

Fractional Factors, Component Factors and Isolated Vertex Conditions in Graphs

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Abstract

For a graph G , a *fractional* $[a, b]$ -factor is a real valued function $h : E(G) \rightarrow [0, 1]$ that satisfies $a \leq \sum_{e \in E_G(v)} h(e) \leq b$ for all $v \in V(G)$, where a and b are real numbers and $E_G(v)$ denotes the set of edges incident with v . In this paper, we prove that the condition $iso(G - S) \leq (k + \frac{1}{2})|S|$ is equivalent to the existence of fractional $[1, k + \frac{1}{2}]$ -factors, where $iso(G - S)$ denotes the number of isolated vertices in $G - S$. Using fractional factors as a tool, we construct component factors under the given isolated conditions. Namely, (i) a graph G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$ -factor if and only if $iso(G - S) \leq \frac{3}{2}|S|$

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for all $S \subset V(G)$; (ii) a graph G has a $\{K_{1,1}, K_{1,2}, \dots, K_{1,k}, \mathcal{T}(2k+1)\}$ -factor ($k \geq 2$) if and only if $iso(G - S) \leq (k + \frac{1}{2})|S|$ for all $S \subset V(G)$, where $\mathcal{T}(3)$ and $\mathcal{T}(2k+1)$ are two special families of trees.

Keywords: component factors, fractional factors, isolated vertices, isolated toughness, Tutte type condition.

1 Introduction

In this paper, we mainly consider finite simple graphs, which have neither loops nor multiple edges. A graph that may have multiple edges but has no loops is referred to as a *multigraph*. When defining notation and definitions, we often call a multigraph as a graph for convenience. Given a graph G , let $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. The number of vertices of G is called its *order* and is denoted by $|G|$. On the other hand, the number of edges in G is called its *size* and is denoted by $\|G\|$.

For a set X , the cardinality of X is denoted by $|X|$ or $\#X$. For a vertex v of a graph G , the degree of v in G is denoted by $\deg_G(v)$ and the set of edges of G incident with v is denoted by $E_G(v)$. For two vertices x and y , an edge joining them is denoted by xy or yx . We denote by $Iso(G)$ the set of isolated vertices of G , and by $iso(G)$ the number of isolated vertices in G . Thus $iso(G) = |Iso(G)|$. For two disjoint vertex sets X and Y of G , the set of edges of G joining X to Y is written as $E_G(X, Y)$ and $e_G(X, Y) := |E_G(X, Y)|$. P_n and C_n are the path and the cycle of order n , respectively. The set of non-negative integers is denoted by $\mathbf{Z}^* = \{0\} \cup \mathbf{Z}^+$.

Let G be a graph, and let $g, f : V(G) \rightarrow \mathbf{Z}^*$ be two integer-valued functions with $g \leq f$, that is, $0 \leq g(x) \leq f(x)$ for all $x \in V(G)$. Then a spanning subgraph F of G is called a (g, f) -factor of G if $g(x) \leq \deg_F(x) \leq f(x)$ for all $x \in V(G)$. For real-valued functions $g, f : V(G) \rightarrow \mathbf{R}$ with $g \leq f$, a *fractional (g, f) -factor* is a function $h : E(G) \rightarrow [0, 1]$ that satisfies the following condition:

$$g(v) \leq \deg^h(v) := \sum_{e \in E_G(v)} h(e) \leq f(v) \quad \text{for all } v \in V(G), \quad (1)$$

where $\deg^h(v)$ is called the *h -degree of v* and $h(e)$ is a real number between 0 and 1 inclusive. If the values of h are 0 and 1 only, then a fractional (g, f) -factor becomes a (g, f) -factor. Many results on fractional factors of graphs can be found in [9].

To study fractional factors, Yang, Ma and Liu [8] introduced a parameter, *isolated toughness* of a graph G , denoted by $I(G)$, which is defined as

$$I(G) = \min \left\{ \frac{|S|}{\text{iso}(G - S)} : S \subseteq V(G), \text{iso}(G - S) \geq 2 \right\}$$

if G is not complete. Otherwise, $I(G) = \infty$. A graph G is called *isolated t -tough* if $I(G) \geq t$, where $t \geq 0$ is a real number.

