Spanning trees with small diameters

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Abstract

A spanning tree with small diameter of a graph has many applications. In this paper we first make the following conjecture and show that the condition is best possible if it is true. If a connected graph G satisfies $\delta(G) \geq 3|G|/(d+2)$, then G has a spanning tree with diameter at most d, where $d \geq 4$ is an integer. We next prove that the conjecture holds if $d \geq 4$ is even or $d \in \{5, 7, 9\}$. Moreover, we prove that if $d \geq 5$ is odd and $\delta(G) \geq 3|G|/(d+1)$, then G has a spanning tree with diameter at most d.

1 Introduction

In this paper we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G and by $N_G(v)$ the neighborhood of v. Thus $\deg_G(v) = |N_G(v)|$. The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

For two vertices x and y of G, $d_G(x, y)$ denotes *the distance between x and y*, which is the minimum length of paths in G connecting x and y. The *diameter* $\mathrm{diam}(G)$ of G is defined as

$$\mathrm{diam}(G) = \max_{x,y \in V(G)} d_G(x, y).$$

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On the other hand, the *radius* radius(G) of G is defined as

$$\text{radius}(G) = \min_{v \in V(G)} \{ \max_{x \in V(G)} d_G(v, x) \},$$

where $\max_{x \in V(G)} d_G(v, x)$ is called the *eccentricity* of v. Thus the radius of G is the minimum value of eccentricities. A vertex whose eccentricity is equal to the radius(G) is called a *central vertex* of G, and the subgraph induced by central vertices is called the *center* of G. It is well-known that the center of a tree consist of one vertex or two adjacent vertices.

There are many research on spanning trees of graphs, for example, graph theoretical results can be found in Chapter 8 of [1] and [6], and algorithms for spanning trees can be found in [7] and [4]. In this paper, we consider graph theoretical results on spanning trees with small diameter of a given graph. Namely, for an integer $d \geq 4$, we give a minimum degree condition for a connected graph to have a spanning tree with diameter at most d. We first make a conjecture on such spanning trees.

Conjecture 1 *Let G be a connected graph and $d \geq 4$ be an integer. If*

$$\delta(G) \geq \frac{3|G|}{d+2}, \tag{1}$$

then G has a spanning tree whose diameter is at most d. Moreover, if this statement holds, then the condition (1) is best possible.

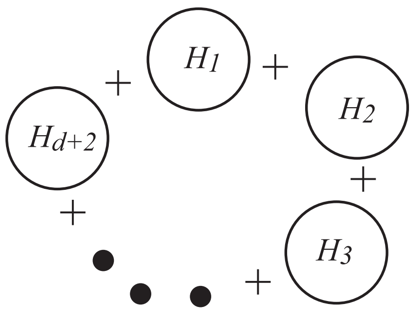

Figure 1: A connected graph G that has no spanning tree with diameter at most d and satisfies $\delta(G) = 3|G|/(d+2) - 1$, where every H_i is a complete graph of order n.

We now show that the condition (1) is sharp if the conjecture is true. Let $H_1, H_2, \ldots, H_{d+2}$ be $d+2$ disjoint copies of the complete graph of order

n, and let G be the graph obtained from $H_1, H_2, \ldots, H_{d+2}$ by joining every vertex of H_i to every vertex of H_{i+1} for all $1 \le i \le d+2$, where $H_{d+3} = H_1$ (see Figure 1). Then, $|G| = (d+2)n$ and $\delta(G) = 3n - 1 = 3|G|/(d+2) - 1$. Let T be any spanning tree of G, and let v be a center of T. Without loss of generality, we may assume that $v \in V(H_1)$. We first assume d is odd. Let $t = (d+3)/2$. Then for two vertices $x \in V(H_t)$ and $y \in V(H_{t+1})$, $d_T(v,x) \ge d_G(v,x) \ge t-1$ and $d_T(v,y) \ge d_G(v,y) \ge t-1$. If the unique path $P_T(x,y)$ in T connecting x and y passes through v, then $d_T(x,y) = d_T(x,v) + d_T(v,y) \ge 2(t-1) = d+1$. Hence the diameter of T is at least $d+1$. If the path $P_T(x,y)$ does not passes through v, then by symmetry, we may assume that $d_T(v,y) \ge d_T(v,x)+1$, and thus $\text{radius}(T) \ge d_T(v,y) \ge t$, which implies $\text{diam}(T) \ge 2 \cdot \text{radius}(T) - 1 \ge d+2$ since T is a tree. Therefore G has no spanning tree with diameter at most d. In the case where d is even, we can similarly show that $\text{radius}(T) \ge (d+2)/2$, and so G has no spanning tree with diameter at most d. Consequently, the minimum degree condition (1) is best possible.

