists a vertex v_a in C_1 such that xv_a has the same color as e. By Claim 4.2 and by symmetry, we may assume that $v_a v_{a+1}$ has the same color as e. If yv_{a+1} has the same color as e, then $T - e - v_a v_{a+1} + xv_a + yv_{a+1}$ is a properly colored spanning tree of K_n, a contradiction. Hence $color(yv_{a+1}) \neq color(e)$. Then $T + yv_{a+1} + (C - x_{a+1} v_{a+2}$ is a properly colored spanning tree of G, a contradiction. □

By Claim 4.1, for every edge e joining T to C_1, there exists a unique edge $f(e)$ in T that has the same color as e and is adjacent to e. Since the total number of edges between T and C_1 is $t(n-t)$ and the number of edges in T is $n-t-1$, there exists an edge e_1 such that at least $t(n-t)/(n-t-1) > t$ edges are mapped to e_1 by f. On the other hand, by Claim 4.3, $k(e_1) \leq t$, which is a contradiction. That completes the proof of Theorem 12. □

The method that we used to prove Theorem 12 can be easily generalized to edge-colored complete balanced bipartite graphs.

Theorem 13. *An edge-colored complete balanced bipartite graph $K_{n,n}$ contains a properly colored spanning tree if and only if it contains a properly colored spanning 1-tree-cycle system.*

Proof. Suppose that $K_{n,n}$ contains a properly colored spanning 1-tree-cycle system $\{T, C_1, C_2, \ldots, C_k\}$ where T is a tree and every C_i is a cycle. As mentioned in the proof of Theorem 12, we assume that $k = 1$ and thus $K_{n,n}$ contains vertex disjoint a tree T and a cycle C_1 with $V(T) \cup V(C_1) = V(K_{n,n})$. We claim that $K_{n,n}$ contains a properly colored spanning tree. Suppose to the contrary that $K_{n,n}$ has no properly colored spanning tree. Let $C_1 = v_1 v_2 \cdots v_{2t-1} v_{2t}$, where the subscript are all elements of $Z/2t$. For any edge For every edge e of T, $k(e)$ denotes the number of edges between T and C_1 which are adjacent to e and colored with $color(e)$.

Claim 4.4. *For every edge xv_i joining $x \in V(T)$ to $v_i \in V(C_1)$, there exists a unique edge xz in T such that $color(xz) = color(xv_i)$.*

Proof. Suppose that there exists an edge xv_i joining T to C_1 such that no edge of T incident with x has the color $color(xv_i)$. Then $T + xv_i + (C_1 - v_i v_{i+1})$ or $T + xv_i + (C_1 - v_i v_{i-1})$ is a properly colored spanning tree of $K_{n,n}$, a contradiction. The uniqueness is obvious since T is properly colored. □

Claim 4.5. *Let $e = xy$ be an edge of T. If an edge xv_i joining T to C_1 has the same color as e, and if $color(v_i v_{i-1}) \neq color(e)$, $color(v_i v_{i+1}) \neq color(e)$, then $k(e) \leq t$.*

Proof. In fact, if $color(xv_j) \neq color(e)$ for every $1 \leq j \leq t$, then it is obvious that $k(e) \leq t$. Assume that there is an edge $xv_a, 1 \leq a \leq t$, that has the same color as e. Then at least one of edges of $v_a v_{a-1}$ and $v_a v_{a+1}\}$ has a distinct color from e, and let $v_a v_b$ the other edge of $v_a v_{a-1}$ and $v_a v_{a+1}$. Then the $T - e + xv_a + (C - v_a v_b)$ is a properly colored spanning tree, a contradiction. □

Claim 4.6. *For every edge $e = xy$ of T, we have $k(e) \leq t$.*

Proof. Let $e = xy$ be an edge of T. Assume that $k(e) > t$. By the same argument given in the proof of Claim 4.2, there exists a vertex v_a in C_1 such that xv_a has the same color as e. By Claim 4.2 and by symmetry, we may assume that $v_a v_{a+1}$ has the same color as e. If yv_{a+1} has the same color as e, then $T - e - v_a v_{a+1} + xv_a + yv_{a+1}$ is a properly colored spanning tree of K_n, a contradiction. Hence $color(yv_{a+1}) \neq color(e)$. Then $T + yv_{a+1} + (C - x_{a+1}v_{a+2}$ is a properly colored spanning tree of G, a contradiction. □

Proof. We first claim that for any $v_j v_{j+1} \in E(C_1)$ with $color(v_j v_{j+1}) = color(e_0)$ and $1 \leq j \leq 2t$, the number of edges between $v_j v_{j+1}$ and e_0 colored by $color(e_0)$ is at most 1, otherwise since $K_{n,n}$ is bipartite, without loss of generality, suppose $color(u_0 v_j) = color(v_j v_{j+1}) = color(u_1 v_{j+1}) = color(e_0)$, then $(C_1 - \{v_j v_{j+1}\}) \cup (T - \{e_0\}) \cup \{u_0 v_j, u_1 v_{j+1}\}$ is a properly colored spanning tree, a contradiction. Now by Claim 4.5, we get that for every edge $u_i v_j (i = 0, 1, j = 1, 2, ..., t)$, there exist an edge $v_j v_{j+1}$(or $v_j v_{j-1}$) such that $color(v_j v_{j+1}) = color(u_i v_j)$(or $color(v_j v_{j-1}) = color(u_i v_j)$)and hence we have $k(e_0) \leq t$. □

Now by Claim 4.5, every edge e between T and C_1 has a unique edge $f(e) \in E(T)$ such that $color(f(e)) = color(e)$ and $f(e)$ is adjacent to e. Since the total number of edges between T and C_1 is $2t(n-t)$ and the number of edges in T is $2n - 2t - 1$, there exists an edge e_1 such that at least $(2t(n-t)/(2n-2t-1) > t$ edges are mapped to e_1 by f. However, by Claim 4.6, $k(e_1) \leq t$, which is a contradiction. That completes the proof of Theorem 13.

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