For a set \mathcal{S} of connected graphs, a spanning subgraph F of G is called an \mathcal{S} -*factor* or a *component factor* if each component of F is isomorphic to an element of \mathcal{S} (see (1) of Figure 1). For a set \mathbb{S} of positive integers, a spanning subgraph F of G is called an \mathbb{S} -*factor* of G if $\deg_F(x) \in \mathbb{S}$ for all vertices x of G . For an integer $k \geq 0$, the set of vertices of G with degree k is denoted by $V_k(G)$, namely, $V_k(G) = \{v \in V(G) : \deg_G(v) = k\}$.

For a tree T , the set of leaves is denoted by $\text{Leaf}(T)$, that is, $V_1(T)$. An edge of T incident with a leaf is called a *pendant edge*. In particular, the number of leaves of T is equal to that of pendant edges of T .

We define a special class of trees $\mathcal{T}(3)$: for any $\{1, 3\}$ -tree R , which is a tree with vertex degrees 1 or 3 only, a new tree T_R is obtained from R by inserting a new vertex of degree 2 into every edge of R , and by adding a new pendant edge to every leaf of R (see (2) and (3) of Figure 1). Then the tree T_R is a $\{1, 2, 3\}$ -tree having $\|R\| + |\text{Leaf}(R)|$ vertices of degree 2 and has the same number of leaves as R . Also there is one-to-one correspondence between $V_3(R)$ and $V_3(T_R)$. The collection of such $\{1, 2, 3\}$ -trees T_R generated from all $\{1, 3\}$ -trees R is denoted by $\mathcal{T}(3)$. A more general class of trees, $\mathcal{T}(2k+1)$ ($k \geq 2$), will be defined in Section 2.

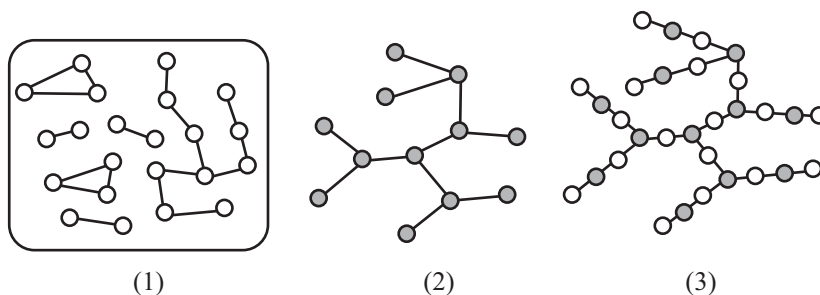


Figure 1: (1) A $\{P_2, C_3, P_5, \mathcal{T}(3)\}$ -factor that contains no P_5 -components; (2) A $\{1, 3\}$ -tree R ; (3) The resulting $\{1, 2, 3\}$ -tree T_R obtained from R .

Tutte [7] established a relationship between isolated 1-tough graphs and $\{K_2, C_n : n \geq 3\}$ -factors.

Theorem 1 (Tutte [7]) *A graph G has a $\{K_2, C_n : n \geq 3\}$ -factor if and only if*

$$iso(G - S) \leq |S| \quad \text{for all } S \subset V(G).$$

Amahashi and Kano [2] extended Theorem 1 and gave a characterization for isolated $1/k$ -tough graphs in terms of star factors.

Theorem 2 (Amahashi and Kano [2]) *Let $k \geq 2$ be an integer. A graph G has a $\{K_{1,j} : 1 \leq j \leq k\}$ -factor if and only if*

$$iso(G - S) \leq k|S| \quad \text{for all } S \subset V(G).$$

Kano, Lu and Yu [5] obtained a sufficient condition for isolated 2-tough graphs to have component factors.

Theorem 3 (Kano, Lu and Yu [5]) *A graph G has a $\{K_{1,2}, K_{1,3}, K_5\}$ -factor if*

$$iso(G - S) \leq \frac{|S|}{2} \quad \text{for all } S \subset V(G).$$

Kano and Saito [6] as well as Zhang, Yan and Kano [10] used isolated k -toughness to ensure the existence of special classes of component factors.