In this paper, we prove the following two theorems, which verify that Conjecture 1 is true for even integers d and for some odd integers d.

Theorem 2 *Let G be a connected graph, and $d \ge 4$ be an even integer. If*

$$\delta(G) \ge \frac{3|G|}{d+2},$$

then G has a spanning tree with diameter at most d.

Theorem 3 *Let G be a connected graph, and let $d \ge 5$ be an odd integer.*
(1) If $d \in \{5,7,9\}$ and

$$\delta(G) \ge \frac{3|G|}{d+2},$$

then G has a spanning tree with diameter at most d.
(2) If

$$\delta(G) \ge \frac{3|G|}{d+1},$$

then G has a spanning tree with diameter at most d.

We now explain a relation of radius of a given graph and the minimum diameter of its spanning trees. It is well-known that the following lemma holds, and we use this fact in the proofs of theorems without mention.

Lemma 4 *A connected graph G with radius r has a spanning tree with diameter at most $2r$.*

Therefore, a research of the radius of a graph is closely related to one of its spanning tree with minimum diameter. We give some known results on the radius of a graph.

Theorem 5 (Erdős, Pach, Pollack and Tuza, [3]) *Let G be a connected graph with $\delta(G) \geq 2$. Then*

$$\text{radius}(G) \leq \frac{3}{2} \cdot \frac{|G| - 3}{\delta(G) + 1} + 5.$$

This result was recently improved as follows.

Theorem 6 (Kim, Rho, Song and Hwang [5]) *Let G be a connected graph with $\delta(G) \geq 2$ and* $\text{radius}(G) \geq 3$*. Then*

$$\text{radius}(G) \leq \frac{3}{2} \cdot \frac{|G|}{\delta(G) + 1}. \tag{2}$$

Note that (2) of Theorem 6 directly implies that if $d \geq 6$ is even and $\delta(G) \geq (3|G|/d) - 1$, then G has a spanning tree with diameter at most d since $\delta(G) \geq (3|G|/d) - 1$ means

$$\text{radius}(G) \leq \frac{3}{2} \cdot \frac{|G|}{\delta(G) + 1} \leq \frac{3}{2} \cdot \frac{|G|}{3|G|/d} = \frac{d}{2}.$$

We conclude this section with remarks about spanning trees with diameter 2 or 3. It is clear that if a connected graph G has a spanning tree T with diameter 2, then T has a vertex v with $\deg_T(v) = |G| - 1$. Hence G has a spanning tree with diameter 2 if and only if G has a vertex of degree $|G| - 1$.

We next show that there is no sufficient condition using $\delta(G) \geq c|G|$ with a constant number $0 < c < 1$ for a graph G to have a spanning tree with diameter 3. Consider a graph G with $\Delta(G) \leq |G| - 2$, which has no spanning tree with diameter 2. It is obvious that G has a spanning tree with diameter 3 if and only if G has an edge uv such that $N_G(u) \cup N_G(v) = V(G)$. Namely, G does not have a spanning tree with diameter 3 if and only if for every edge xy, there is a vertex z such that $z \notin N_G(x) \cup N_G(y)$. In $\overline{G}$, the complement of G, this situation on x, y, z is equivalent to $z \in N_{\overline{G}}(x) \cap N_{\overline{G}}(y) \neq \emptyset$ for

$xy \notin E(\overline{G})$. So if $\mathrm{diam}(\overline{G}) = 2$, then this situation holds in $\overline{G}$, and hence G does not have a spanning tree with diameter 3.

