Weight-Equitable Subdivision of Red and Blue Points in the Plane

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Abstract

Let R and B be two disjoint sets of red points and blue points, respectively, in the plane in general position. Assign a weight α to each red point and a weight β to each blue point, where α and β are positive integers. Define the weight of a region in the plane as the sum of the weights of red and blue points in it. We characterize the existence of a line that bisects the weight of the plane whenever the total weight $\alpha|R| + \beta|B|$ is 2ω for some integer $\omega \geq 1$. Moreover, we look closely into the special case where $\alpha = 2$ and $\beta = 1$ since this case is essentially important for a weight-equitable subdivision of the plane. Among other results, we show that for any configuration of $R \cup B$ with total weight $2|R| + |B| = n\omega$ for some integer $n \geq 2$ and odd integer $\omega \geq 1$, the plane can be subdivided into n convex regions of weight ω if and only if $|B| \geq n$.

Keywords: weight-equitable subdivision, equitable subdivision, red and blue points in the plane

1 Introduction

Let R be a set of red points and B be a set of blue points in the plane. If no three points of $R \cup B$ are collinear, then we say that $R \cup B$ is *in*

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general position. For a line l in the plane, l has a direction and the open half-plane to the left of l is denoted by $left(l)$, while the open half-plane to the right of l is denoted by $right(l)$. In particular, a plane is partitioned into $left(l) \cup right(l) \cup l$. Note that if l^* is a line on l with opposite direction, then $left(l^*) = right(l)$ and $right(l^*) = left(l)$. In some cases, we consider closed half-planes by specifying them.

Subdivision problems of red and blue points in the plane have been widely studied and the survey paper of Kaneko and Kano [5] lists some of the findings in this area. One classical result is the discrete version of the Ham-sandwich Theorem.

Theorem 1 (Ham-sandwich Theorem, [9]) *Given a red point set R and a blue point set B in the plane in general position, there exists a line l such that $|left(l) \cap R| = |right(l) \cap R|$, $|l \cap R| \leq 1$, $|left(l) \cap B| = |right(l) \cap B|$, and $|l \cap B| \leq 1$. If both $|R|$ and $|B|$ are even, then l passes through no red point and no blue point, thus l bisects R and B.*

A generalization of the above Theorem 1 is called the Equitable Subdivision Theorem, which was proven independently by Bespamyatnikh et al. [1], Sakai [8], and Ito et al., [3]. In particular, Theorem 2 was proved by Bespamyatnikh et al. [1] using a 3-Cutting Theorem, which is mentioned in Section 5.

Theorem 2 ([1], [8], [3]) *Assume that ag red points and bg blue points are in the plane in general position, where a, b and g are positive integers. Then there exists a subdivision of the plane into g convex regions each of which contains exactly a red and b blue points.*

The above Theorem 2 was conjectured by Kaneko and Kano [4], who also gave the proof for the case $a = 1, 2$. Some generalizations and related results to Theorem 2 were also obtained by Kano and Uno [6], [7].

In this paper, we assign weights to the red and blue points. Each red point has weight α and each blue point has weight β, where α and β are two distinct positive integers. Define the *weight* of a region in the plane as the total weight of the points in it. The goal is to find equitable subdivision theorems that partition the total weight of the points in the plane.

In the next section, we prove a characterization of the existence of a *line bisector*, which we define as a line not passing through a red or blue point that subdivides the plane into two regions with the same weight. In the succeeding sections, we examine closely the special case where $\alpha = 2$ and $\beta = 1$ since this is essentially important for a weight-equitable subdivision of the plane and prove results similar to the Equitable Subdivision and 3-Cutting theorems given by Bespamyatnikh et al. [1]. In the last of this paper, we explain polynomial time algorithms for finding the above equitable subdivisions, which are directly obtained from their proofs and not so fast though.

2 Red and blue points with arbitrary weights

Suppose that a red point set R and a blue point set B are given in the plane in general position such that each red point has weight α and each blue point has weight β, where α and β are distinct positive integers. For a region X in the plane, the weight of X is denoted and given by $\text{wt}(X) = \alpha|X \cap R| + \beta|X \cap B|$. Unless we specify, all lines do not pass through any red or blue point.