Theorem 4 (Kano and Saito [6]) *Let $k \geq 2$ be an integer. A graph G has a $\{K_{1,j} : k \leq j \leq 2k\}$ -factor if*

$$iso(G - S) \leq \frac{|S|}{k} \quad \text{for all } S \subset V(G).$$

Theorem 5 (Zhang, Yan and Kano [10]) *Let $k \geq 2$ be an integer. A graph G has a $\{K_{1,j}, K_{2k} : k \leq j \leq 2k - 1\}$ -factor if*

$$iso(G - S) \leq \frac{|S|}{k} \quad \text{for all } S \subset V(G).$$

In this paper, we carry on the investigations along the same direction mentioned above and obtain the factor characterizations of $I(G) = \frac{2}{3}$ and $I(G) = \frac{2}{2k+1}$ ($k \geq 2$):

Theorem 6 *A graph G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$ -factor if and only if*

$$iso(G - S) \leq \frac{3}{2}|S| \quad \text{for all } S \subset V(G). \quad (2)$$

Theorem 7 *Let $k \geq 2$ be an integer. Then a graph G has a $\{K_{1,1}, K_{1,2}, \dots, K_{1,k}, \mathcal{T}(2k+1)\}$ -factor if and only if*

$$iso(G - S) \leq \left(k + \frac{1}{2}\right)|S| \quad \text{for all } S \subset V(G). \quad (3)$$

2 Proofs of Theorems 6 and 7

For a function $f : V(G) \rightarrow \mathbf{Z}^*$ and a set X of vertices of G , we write

$$f(X) := \sum_{x \in X} f(x), \quad \text{in particular,} \quad \deg_G(X) = \sum_{x \in X} \deg_G(x).$$

The tools for proving Theorems 6 and 7 are fractional factors. We first describe the condition (3) in terms of fractional $[1, k + \frac{1}{2}]$ -factors (Theorem 9), and then show that the minimal fractional factors are the desired component factors. In establishing the link between the condition (3) and fractional factors, we need the following result.

Theorem 8 (Anstee [3], Heinrich et al. [4]) *Let G be a multigraph and $g, f : V(G) \rightarrow \mathbf{Z}^*$ with $0 \leq g(x) < f(x)$ for all $x \in V(G)$. Then G has a (g, f) -factor if and only if*

$$g(T) - \deg_{G-S}(T) \leq f(S) \quad \text{for all } S \subset V(G), \quad (4)$$

where $T = \{v \in V(G) - S : \deg_{G-S}(v) < g(v)\}$.

If two functions $g, f : V(G) \rightarrow \mathbf{R}$ take constant real values $g(x) = a$ and $f(x) = b$ for every vertex x , then fractional (g, f) -factors are called *fractional $[a, b]$ -factors*.

Theorem 9 *Let $k \geq 1$ be an integer and let G be a graph. Then G has a fractional $[1, k + \frac{1}{2}]$ -factor h with values in $\{0, \frac{1}{2}, 1\}$ if and only if*

$$\text{iso}(G - S) \leq \left(k + \frac{1}{2}\right)|S| \quad \text{for all } S \subset V(G). \quad (5)$$

Proof. Assume that G satisfies (5). Let G^* denote the multigraph obtained from G by replacing each edge e of G by two parallel edges $e(1)$ and $e(2)$. Then $V(G^*) = V(G)$, and $\deg_{G^*}(v) = 2 \deg_G(v)$ for every $v \in V(G^*)$. Define two functions $g, f : V(G^*) \rightarrow \mathbf{Z}^*$ as

$$g(x) = 2 \quad \text{and} \quad f(x) = 2k + 1 \quad \text{for all } x \in V(G^*).$$

Then $g < f$, and for any $S \subset V(G^*)$, we have

$$\begin{aligned} T &= \{v \in V(G^*) - S : \deg_{G^*-S}(v) < g(v) = 2\} \\ &= \{v \in V(G^*) - S : \deg_{G^*-S}(v) = 0\}. \\ &= \text{Iso}(G - S). \end{aligned}$$

Thus it follows from the above equality and (5) that

$$\begin{aligned} g(T) - \deg_{G^*-S}(T) &= 2 \cdot \text{iso}(G - S) - 0 \\ &\leq 2 \cdot \left(k + \frac{1}{2}\right) |S| = (2k + 1) |S| = f(S). \end{aligned}$$

Hence by Theorem 8, G^* has a (g, f) -factor F . Now we construct a fractional $[1, k + \frac{1}{2}]$ -factor $h : E(G) \rightarrow \{0, \frac{1}{2}, 1\}$ as follows: for an edge e of G , (i) if F contains both edges $e(1)$ and $e(2)$ of G^* , then define $h(e) = 1$; (ii) if F contains exactly one of $e(1)$ and $e(2)$, then define $h(e) = \frac{1}{2}$; (iii) otherwise, let $h(e) = 0$. It is easy to see that h is the desired fractional $[1, k + \frac{1}{2}]$ -factor with values in $\{0, \frac{1}{2}, 1\}$.