In [2], Brown showed that there is a graph H with $|H| = q^2 + q + 1$, $\Delta(H) = q + 1$ and $\mathrm{diam}(H) = 2$ for a prime power q. Then $\overline{H}$ does not have a spanning tree with diameter 3 by $H = \overline{\overline{H}}$, and satisfies $\delta(\overline{H}) = |\overline{H}| - \Delta(H) - 1 = q^2 - 1 = |\overline{H}|(q^2-1)/(q^2+q+1)$. Since $\lim_{q\to\infty}(q^2-1)/(q^2+q+1) = 1$, it is impossible to give a sufficient condition using $\delta(\overline{H}) \geq c|\overline{H}|$ with a constant number $0 < c < 1$ for a graph $\overline{H}$ to have a spanning tree with diameter 3.

2 Proofs of Theorems 2 and 3

We first prove Theorem 2 by using Theorem 6.

Proof of Theorem 2. The case of $d = 4$ will be proved in Proposition 8. Assume that $d \geq 6$ is an even integer, and G satisfies $\delta(G) \geq 3|G|/(d+2)$. Then by (2), we have

$$\begin{aligned} \mathrm{radius}(G) &\leq \frac{3}{2} \cdot \frac{|G|}{\delta(G)+1} \leq \frac{3}{2} \cdot \frac{|G|}{\frac{3|G|}{d+2}+1} \\ &= \frac{(d+2)}{2} \cdot \frac{3|G|}{3|G|+d+2} \\ &< \frac{d+2}{2} = \frac{d}{2} + 1. \end{aligned}$$

Since d is even, the above inequality implies $\mathrm{radius}(G) \leq d/2$. Therefore G has a spanning tree with diameter at most d. □

Proof of (2) of Theorem 3. Let $d \geq 5$ be an odd integer. Assume that $\delta(G) \geq 3|G|/(d+1)$. Then $d' = d-1$ is even and G satisfies $\delta(G) \geq 3|G|/(d'+2)$, and thus by Theorem 2, G has a spanning tree with diameter at most $d' = d-1$.

In order to prove (1) of Theorem 3, we need some notations. Let G be a connected graph. For disjoint vertex sets X and Y of G, we write $E_G(X, Y)$ for the set of edges of G joining a vertex of X to a vertex of Y. For vertices u and v, a vertex set X, and a positive integer m, let us define notations as follows.

$$\begin{aligned} d_G(X, v) = d_G(v, X) &:= \min_{x\in X} d_G(x, v) \\ Disk^m(v) &:= \{x \in V(G) : \ d_G(v, x) \leq m\} \\ Disk^m(X) &:= \{y \in V(G) : \ d_G(X, y) \leq m\} \\ Circle^m(v) &: \{x \in V(G) : \ d_G(v, x) = m\} \end{aligned}$$

The following lemma is easy, but useful, and we often use it without mention. Moreover, it is clear that $Disk^r(v) = V(G)$ if and only if the eccentricity of v is at most r.

Lemma 7 *Let r, s and d be positive integers. Let G be a connected graph, and let u be a vertex and vw be an edge of G. Then the following two statements hold.*

(i) If $Disk^r(u) = V(G)$, then G has a spanning tree with diameter at most $2r$. In particular, if G has no spanning tree with diameter at most $2d$, then for any vertex x, G has a vertex y such that $d_G(x, y) \geq d + 1$.

(ii) if $Disk^s(\{v, w\}) = V(G)$, then G has a spanning tree with diameter at most $2s + 1$.

We begin with the proof of the case of $d = 4$ in Theorem 2.

Proposition 8 *If a connected graph G satisfies $\delta(G) \geq 3|G|/6 = |G|/2$, then G has a spanning tree with diameter at most 4.*

Proof. Assume $\delta(G) \geq |G|/2$. Let v be a vertex of G, and x be any vertex of $V(G) - N_G(v)$, if any. Then $N_G(v) \cap N_G(x) \neq \emptyset$ by $\delta(G) \geq |G|/2$, and so $d_G(v, x) = 2$. Hence $Disk^2(v) = V(G)$, and thus G has a spanning tree with diameter at most 4 by Lemma 7. □

Proposition 9 *If a connected graph G satisfies $\delta(G) \geq 3|G|/7$, then G has a spanning tree with diameter at most 5.*

Proof. Suppose that a connected graph G satisfies $\delta(G) \geq 3|G|/7$, but has no spanning tree with diameter at most 5. The next claim follows from (i) of Lemma 7.