It is easy to see that if $|R| = |B| = 3$, $\alpha = 5$, and $\beta = 3$, then no line bisector exists because the total weight is $5|R| + 3|B| = 24$ and 12 cannot be expressed as a linear combination of weights 5 and 3. The problem therefore of characterizing the existence of a line bisector is not trivial. Observe first that if $|R|$ and $|B|$ are both even, then the existence of a bisector line is immediate from the Ham-sandwich Theorem. Suppose then that a bisector line l exists and $|R|$ and $|B|$ are not both even. Without loss of generality, suppose that $|R|$ is odd. Denote the sets of red points to the left and right of l by R_1 and R_2, respectively. Similarly, denote the sets of blue points to the left and right of l by B_1 and B_2, respectively. Since $|R_1| \neq |R_2|$, by symmetry we may assume $|R_1| < |R_2|$. Then $|B_1| > |B_2|$. Since l bisects the plane into two regions of the same weight, we have $\alpha|R_1| + \beta|B_1| = \alpha|R_2| + \beta|B_2|$. Solving for α, we have

$$\alpha = \frac{|B_1| - |B_2|}{|R_2| - |R_1|}\beta = \frac{h\beta}{k}, \tag{1}$$

where $h = |B_1| - |B_2| \geq 1$ and $k = |R_2| - |R_1| \geq 1$.

Theorem 3 *Let α, β and ω be positive integers. Let R and B be disjoint non-empty sets of red points and blue points in the plane in general position, respectively, and each red point has weight α and each blue point has weight β. Assume that $\alpha > \beta$ and $\alpha|R| + \beta|B| = 2\omega$. Then for any configuration of $R \cup B$ there exists a line that subdivides the plane into two regions of weight ω if and only if either both $|R|$ and $|B|$ are even or $\alpha = 2\beta$ and $|B|$ is even.*

Proof. The Ham-sandwich Theorem guarantees the existence of a line bisector whenever $|B|$ and $|R|$ are both even. Suppose then that $\alpha = 2\beta$ and

$|B|$ is even. We may assume that $|R|$ is odd. Then by the Ham-sandwich Theorem there is a line l, passing through a red point, say u, such that $\mathrm{wt}(\mathit{left}(l)) = \mathrm{wt}(\mathit{right}(l)) = (2\omega - \alpha)/2 = \omega - \beta$. Then we rotate l on u until we hit another point, say v.

If v is blue, then the desired line l' is obtained by a slight rotation of l so that u and v are on different sides of l'. Notice that the weight of $\mathit{left}(l')$ is ω since we either add v to $\mathit{left}(l)$ (Figure **??** (a)), or add u and take out v (Figure **??** (b)). If v is red, whether it is on the left or right of l, we rotate l further on v until we hit a blue point (Figure **??** (c) and (d)). Note that eventually we will hit a blue point.

Figure 1

Conversely, we show that if at least one of $|R|$ and $|B|$ is odd, and either $\alpha \neq 2\beta$ or $|B|$ is odd, then we can always construct an example with no possible bisector line. Note first that if $\alpha = 2\beta$, then $h = |B_1| - |B_2|$ is even by (1). This means that $|B_1|$ and $|B_2|$ must have the same parity, and so $|B|$ has to be even. Equivalently, if $|B|$ is odd, then $\alpha \neq 2\beta$.

Assume that there exists a bisector line l and suppose that $\alpha \neq 2\beta$. Consider the following configuration: all red points are placed uniformly on a very small circle and all the blue points are placed uniformly on a very large circle with the same center as the small circle (Figure **??**). Let $k = |R_2| - |R_1| \geq 0$ and $h = |B_1| - |B_2|$ as above. If $k = 0$, then $|B_1| = |B_2|$, and so both $|R|$ and $|B|$ are even,which contradicts the assumption. Hence $k \geq 1$. Since $\alpha \neq 2\beta$ and $\alpha > \beta$, we have $h \neq 2k$ and $h > k \geq 1$ by (1); thus $h \geq 3$. Moreover, if $|B|$ is even, then $h \equiv 0 \pmod 2$ and so $h \geq 4$. In this configuration, observe that if $|B|$ is odd then any line l satisfying $h = |B_1| - |B_2| \geq 3$ will give us $\mathit{right}(l) \cap R = \emptyset$ (Figure **??** (a)). Similarly, if $|B|$ even then any line l satisfying $h = |B_1| - |B_2| \geq 4$ will give us $\mathit{right}(l) \cap R = \emptyset$ (Figure **??**(b)). For both cases, we get $k < 0$, which contradicts (1). Therefore, there could never be a line subdividing the plane into two regions with the same weight. □

The process of hitting a blue point in the proof of sufficiency (as illustrated in Figure **??**) is mentioned often in this paper, and so we formalize it below as an observation.

Observation 4 *If $\alpha = 2\beta$ and there exists a line l that passes through a red point and satisfies $\mathrm{wt}(\mathit{left}(l)) = \omega - \beta$, then we can move l to another line l' such that $\mathrm{wt}(\mathit{left}(l')) = \omega$ and $|\mathit{left}(l') \cap B| = |\mathit{left}(l) \cap B| \pm 1$.*

3 The case where $\alpha = 2$ and $\beta = 1$

In bisecting the total weight of the plane, Theorem 3 shows that the case where $\alpha = 2\beta$ is important. This is essentially equivalent to the case where $\alpha = 2$ and $\beta = 1$, and so we explore this case further. Ultimately, we want to subdivide the plane into disjoint convex regions each of equal weights. This is what we call a *weight-equitable subdivision.* Among other results, we prove two Weight-Equitable Subdivision Theorems, namely Theorem 5 for regions of even weights and Theorem 6 for odd weights.