Next assume that G has a fractional $[1, k + \frac{1}{2}]$ -factor h with values in $\{0, \frac{1}{2}, 1\}$. Let $S \subset V(G)$, and let F be the spanning subgraph of G induced by $\{e \in E(G) : h(e) \in \{\frac{1}{2}, 1\}\}$. Clearly, the neighbors of each isolated vertex u of $G - S$ are contained in S and $\deg^h(u) \geq 1$, thus we have

$$\begin{aligned} \text{iso}(G - S) &\leq \sum_{e \in E_F(\text{Iso}(G-S), S)} h(e) \\ &\leq \sum_{x \in S} \deg^h(x) \leq \left(k + \frac{1}{2}\right) |S|. \end{aligned}$$

Hence $\text{iso}(G - S) \leq (k + \frac{1}{2}) |S|$, or (5) holds. \square

Proof of Theorem 6. We first show that every tree $T \in \mathcal{T}(3)$ satisfies the condition (2). Define a function $h : E(T) \rightarrow \{\frac{1}{2}, 1\}$ as follows: for every pendant edge e_1 of T , let $h(e_1) = 1$ and for any other edge e_2 , let $h(e_2) = \frac{1}{2}$. Since T is a $\{1, 2, 3\}$ -tree and no pendant edge is incident with a vertex of degree 3, h is a fractional $[1, \frac{3}{2}]$ -factor. Hence, by Theorem 9, T satisfies the condition (2).

Assume that G has a $\{P_2, C_3, P_5, \mathcal{T}(3)\}$ -factor F . Let D_1, D_2, \dots, D_m be the components of F . Then each D_i is P_2, C_3, P_5 or a tree in $\mathcal{T}(3)$, and thus $\text{iso}(D_i - X_i) \leq \frac{3}{2} |X_i|$ for every $X_i \subset V(D_i)$. Then for any $S \subset V(G)$, we have

$$\begin{aligned} \text{iso}(G - S) &\leq \text{iso}(F - S) = \sum_{i=1}^m \text{iso}(D_i - (S \cap V(D_i))) \\ &\leq \sum_{i=1}^m \frac{3}{2} |S \cap V(D_i)| = \frac{3}{2} |S|. \end{aligned}$$

Hence the necessity is proved.

Next we prove the sufficiency. By Theorem 9, G has a fractional $[1, \frac{3}{2}]$ -factor h with values in $\{0, \frac{1}{2}, 1\}$. We call an edge e_1 with $h(e_1) = 1$ a *red edge* and an edge e_2 with $h(e_2) = \frac{1}{2}$ a *blue edge*. Let F be the subgraph of G induced by the set of all red and blue edges. Namely, F is obtained from G by removing all the edges e_3 with $h(e_3) = 0$. Since h is a fractional $[1, \frac{3}{2}]$ -factor, F is a spanning subgraph of G , and for every vertex v of G , one of the following two statements holds:

- (i) no red edge is incident with v and two or three blue edges are incident with v ; or
- (ii) exactly one red edge is incident with v and at most one blue edge is incident with v .

Choose a fractional $[1, \frac{3}{2}]$ -factor h so that the number of edges in F is as small as possible. For the convenience, we also call F a fractional $[1, \frac{3}{2}]$ -factor. It is clear that $\deg_F(v) \in \{1, 2, 3\}$. On the other hand, the degree $\deg^h(v)$ of v in h is 1 or $\frac{3}{2}$.

Claim 1. *Every cycle of F is a C_3 -component in F .*

Proof. Assume that F contains a cycle C . First assume that C is of even order. Take a perfect matching M of C , and recolor all the edges of M red, and remove all the edges in $C - M$. Then the resulting subgraph is a new fractional $[1, \frac{3}{2}]$ -factor with red and blue edges, but its size is smaller than F , which contradicts the choice of F . Hence C is of odd order.