Claim 1 *For each vertex v, G has a vertex u such that $d_G(v, u) = 3$.*

Claim 2 *For two adjacent vertices u and v, $|N_G(u) \cap N_G(v)| \geq 2|G|/7$.*

Proof. Assume that there exist two adjacent vertices x and y such that $|N_G(x) \cap N_G(y)| < 2|G|/7$. Then

$$|N_G(x) \cup N_G(y)| = |N_G(x)| + |N_G(y)| - |N_G(x) \cap N_G(y)| > \frac{4|G|}{7}.$$

Hence for every vertex $z \in V(G) - (N_G(x) \cup N_G(y))$, $N_G(z) \cap (N_G(x) \cup N_G(y)) \neq \emptyset$. This implies that $Disk^2(\{x, y\}) = V(G)$. Hence G has a spanning tree with diameter at most 5 by Lemma 7. This is a contradiction. Therefore the claim holds.

By Claim 1, we can take a path (v_1, v_2, v_3, v_4) of length 3 such that $d_G(v_1, v_4) = 3$. Then $(N_G(v_1) \cap N_G(v_2)) \cap (N_G(v_3) \cap N_G(v_4)) = \emptyset$ by $d_G(v_1, v_4) = 3$. Let $W = (N_G(v_1) \cap N_G(v_2)) \cup (N_G(v_3) \cap N_G(v_4))$. Then it follows from Claim 2 that $|W| \geq 2 \times (2|G|/7) = 4|G|/7$. Thus for every vertex $x \in V(G) - (W \cup \{v_1, v_2, v_3, v_4\})$, we have $N_G(x) \cap W \neq \emptyset$, which implies $Disk^2(\{v_2, v_3\}) = V(G)$. Hence by Lemma 7, G has a spanning tree with diameter at most 5. This is a contradiction, and therefore the proposition is proved. □

Proposition 10 *If a connected graph G satisfies $\delta(G) \geq 3|G|/9 = |G|/3$, then G has a spanning tree with diameter at most 7.*

Proof. Suppose that a connected graph G satisfies $\delta(G) \geq |G|/3$, but has no spanning tree with diameter at most 7. The next claim holds as before.

Claim 1 *For each vertex v of G, there exists a vertex u such that $d_G(v, u) = 4$.*

By Claim 1, we can take a path (v_1, v_2, v_3, v_4) of length 3 such that $d_G(v_1, v_4) = 3$. Then $N_G(v_1) \cap N_G(v_4) = \emptyset$, and so we obtain

$$|N_G(v_1) \cup N_G(v_4)| \geq 2 \times \frac{|G|}{3} = \frac{2|G|}{3}.$$

Hence for every $x \in V(G) - (N_G(v_1) \cup N_G(v_4))$, $N_G(x) \cap (N_G(v_1) \cup N_G(v_4)) \neq \emptyset$, which implies $Disk^3(\{v_2, v_3\}) = V(G)$, and thus G has a spanning tree with diameter at most 7, a contradiction. Therefore the proposition holds. □

Proposition 11 *If a connected graph G satisfies $\delta(G) \geq 3|G|/11$, then G has a spanning tree with diameter at most 9.*

Proof. Suppose that a connected graph G satisfies $\delta(G) \geq 3|G|/11$, but has no spanning tree with diameter at most 9. By Lemma 7, the next claim holds.

Claim 1 *For every vertex v, there exists a vertex u such that $d_G(v, u) = 5$.*

Claim 2 *For every path $(v_1, v_2, v_3, v_4, v_5, v_6)$ with $d_G(v_1, v_6) = 5$, it follows that $|N_G(v_3) \cap N_G(v_4)| < 2|G|/11$ and $|N_G(v_3) \cup N_G(v_4)| > 4|G|/11$.*

Proof. Assume that $|N_G(v_3) \cap N_G(v_4)| \geq 2|G|/11$. Then since $N_G(v_1)$, $N_G(v_3) \cap N_G(v_4)$ and $N_G(v_6)$ are pairwise disjoint, $U = N_G(v_1) \cup (N_G(v_3) \cap N_G(v_4)) \cup N_G(v_6)$ satisfies $|U| \geq 8|G|/11$. Hence for every vertex x, it follows that $N_G(x) \cap U \neq \emptyset$, which implies $Disk^4(\{v_3, v_4\}) = V(G)$, and so G has a spanning tree with diameter at most 9, a contradiction. Thus $|N_G(v_3) \cap N_G(v_4)| < 2|G|/11$.