Theorem 5 *Let R and B be disjoint sets of red points and blue points, respectively, in the plane in general position. Assume that every red point has weight 2 and every blue point has weight 1. Let $\omega \geq 2$ be an even integer and $n \geq 1$ be an integer. Then for any configuration of $R \cup B$ with total weight $2|R| + |B| = n\omega$, there exists a subdivision of the plane into n convex regions each with weight ω.*

Theorem 6 *Let R and B be disjoint sets of red points and blue points, respectively, in the plane in general position. Assume that every red point has weight 2 and every blue point has weight 1. Let $\omega \geq 1$ be an odd integer and $n \geq 1$ be an integer. Then for any configuration of $R \cup B$ with total weight $2|R| + |B| = n\omega$, there exists a subdivision of the plane into n convex regions each with weight ω if and only if $|B| \geq n$.*

If ω is an odd integer and the plane is subdivided into n convex regions with weight ω, then each region contains at least one blue point, and thus the number of blue points has to be at least n. Hence, Theorem 6 shows that this necessary condition is sufficient for the existence of a weight-equitable subdivision of the plane. The proof of Theorem 6 is stated at the end of Section 5. On the other hand, Theorem 5 follows from the following Lemma 7 by induction on n.

Lemma 7 *Let ω be an even integer. Then for any configuration of $R \cup B$ with total weight $2|R| + |B| = n\omega$, there exists a line l with $\mathrm{wt}(\mathit{left}(l)) = \omega$.*

Proof. We may assume that all points have distinct x-coordinates by a suitable rotation of the plane, and let l_1 be a vertical line directed upward at the left of the $conv(R \cup B)$. Move l_1 horizontally to the right. If we attain a weight ω to the left of some line l_2 in this procedure, we are done. Otherwise, we attain a weight $\omega - 1$ to the left of l_2 and the next point to the right is a red point v. Thus there is a vertical line l that passes through the red point v and satisfies $\mathrm{wt}(\mathit{left}(l)) = \omega - 1$. By Observation 4, there exists a line l'

with the desired weight ω on its left half-plane. $\square$

The remaining sections in this paper is devoted to the development of a proof of Theorem 6.

4 Weight-Equitable 2-Cutting

Assume that the total weight of the plane is $2|R| + |B| = n\omega$, where ω is odd and $|B| \geq n$. If $\omega = 1$, then all points are blue, and Theorem 6 holds. Thus we assume hereafter that $\omega \geq 3$. Let $n = n_1 + n_2$, where n_1, n_2 are positive integers. An (n_1, n_2) 2-*cutting* is a partition of the plane by a line such that one half-plane has at least n_1 blue points and the other has at least n_2 blue points. Moreover, an (n_1, n_2) 2-cutting is called *weight-equitable* if the half-plane containing at least n_1 blue points has weight exactly $n_1\omega$ and the other, which contains at least n_2 blue points, has weight exactly $n_2\omega$.

We need the following results.

Lemma 8 (Kaneko and Kano [4]) *If there exist two lines l_1 and l_2 such that $|left(l_1) \cap R| = |left(l_2) \cap R|$ and $|left(l_1) \cap B| < |left(l_2) \cap B|$, then for every integer k, $|left(l_1) \cap B| \leq k \leq |left(l_2) \cap B|$, there exists a line l_3 such that $|left(l_3) \cap R| = |left(l_1) \cap R|$ and $|left(l_3) \cap B| = k$.*

Corollary 9 *Let $n = n_1 + n_2$. Assume that there exist two lines l_1 and l_2 such that $|left(l_1) \cap B| = |left(l_2) \cap B| = n_1$ and $\mathrm{wt}(left(l_1)) \leq n_1\omega \leq \mathrm{wt}(left(l_2))$. Then there exists an (n_1, n_2) weight-equitable 2-cutting of the plane.*

Proof. It is clear that $\mathrm{wt}(left(l_1) \cap R) \leq n_1\omega - n_1$, which implies that $|left(l_1) \cap R| \leq n_1(\omega - 1)/2$. Similarly, we have $|left(l_2) \cap R| \geq n_1(\omega - 1)/2$. Since ω is odd, $n_1(\omega - 1)/2$ is an integer. Thus by Lemma 8, there is a line l_3 with $|left(l_3) \cap B| = n_1$ and $|left(l_3) \cap R| = n_1(\omega - 1)/2$, which implies $\mathrm{wt}(left(l_3)) = n_1\omega$. Therefore l_3 determines an (n_1, n_2) weight-equitable 2-cutting, as desired. $\square$