Assume that C has two adjacent vertices u_1 and u_2 with degree 3 in F . Then $F - u_1u_2$ is a new fractional $[1, \frac{3}{2}]$ -factor with fewer edges than F , which contradicts the choice of F . Hence if a vertex v of C has degree 3 in F , then the two neighbors of v in C have degree 2 in F . Assume that C has a vertex v with $\deg_F(v) = 3$. Let u_1 and u_2 be the two neighbors of v in C . Take a perfect matching M of $C - v$. Recolor the edges of M red, and remove all the edges of $(C - v) - M$ and vu_2 . Since vu_1 and vu_2 are both blue edges, we obtain a new fractional $[1, \frac{3}{2}]$ -factor with fewer edges than F , which is a contradiction. Hence C is a component in F .

Moreover, it is easy to see that an odd cycle of C order at least 5 has a $\{P_2, P_5\}$ -factor F_C . Remove all the edges of C not contained in F_C , recolor the edges contained in P_2 of F_C red, and two pendant edges of P_5 of F_C red and the remaining two edges of P_5 of F_C blue. Then we obtain a new fractional $[1, \frac{3}{2}]$ -factor with fewer edges than F , a contradiction. Therefore every cycle contained in F is C_3 . Consequently Claim 1 is proved.

For the simplicity of statements, from now on, we will use “a smaller fractional $[1, \frac{3}{2}]$ -factor” to substitute the phrase “a new fractional $[1, \frac{3}{2}]$ -factor with fewer edges than F ”.

Claim 2. *Every non-cycle component of F is P_2 , P_5 or a tree of $\mathcal{T}(3)$.*

Proof. Let x and y be two vertices of degree 3 in F such that they are adjacent or connected by a path whose inner vertices all have degree 2 in F . If x and y are adjacent in F , then $F - xy$ is a smaller fractional $[1, \frac{3}{2}]$ -factor, a contradiction. Assume that x and y are connected by a path $(x, u_1, u_2, \dots, u_n, y)$ of length at least 3 ($n \geq 2$) such that every u_i has degree 2 in F . We remove $u_n y$, recolor $u_{n-1} u_n$ red, and recolor all remaining edges of the path blue. Then the resulting subgraph is a smaller fractional $[1, \frac{3}{2}]$ -factor, a contradiction. Therefore,

(a) *if two vertices of F with degree 3 are connected by a path in F whose inner vertices all have degree 2 in F , then the length of the path is 2.*

Let z be a leaf of F and let x be a vertex of degree 3 in F . If z and x are adjacent, then the edge xz is red and so $\deg^h(x) \geq 2$, which is impossible. Hence z and x are not adjacent. Assume that z and x are connected by a path $(z, u_1, u_2, \dots, u_n, x)$ with $\deg_F(u_i) = 2$ for every i . Then zu_1 is red. First assume that $n \geq 3$. Then remove $u_n x$, recolor $u_{n-1} u_n$ red, and recolor all the remaining edges of the path except zu_1 blue. Then the resulting subgraph is a smaller fractional $[1, \frac{3}{2}]$ -factor, a contradiction. Next assume $n = 1$. Then by removing $u_1 x$, we obtain a smaller fractional $[1, \frac{3}{2}]$ -factor, a contradiction again. Therefore,

(b) *if a leaf z is contained in a component D of F with $\Delta(D) \geq 3$, then there is a path (z, u_1, u_2, x) in F such that $\deg_F(u_i) = 2$ and $\deg_F(x) = 3$.*

Consequently, if a component D of F contains at least two vertices of degree 3, then by (a) and (b), D is a tree of $\mathcal{T}(3)$. If D has exactly one vertex of degree 3, then by (b), D is also a tree of $\mathcal{T}(3)$. If D has no vertex of degree 3, then D is a path. It is obvious that P_3 has no fractional $[1, \frac{3}{2}]$ -factors, and so D is not P_3 . If D is a path of even order, then D has a P_2 -factor, and so it contradicts the minimality of F . If D is a path P_n of odd order with $n \geq 7$, then D has a $\{P_2, P_5\}$ -factor and thus contradicts the minimality of F . Hence Claim 2 holds.