It follows from $|N_G(v_3) \cap N_G(v_4)| < 2|G|/11$ that $|N_G(v_3) \cup N_G(v_4)| = |N_G(v_3)| + |N_G(v_4)| - |N_G(v_3) \cap N_G(v_4)| > 4|G|/11$.

Claim 3 *If there is a path $(v_1, v_2, \ldots, v_6)$ with $d_G(v_1, v_6) = 5$, then either $|N_G(v_1) \cup N_G(v_2)| < 4|G|/11$ or $|N_G(v_5) \cup N_G(v_6)| < 4|G|/11$.*

Proof. Assume that there exists a path $(v_1, v_2, \ldots, v_6)$ such that $d_G(v_1, v_6) = 5$, $|N_G(v_1) \cup N_G(v_2)| \geq 4|G|/11$ and $|N_G(v_5) \cup N_G(v_6)| \geq 4|G|/11$. Let $U = (N_G(v_1) \cup N_G(v_2)) \cup (N_G(v_5) \cup N_G(v_6))$. Then $|U| \geq 8|G|/11$, and thus for every vertex x of G, $N_G(x) \cap U \neq \emptyset$, which implies $Disk^4(\{v_3, v_4\}) = V(G)$. Hence G has a spanning tree with diameter at most 9, a contradiction. Therefore the claim holds.

Claim 4 *There exist no two vertices v and w such that $d_G(v, w) = 6$.*

Proof. Assume that there exist two vertices v and w such that $d_G(v, w) = 6$. Let $(v = v_1, v_2, v_3, \ldots, v_7 = w)$ be a path of length 6 connecting v and w. Then $N_G(v_1)$, $N_G(v_4)$ and $N_G(v_7)$ are pairwise disjoint. So $X = N_G(v_1) \cup N_G(v_4) \cup N_G(v_7)$ satisfies $|X| \geq 9|G|/11$. Thus every vertex x of G satisfies $N_G(x) \cap X \neq \emptyset$, which implies $d_G(x, v_4) \leq 5$. Moreover, if $d_G(x, v_4) = 5$, then there exists a path of length 5 that connects x and v_4 and passes through v_1 or v_7. If $d_G(x, v_4) \leq 4$ for every $x \in V(G)$, then G has a spanning tree with diameter at most 8. If for every vertex z with $d_G(z, v_4) = 5$, there is a path of length 5 that connects z and v_4 and passes through v_1, then G has a spanning tree T with diameter 9, in which $d_T(z, v_1) = 2$ for every vertex z with $d_G(z, v_4) = 5$. By the symmetry of v_1 and v_7, we may assume that there exist two vertices x_1 and y_1 such that $d_G(x_1, v_4) = 5$, $d_G(y_1, v_4) = 5$, there is a path $P(x_1, v_4)$ of length 5 connecting x_1 and v_4 and passing through v_1, and there is a path $P(y_1, v_4)$ of length 5 connecting y_1 and v_4 and passing through v_7.