Since we may assume that there exists no weight-equitable 2-cutting, by Corollary 9 we can introduce the sign of an integer as follows. For an integer $1 \leq x \leq n - 1$, define the *sign* of x, denoted by $sign(x)$, as follows.

$$sign(x) = \begin{cases} + & \text{, if } \mathrm{wt}(left(l)) > x\omega \text{ for all } l \text{ with } |left(l) \cap B| = x; \\ - & \text{, if } \mathrm{wt}(left(l)) < x\omega \text{ for all } l \text{ with } |left(l) \cap B| = x. \end{cases}$$

Lemma 10 *Assume $n = n_1 + n_2$. If $sign(n_1) = sign(n_2)$, then $sign(n_1) = sign(n_2) = -$, and an (n_1, n_2) weight-equitable 2-cutting of the plane exists.*

Proof. If $sign(n_1) = sign(n_2) = +$ then, by taking two vertical lines l_1 and l_2 with $|left(l_1) \cap B| = n_1$, $|right(l_2) \cap B| = n_2$, and $l_2 \in right(l_1)$, we have $\mathrm{wt}(left(l_1)) + \mathrm{wt}(right(l_2)) > (n_1 + n_2)\omega = n\omega$, which exceeds the total weight of the plane. Therefore, if $sign(n_1) = sign(n_2)$, then $sign(n_1) = sign(n_2) = -$.

Now suppose that $sign(n_1) = sign(n_2) = -$. We may assume that the points of $R \cup B$ have different x-coordinates. Let l_1 be a vertical line directed upward such that $|left(l_1) \cap B| = n_1$ and $\mathrm{wt}(left(l_1))$ is maximal subject to $|left(l_1) \cap B| = n_1$. Then $\mathrm{wt}(left(l_1)) < n_1\omega$ by $sign(n_1) = -$ and l_1 is close to a blue point $b_1 \in right(l_1)$ (i.e. if we move l_1 horizontally to the right we first hit the blue point b_1). If $|right(l_1) \cap B| = n_2$, then $\mathrm{wt}(right(l_1)) < n_2\omega$ by $sign(n_2) = -$. This contradicts the total weight of the plane, and so we must have $|right(l_1) \cap B| > n_2$. Let l_2 be a vertical line to the right of l_1 which satisfies $|right(l_2) \cap B| = n_2$. Moreover, we choose l_2 so that $right(l_2)$ has maximal weight under the condition $|right(l_2) \cap B| = n_2$. So l_2 is close to a blue point $b_2 \in left(l_2)$.

If all the points in $right(l_1) \cap left(l_2)$ are blue then, by moving l_1 to the right, we find a vertical line l_3 to the left of l_2 such that $|left(l_3) \cap B| \geq n_1$ and $\mathrm{wt}(left(l_3)) = n_1\omega$. Thus the desired (n_1, n_2) weight-equitable 2-cutting is obtained. Now, suppose not all points in $right(l_1) \cap left(l_2)$ are blue, that is, $right(l_1) \cap left(l_2)$ contains at least one red point. The last statement implies that $b_1 \neq b_2$. By moving l_1 horizontally to the right, we can find a vertical line l_4 such that $left(l_4)$ contains b_1 and satisfies either (i) $\mathrm{wt}(left(l_4)) = n_1\omega$ or (ii) $\mathrm{wt}(left(l_4)) = n_1\omega - 1$ and passing through a red point. In the latter case, by Observation 4, we can find a line l_5 such that $\mathrm{wt}(left(l_5)) = n_1\omega$ and $|left(l_5) \cap B| \geq n_1$. Therefore the lemma holds. □

To apply Lemma 10 we need a pair (n_1, n_2) with the same sign, which does not always exist (see Figure **??**).

Figure 4

Caption: This configuration has weight 27 and does not have a weight-equitable 2-cutting with $\omega = 9$, but has an equitable 3-cutting.

The following lemma tells us what happens if a pair with the same sign does not occur. We present it here with the proof given by Bespamyatnikh et al. [1].

Lemma 11 *For any sequence of signs $sign(1), sign(2), ..., sign(n-1)$, there is either a pair $(m, n-m)$ both with negative signs or a triple $(k-1, n-k, 1)$ with $sign(k-1) = sign(n-k) = sign(1)$, for some integer $2 \leq k \leq n-1$.*

Proof. If there is an integer $1 \leq m < n$ with $sign(m) = sign(n-m)$, then by Lemma 10 $sign(m) = sign(n-m) = -$, and we are done. Suppose now that $sign(i) \neq sign(n-i)$ for all $1 \leq i \leq n-1$. In particular, suppose $sign(1) \neq sign(n-1)$. Let k be the smallest index with $sign(k) \neq sign(1)$. Then $sign(k-1) = sign(n-k) = sign(1)$ since $sign(k) \neq sign(n-k)$. And so $(k-1, n-k, 1)$ is the desired triple. □

Since a pair with the same sign may not always exist, consequently, a weight-equitable 2-cutting in the plane may not always happen.