Clearly, Claims 1 and 2 imply the sufficiency. \square

To state Theorem 7, we need a new class of trees $\mathcal{T}(2k + 1)$. Let $k \geq 2$ be an integer and let R be a tree that satisfies the following conditions: for every vertex $v \in V(R)$,

- (i) $\deg_{R-\text{Leaf}(R)}(v) \in \{1, 3, \dots, 2k + 1\}$, and
 - (ii) $2 \cdot (\text{the number of leaves adjacent to } v \text{ in } R) + \deg_{R-\text{Leaf}(R)}(v) \leq 2k + 1$ in R .
- (6)

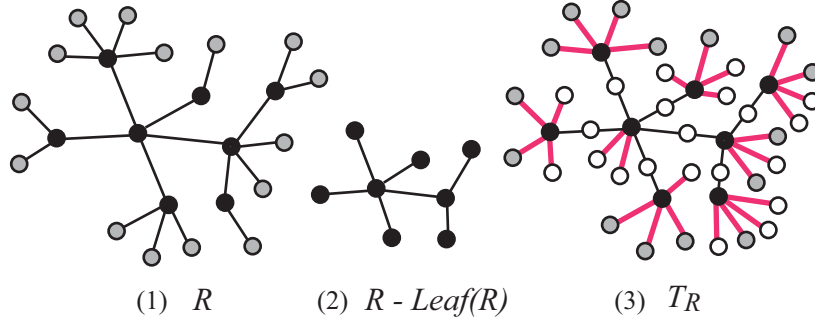


Figure 2: (1) A tree R that satisfies (6) with $k = 4$; (2) The tree $R - Leaf(R)$; (3) The tree T_R obtained from R , where all red edges e_1 have value $h(e_1) = 1$ and all other edges e_2 have value $h(e_2) = \frac{1}{2}$.

For such a tree R , we obtain a new tree T_R as follows:

- (iii) insert a new vertex of degree 2 into each edge of $R - Leaf(R)$, and
- (iv) for each vertex v of $R - Leaf(R)$ with $\deg_{R - Leaf(R)}(v) = 2r + 1 < 2k + 1$ in R , add $k - r -$ (the number of leaves adjacent to v in R) pendant edges to v (see (3) of Figure 2).

Then the resulting tree T_R has a fractional $[1, k + \frac{1}{2}]$ -factor h such that every pendant edge e_1 of T_R has $h(e_1) = 1$, all other edges e_2 have $h(e_2) = \frac{1}{2}$. Moreover, this fractional $[1, k + \frac{1}{2}]$ -factor h satisfies that every vertex v of $R - Leaf(R)$ has $\deg^h(v) = k + \frac{1}{2}$, and every leaf x of T_R and every inserted vertex y of degree 2 have $\deg^h(x) = \deg^h(y) = 1$ (see (3) of Figure 2). The set of such trees T_R for all trees R satisfying (6) is denoted by $\mathcal{T}(2k + 1)$. Note that the construction of $\mathcal{T}(3)$ and that of $\mathcal{T}(2k + 1)$ with $k \geq 2$ are similar, but adding pendant edges to some vertices of $R - Leaf(R)$ is not defined in the construction of $\mathcal{T}(3)$.

Proof of Theorem 7. For any tree $T \in \mathcal{T}(2k + 1)$, since T has a fractional $[1, k + \frac{1}{2}]$ -factor h of values $\{\frac{1}{2}, 1\}$, T satisfies (3) by Theorem 9.

Assume that G has a $\{K_{1,1}, K_{1,2}, \dots, K_{1,k}, \mathcal{T}(2k + 1)\}$ -factor F . Let D_1, D_2, \dots, D_m be the components of F . Then each D_i is $K_{1,s}$ for some $1 \leq s \leq k$, or a tree in $\mathcal{T}(2k + 1)$. Thus $iso(D_i - X_i) \leq (k + \frac{1}{2})|X_i|$ for every

$X_i \subset V(D_i)$. Then, for $S \subset V(G)$, we have

$$\begin{aligned} iso(G - S) &\leq iso(F - S) = \sum_{i=1}^m iso(D_i - (S \cap V(D_i))) \\ &\leq \sum_{i=1}^m \left(k + \frac{1}{2}\right) |S \cap V(D_i)| = \left(k + \frac{1}{2}\right) |S|. \end{aligned}$$

Hence the necessity is proved.