Let x_2 be a vertex in the path $P(x_1, v_4)$ adjacent to both x_1 and v_1, and y_2 be a vertex in the path $P(y_1, v_4)$ adjacent to both y_1 and v_7. We shall show that $N_G(x_1)$, $N_G(v_2)$, $N_G(v_5)$ and $N_G(y_2)$ are pairwise disjoint. It is obvious that $d_G(x_1, v_2) = 3$, $d_G(v_2, v_5) = 3$ and $d_G(v_5, y_2) = 3$. Moreover, it follows that $d_G(x_1, v_5) \geq d_G(x_1, v_4) - d_G(v_4, v_5) = 4$ and $d_G(v_2, y_2) \geq d_G(v_2, v_7) - d_G(v_7, y_2) = 4$. If $d_G(x_1, y_2) \leq 2$, then $6 = d_G(v_1, v_7) \leq d_G(v_1, x_1) + d_G(x_1, y_2) + d_G(y_2, v_7) = 5$, a contradiction. Hence $d_G(x_1, y_2) \geq 3$. Therefore $N_G(x_1)$, $N_G(v_2)$, $N_G(v_5)$ and $N_G(y_2)$ are pairwise disjoint. This is a contradiction since every set contains at least $3|G|/11$ vertices of G.

Claim 5 *Let u and v be two adjacent vertices for which there is a path $(s_1, s_2, u, v, t_2, t_1)$ with $d_G(s_1, t_1) = 5$. Then there exists no vertex w such that $d_G(u, w) = 5$ and $d_G(v, w) = 5$.*

Proof. Assume that there exists a vertex w such that $d_G(u, w) = 5$ and $d_G(v, w) = 5$. Let $(u = u_1, u_2, u_3, \ldots, u_6 = w)$ be a path of length 5 connecting u and w. Since $d_G(v, w) = 5 \leq d_G(v, u_4) + d_G(u_4, w)$, it follows that $d_G(v, u_4) \geq 3$. It follows from $d_G(s_1, t_1) = 5$ that either $d_G(s_1, u_4) \geq 3$ or $d_G(t_1, u_4) \geq 3$. Assume first $d_G(s_1, u_4) \geq 3$. Then $N_G(s_1)$, $N_G(v)$ and $N_G(u_4)$ are pairwise disjoint. Thus for every vertex $x \in V(G)$, it follows that $N_G(x) \cap (N_G(s_1) \cup N_G(v) \cup N_G(u_4)) \neq \emptyset$, which implies $Disk^4(\{u, u_2\}) = V(G)$. Therefore G has a spanning tree with diameter at most 9. Hence $d_G(s_1, u_4) \leq 2$ and $d_G(t_1, u_4) \geq 3$.

By $d_G(t_1, u_4) \geq 3$, it holds that $N_G(u)$, $N_G(u_4)$ and $N_G(t_1)$ are pairwise disjoint, and for every vertex x, it follows that $N_G(x) \cap (N_G(u) \cup N_G(u_4) \cup N_G(t_1)) \neq \emptyset$. If every vertex y with $N_G(t_1) \cap N_G(y) \neq \emptyset$ satisfies $d_G(y, u) \leq 4$, then $Disk^4(\{u, u_2\}) = V(G)$, and so G has a spanning tree with diameter at most 9. Hence we may assume that there exists a vertex y_1 such that $d_G(y_1, u) = 5$ and there is a path of length 5 that connects y_1 and u and passes through t_1. By Claim 1, it follows that $|N_G(t_1) \cup N_G(t_2)| > 4|G|/11$.

Let $(v = v_1, v_2, v_3, \ldots, v_6 = w)$ be a path of length 5 connecting v and w. By the symmetry of u and v, we can similarly show that there exists a vertex x_1 such that $d_G(x_1, s_1) = 2$, $d_G(x_1, v) = 5$ and $|N_G(s_1) \cup N_G(s_2)| > 4|G|/11$.

Since $d_G(s_1, t_1) = 5$, the above fact that $|N_G(t_1) \cup N_G(t_2)| > 4|G|/11$ and $|N_G(s_1) \cup N_G(s_2)| > 4|G|/11$ contradicts Claim 3. Therefore the claim is proved.

By Claim 1, there exist two adjacent vertices u and v such that there exists a path $(s_1, s_2, u, v, t_2, t_1)$ of length 5 and $d_G(s_1, t_1) = 5$. If $Disk^4(\{u, v\}) = V(G)$, then G has a spanning tree with diameter 9. Thus there exists a vertex w such that $d_G(u, w) \geq 5$ and $d_G(v, w) \geq 5$. By Claim 3, we may assume that w satisfies $d_G(u, w) = 5$ and $d_G(v, w) = 5$. But this contradicts Claim 5. Consequently Proposition 11 is proved. □

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