Corollary 12 *If $sign(1) = -$, then there exists a weight-equitable 2-cutting in the plane.*

Proof. We prove this by contradiction. Suppose, on the contrary, that no weight-equitable 2-cutting exists in the plane. By Lemma 10, no pair with the same sign exists. Consequently, as in the proof of Lemma 11, a triple $(k-1, n-k, 1)$ with the same sign exists, where k is defined to be the smallest positive integer whose sign differs from 1. By assumption, $sign(1) = -$; hence, $sign(k) = +$ and $sign(k-1) = -$. By some rotation of the plane, we may regard all red and blue points to have different x-coordinates. Let l_1 be a vertical line, not passing through any red or blue point, such that $|left(l_1) \cap B| = k-1$. In addition, choose l_1 so that $\mathrm{wt}(left(l_1))$ is maximal. This means that the next point to the right of l_1 is a blue point, say v. Let l_2 be a vertical line slightly to the right of v. Then $|left(l_2) \cap B| = |left(l_1) \cap B| + 1 = k$. Observe that $k\omega < \mathrm{wt}(left(l_2)) = \mathrm{wt}(left(l_1)) + \mathrm{wt}(v) < (k-1)\omega + 1$, which implies that $\omega < 1$. We arrive at a contradiction since $\omega \geq 3$. □

5 Weight-Equitable 3-Cutting

In this section we show how to cut the plane into three convex regions, and we call this the Weight-Equitable 3-Cutting Theorem. First we state and explain a result on equitable 3-cuttings of red points and blue points in the plane since our weight-equitable 3-cutting theorem, together with its proof, is crafted in a similar manner. Let R be a set of ag red points and B be a set of bg blue points in the plane in general position, where a, b, g are positive integers. For every line l such that $left(l)$ contains exactly ak red points, if $left(l)$ always contains more than bk blue points, then we write $s(k) = +$, and if *left*(l) always contains less than bk blue points, then we write $s(k) = -$. Let g_1, g_2, g_3 be positive integers such that $g_1+g_2+g_3 = g$. A (g_1, g_2, g_3)-equitable 3-cutting is a partition of the plane into three convex wedges by three rays emanating from the same point called its *apex*, where the ith wedge contains exactly ag_i red points and bg_i blue points, for $i = 1, 2, 3$.

Theorem 13 (Bespamyatnikh et al. [1]) *Let R be a set of ag red points and B be a set of bg blue points in the plane in general position. Let g_1, g_2 and g_3 be positive integers with $g_1 + g_2 + g_3 = g$. If $s(g_1) = s(g_2) = s(g_3)$, then there exists a (g_1, g_2, g_3)-equitable 3-cutting.*

We now define a weight-equitable 3-cutting of the plane. Assume that the total weight of the plane in $2|R| + |B| = n\omega$, where ω is odd and $|B| \geq n$. If $n = 2$ then by Theorem 3, a weight-equitable subdivision of the plane exists. Assume from here on that $n \geq 3$. Let $n = n_1 + n_2 + n_3$, where n_1, n_2, n_3 are positive integers. A region determined by two rays emanating from the same point on the plane is called a *wedge*. An (n_1, n_2, n_3) *weight-equitable* 3-*cutting* is a partition of the plane into three convex wedges $W_1 \cup W_2 \cup W_3$ determined by 3 rays, not containing a red or blue point and emanating from a common point called *apex*, such that W_i has weight $n_i\omega$ and contains at least n_i blue points for $i = 1, 2, 3$.

Figure 5

Caption: add " $\cdots$ of the plane, where $\omega = 5$. "

Since we prove Theorem 6 by induction on n, we may assume that there exists no weight-equitable 2-cutting. If n is even, then $n = (n/2) + (n/2)$, and so there exists a weight-equitable 2-cutting by Lemma 10. Thus n is odd. Moreover, Corollary 12 tells us that $sign(1) = +$. Therefore, by Lemma 11, $sign(k-1) = sign(n-k) = sign(1) = +$. The following Theorem 14 is our weight-equitable 3-cutting theorem.

Theorem 14 (Weight-equitable 3-cutting theorem) *Let $n \geq 3$ be an odd integer and $2 \leq k \leq n-1$ be an integer. Assume that $sign(k-1) = sign(n-k) = sign(1) = +$. Then then there exists an $(k-1, n-k, 1)$ weight-equitable 3-cutting of the plane.*

In Figure **??** the total weight of the plane is $2|R|+|B| = 30$. Choose $n = 6$, $\omega = 5$, and $k = 4$. Then the 3-cutting given in Figure **??** is a $(k-1, n-k, 1)$ weight-equitable 3-cutting of the plane. For the proof of Theorem 14, we aim to construct an $(k-1, n-k, 1)$ weight-equitable 3-cutting with one ray going down, which we call the *downward ray.* The two other rays are called *left ray* and *right ray.* Similarly, we call the wedges adjacent to the downward ray as *left wedge* and *right wedge*, while the remaining wedge is called the *upper wedge.* We now give some new notations, definitions, and explanation for the proof of Theorem 14.