Next we prove the sufficiency. Assume that G satisfies (3). By Theorem 9, G has a fractional $[1, k + \frac{1}{2}]$ -factor h with values in $\{0, \frac{1}{2}, 1\}$. We call an edge e_1 with $h(e_1) = 1$ a *red edge* and an edge e_2 with $h(e_2) = \frac{1}{2}$ a *blue edge*. Let F be the subgraph of G induced by the set of all red and blue edges. Namely, F is obtained from G by removing all the edges e_3 with $h(e_3) = 0$, and since h is a fractional $[1, k + \frac{1}{2}]$ -factor, F is a spanning subgraph. Choose a fractional $[1, k + \frac{1}{2}]$ -factor h of G so that the number of edges in F is as small as possible. It is clear that $1 \leq \deg_F(v) \leq 2k + 1$. On the other hand, the h -degree of v can be expressed as $\deg^h(v) = 1 + \frac{1}{2}t$ for some integer t , $0 \leq t \leq 2k - 1$.

Claim 1. F contains no cycles, and thus F is a forest.

Proof. Suppose that F contains a cycle C . First assume that C is of even order. Take a perfect matching M of C , and recolor all edges of M red, and remove all edges in $C - M$. Then the resulting subgraph is a new fractional $[1, k + \frac{1}{2}]$ -factor of G , but its size is smaller than F , a contradiction. Hence C is of odd order.

As in the proof of Theorem 6, we substitute “a new fractional $[1, k + \frac{1}{2}]$ -factor with less edges than F ” by “a smaller fractional factor” for the simplicity in the rest of the proof.

Assume that C has two adjacent vertices v_1 and v_2 with degrees at least 3 in F . Then $F - v_1v_2$ is a smaller fractional factor, a contradiction. Hence if a vertex v of C has degree at least 3 in F , then the two neighbors of v in C have degree 2 in F . Assume that C has a vertex v with $\deg_F(v) \geq 3$. Let u_1 and u_2 be the two neighbors of v in C . Take a perfect matching M of $C - v$, and recolor all edges of M red, and remove all edges of $(C - v) - M$ and the edge vu_2 . Then we obtain a smaller fractional factor, a contradiction. Hence C is a component of F .

It is easy to see that C has a $\{P_2, P_3\}$ -factor F_C . We recolor all edges of P_2 -components and P_3 -components of F_C red, and remove all other edges of C . Then we obtain a smaller fractional factor, a contradiction. Therefore F has no cycles, and the claim is proved.

Claim 2. *Let x and y be two vertices of degrees at least 3 in F . Then x and y are not adjacent in F . If x and y are connected by a path whose inner vertices all have degree 2 in F , then the length of the path is 2, and $\deg^h(x) = \deg^h(y) = k + \frac{1}{2}$ and the two edges in the path are blue edges.*

Proof. If x and y are adjacent in F , then $F - xy$ is a smaller fractional factor, a contradiction. Assume that x and y are connected by a path $(x, u_1, u_2, \dots, u_n, y)$ with $\deg_F(u_i) = 2$ ($1 \leq i \leq n$). If $n \geq 2$, then by removing $u_n y$ and recoloring $u_{n-1} u_n$ red and all the remaining edges of the path blue, the resulting subgraph is a smaller fractional factor, a contradiction. Therefore $n = 1$, and the path is (x, u_1, y) .

If $\deg^h(x) < k + \frac{1}{2}$, then by removing $u_1 y$ and recoloring xu_1 red, we obtain a smaller fractional factor, a contradiction. Hence $\deg^h(x) = \deg^h(y) = k + \frac{1}{2}$ by the symmetry. If xu_1 is red, then removing $u_1 y$ we obtain a smaller fractional factor, a contradiction. Therefore xu_1 and yu_1 are blue edges, and the claim holds.

Claim 3. *A leaf z in F is either contained in a star component, or adjacent to a vertex x with $\deg_F(x) \geq 3$ and $\deg^h(x) = k + \frac{1}{2}$.*

Proof. Let z be a leaf of F , and D be the component of F containing z . Assume that D is a path $(z, u_1, u_2, \dots, u_n, y)$ such that $\deg_F(u_i) = 2$ ($1 \leq i \leq n$) and $\deg_F(y) = 1$. If $n \geq 2$, then by removing $u_1 u_2$ and recoloring $u_2 u_3$ red, we obtain a smaller fractional factor, a contradiction. If $n = 1$, then $D = P_3 = K_{1,2}$, which is a star.