We often write n_1, n_2, n_3 for $k-1, n-k, 1$, respectively. Let q be a point in the plane such that

$$\text{the vertical line passing through } q \text{ contains no point of } R \cup B \quad (2)$$

$$\text{and no three points of } R \cup B \cup \{q\} \text{ are collinear.} \quad (3)$$

First we define a *canonical blue 3-cutting* with apex at point q. Let $p_1, p_2, ..., p_{|B|}$ be the list of blue points in clockwise order starting from the downward ray from q. The left ray of the canonical blue 3-cutting passes through the n_1th blue point p_{n_1}. On the other hand, the right ray of the canonical blue 3-cutting passes through $p_{|B|-n_3+1}$, which is the n_3rd blue point in counterclockwise order starting from the downward ray from q. Consequently, the upper wedge of the canonical blue 3-cutting contains at least n_2 blue points.

From here on, we replace all red points by very small red circles centered at the original position of the red points in such a way that these red circles and blue points are also in general position. Thus, if a line l passes through a red circle u, then we distinguish whether we attribute a weight of 1 to both half-planes induced by l or whole u is included in one of the half-planes induced by l. The contributed weight of a red point, say u, to a certain half-plane is denoted by $\mathrm{wt}(u)$.

Now we consider a *canonical weight 3-cutting.* In a canonical weight 3-cutting, the left ray passes through a point x such that the weight of the left wedge including x is exactly $n_1\omega$. If x is red, we distinguish whether or not $\mathrm{wt}(x) = 1$ or 2. On the other hand, the right ray passes through the point y such that the weight of the right wedge including y is exactly $n_3\omega$. Again, if y is red, we distinguish whether or not $\mathrm{wt}(y) = 1$ or 2. Consequently, the upper wedge will have weight exactly $n_2\omega$.

Figure 6

Caption: $\omega = 7$ (a) A (1,2,2) canonical blue

Assume that all points have distinct x-coordinates. Let $\mathcal{R}$ consist of those points q satisfying conditions (2), (3) and the following (4).

$$\text{The } (n_1, n_2, n_3) \text{ canonical blue 3-cutting with apex at } q \text{ is convex.} \qquad (4)$$

Let $p_1, p_2, ..., p_{|B|}$ be the list of blue points sorted by increasing x-coordinates. Consider two vertical lines l_1 and l_2 passing through the blue points p_{n_1} and $p_{|B|-n_3+1}$, respectively. Hence for any point q_1 to the left of l_1, the left wedge of a canonical blue 3-cutting with apex q_1 is non-convex. Symmetrically, for any point q_2 on the right of l_2, the right wedge of a canonical blue 3-cutting with apex q_2 is non-convex. Therefore the region $\mathcal{R}$ is bounded by the lines l_1 and l_2. Since $sign(n_1) = sign(n_2) = sign(n_3) = +$, the closed half-plane to the left of l_1 has weight greater than $n_1\omega$ and the closed half-plane to the right of l_2 has weight greater than $n_3\omega$.

Consider now a vertical line l between l_1 and l_2. For any point q in l, the angles of the left and right wedges of the canonical blue 3-cutting with apex at q are both less than π. Furthermore, each such angle is a monotone function of the y-coordinate of the point q, that is, the angles of both wedges decrease when q goes up the vertical line l. This implies that the angle of the upper wedge is also a monotone function. If q has the minimum y-coordinate among all the red and blue points, then both the left and right rays go up and the upper wedge is convex. Now, for a point q in l with the maximum y-coordinate among all red and blue points, the upper wedge of the canonical blue 3-cutting has angle greater than π, making it non-convex. Hence there must be a point q in l such that the points above q do not belong to $\mathcal{R}$.

Draw lines passing through pairs of points of $R \cup B$ as well as vertical lines passing through the points of $R \cup B$ forming an arrangement $\mathcal{A}$ of regions bounded by these lines which we call *faces*. For any point x on the interior of a face $F \in \mathcal{A}$, the canonical blue and canonical weight 3-cuttings with apex at x are essentially the same. We assign label B to the two rays of the canonical blue 3-cutting and label W to the two rays of the canonical weight 3-cutting. Assign a label $1, 2, 3$ or 4 to a point x on a face according to the clockwise order of left rays and the clockwise order of right rays (see Figure **??**) as follows:

1=WB-WB, 2=WB-BW, 3=BW-BW, and 4=BW-WB.