Next assume that F contains a path $(z, u_1, u_2, \dots, u_n, x)$ such that $\deg_F(u_i) = 2$ ($1 \leq i \leq n$) and x has degree at least 3 in F . If $n \geq 2$, then by removing $u_n x$, and recoloring $u_{n-1} u_n$ and zu_1 red, and recoloring all other remaining edges (if any) of the path blue, we obtain a smaller fractional factor, a contradiction. If $n = 1$, then $F - u_1 x$ is a smaller fractional $[1, k + \frac{1}{2}]$ -factor, a contradiction. Therefore z and x are adjacent.

Moreover, if D contains exactly one vertex x of degree at least 3, by the same argument given above, we see that every leaf of D is adjacent to x and thus D is a star; otherwise, D contains another vertex y of degree at least 3, then by Claim 2, $\deg^h(x) = k + \frac{1}{2}$. Thus the claim is proved.

Claim 4. (i) *If uv is an edge such that $\deg_F(u) \geq 3$ and $\deg_F(v) = 2$, then $\deg^h(u) = k + \frac{1}{2}$ and uv is a blue edge.* (ii) *If xy is a red edge, then one of x and y is a leaf of F .*

Proof. Let uv be an edge of F such that $\deg_F(u) \geq 3$, $\deg_F(v) = 2$ and $\deg^h(u) < k + \frac{1}{2}$. Let z_1 be a vertex adjacent to v . If $\deg_F(z_1) \geq 3$, then by Claim 2, we have $\deg^h(u) = k + \frac{1}{2}$, a contradiction. If $\deg_F(z_1) = 1$, then it contradicts Claim 3. Hence $\deg_F(z_1) = 2$. By removing uv and recoloring

uz_1 red, we obtain a smaller fractional factor, a contradiction. Hence if uv is an edge with $\deg_F(u) \geq 3$ and $\deg_F(v) = 2$, then $\deg^h(u) = k + \frac{1}{2}$. It is sufficient to prove (ii) to deduce that uv is a blue edge.

Assume to the contrary that xy is a red edge and neither x nor y is a leaf. If $\deg_F(x) \geq 3$ and $\deg_F(y) \geq 3$, then $F - xy$ is a smaller fractional factor, a contradiction. Hence, by symmetry, we may assume that $\deg_F(x) = 2$ and $\deg_F(y) \geq 3$. Let z_1 be another vertex adjacent to x . If $\deg_F(z_1) \geq 3$, then $F - xz_1$ is a smaller fractional factor since xy is a red edge, a contradiction. By Claim 3, $\deg_F(z_1) = 2$. From Claim 2, F has no path connecting two vertices of degrees at least 3 and passing through (z_1, x, y) . Hence we may assume that there is a path $(z, u_1, \dots, u_n = z_1, x, y)$ with $\deg_F(z) = 1$, $\deg_F(u_i) = 2$ ($1 \leq i \leq n$) and $\deg_F(x) = 2$. By Claim 3, $u_1 = x$, which is a contradiction since $u_n = z_1 \neq x$. Therefore Claim 4 holds.

Claim 5. *Every component D of F is isomorphic to a graph in $\{K_{1,1}, K_{1,2}, \dots, K_{1,k}, \mathcal{T}(2k+1)\}$.*

Proof. If D is a star with center u , then $\deg^h(u) \leq k + \frac{1}{2}$, which implies $\deg_D(u) \leq k$ since every pendant edge of D is red. So we may assume that D is not a star, which implies that D has at least two vertices of degrees at least 3 by Claim 3.

Let v be a vertex of degree at least 3. If xv is a red edge, then x is a leaf and vice versa. If vy is a blue edge, then $\deg_F(y) = 2$ and vice versa. Consequently, D is a tree in $\mathcal{T}(2k+1)$.

With Claim 5, we complete the proof. □

Acknowledgement

The research was carried out during the first and the third authors' visits to Xi'an Jiaotong University. The support and hospitality from the university were greatly appreciated. We are grateful to the anonymous referees for their constructive comments.

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