Figure 7

If the rays of a canonical blue and canonical weight 3-cutting coincide, we first write label B and then label W (i.e. the label of the canonical blue 3-cutting has priority). Since all points in the interior of a face of $\mathcal{A}$ have the same label, we take that to be the label of that face.

Consider the labels on the boundary of $\mathcal{R}$. At $y = -\infty$ (the farthest points below conv$(R \cup B)$) all labels are 2=WB-BW due to the signs of n_1 and n_3. The faces forming the left boundary of $\mathcal{R}$, all of which are incident and on the right of l_1, do not have label 4=BW-WB as the following claim. For convenience, we call red and blue points *data points*, and the convex wedge determined by two rays r_a and r_b emanating from a common apex is denoted by $wedge\{r_a, r_b\}$.

Claim 1 *A face forming the left boundary of $\mathcal{R}$, which is incident and on the right of l_1, does not have label 4=BW-WB.*

Proof. Assume that there exists a point q which lies in a face forming the left boundary of $\mathcal{R}$ and has label 4=BW-WB. If q lies below p_{n1}, then q has label 1=WB-BW or 2=WB-WB by $sign(n_1) = +$. So we now assume that q lies above p_{n1} (see Figure 1).

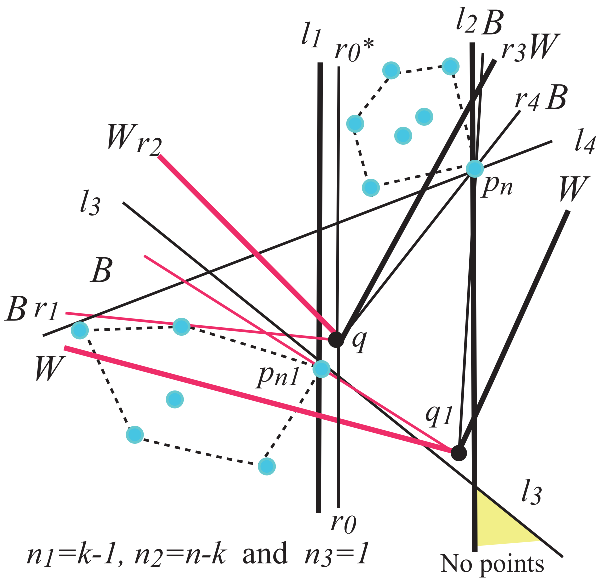

Figure 1: A point q has label 4=BW-WB and q_1 has label 2=WB-BW.

Let r_0 and r_0^* be the vertical rays emanating from q going downward and upward, respectively, and let r_1, r_4 and r_2, r_3 be the rays emanating from q that determine the canonical blue 3-cutting and the canonical weight 3-

cutting, respectively (see Figure 1). Then an open wedge $wedge\{r_0, r_4\}$ contains no blue point since $n_3 = 1$, and an open wedge $wedge\{r_1, r_0^*\}$ contains no blue point since an closed wedge $wedge\{r_0, r_1\}$ contains n_1 blue points. Moreover the weight of $wedge\{r_0, r_4\}$ is less than $\omega - 1$ since the label of q is 4=BW-WB.

There is a line l_3 touching the convex hull $conv(left(l_1) \cap B)$ such that the region below l_3 and right l_2 contains no red and blue points. Then for a point q_1 in a face attaching the intersection of l_2 and l_3, the label of q_1 is 2=WB-BW since $sign(n_1) = sign(n_3) = +$ and $|left(l_3) \cap B| = n_1$. Then we shall show that there exists an $(k-1, n-k, s)$-weight-equitable 3-cutting with apex between l_1 and l_2, above l_3 and below r_4.

We now consider the region X surrounded by three lines l_1, l_2, l_3 and a l_4, where l_4 is a line touching both $conv(left(l_1) \cap B)$ and $conv(right(l_1) \cap B)$ (see Figure 1). We divide this region X by lines passing through a blue vertex of $conv(left(l_1) \cap B)$ and a red point either in $left(l_1)$ and below l_3 or in $right(l_1) \cap left(l_2)$, lines passing through the blue point p_n and a red point either in $right(l_2)$ and above l_4 or in $right(l_1) \cap left(l_2)$ (see Figure 2).

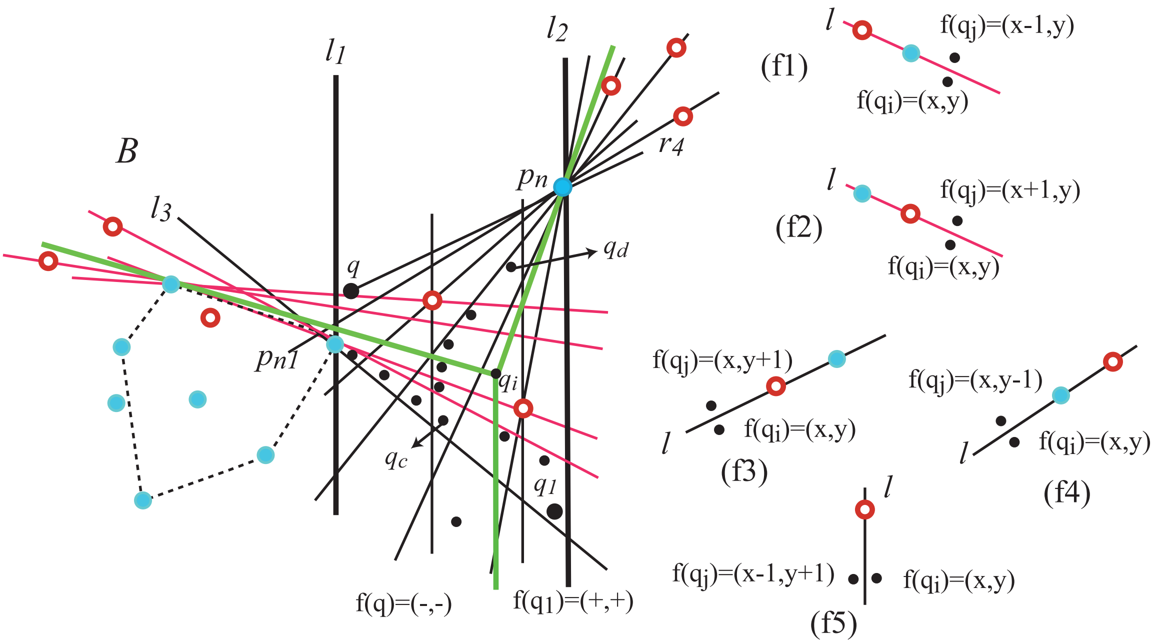

Figure 2: A Subdivision of X, where a point q has label 4=BW-WB and q_1 has label 2=WB-BW.

For any point q' in a face of X determined as above, we define a function f as follows.

$$f(q') = \Big((\text{the number of red points in the left wedge of } q') - n_1(\omega - 1)/2,$$
$$(\text{the number of red in left wedge of } q') - (\omega - 1)/2\Big).$$

Then for two points q_i and q_j in X, which are contained in two faces touching a common line l, the following holds.

(f1) If l passes through a blue point of $conv(left(l_1) \cap B)$ and a red point in $left(l_1)$ and q_i is below l, then $f(q_j) = (x - 1, y)$, where $f(q_i) = (x, y)$ (see Figure 2).

(f2) If l passes through a blue point of $conv(left(l_1) \cap B)$ and a red point in $right(l_1) \cap left(l_2)$ and q_i is below l, then $f(q_j) = (x + 1, y)$, where $f(q_i) = (x, y)$.

(f3) If l passes through a blue point of $conv(right(l_1) \cap B)$ and a red point in $right(l_1) \cap left(l_2)$ and q_i is below l, then $f(q_j) = (x, y + 1)$, where $f(q_i) = (x, y)$.

(f4) If l passes through a blue point of $conv(right(l_1) \cap B)$ and a red point in $right(l_2)$ and q_i is below l, then $f(q_j) = (x, y - 1)$, where $f(q_i) = (x, y)$.

(f5) If l is a vertical line passing through a red point in $right(l_1) \cap left(l_2)$, q_i is to the left of l and q_j is to the right of l, then $f(q_j) = (x - 1, y + 1)$, where $f(q_i) = (x, y)$ (see Figure 2).

We first show that if there is a point q_a such that $f(q_a) = (0, 0)$, then the wedges determined by q_a form the desired $(k - 1, n - k, s)$-weight-equitable 3-cutting since the left wedge of q_a contains n_1 blue points and has weight $n_1 + n_1(\omega - 1) = n_1\omega$, and the right wedge of q_a contains one blue point and has weight $1 + \omega - 1 = \omega$.

When we move q' from q_c to q_d along a vertical lines and lines of (f1) or (f2), then its y-coordinate is constant and its x-coordinate is changed one by one one. Similarly, if we move q' along a line of (f1) or (f2), then its x-coordinate is constant and its y-coordinate is changed one by one one. By the above properties and by $f(q_1) = (-, -)$ and $f(q_1) = (+, +)$, there is a point q for which $f(q) = (0, 0)$. Hence Claim 1 holds.

Claim 2 *If F_x is a top and left face of $\mathcal{R}$, then it has label 1=WB-WB.*

Proof. Let q be a point in F_x. If q lies below P_{n1}, then the claim holds. So we may assume that q lies above P_{n1}. By Claim 1, q does not have a label 4=BW-WB. If q has a label 3=BW-BW, then the canonical weight 3-cutting is the desired $(k-1, n-k, 1)$ weight-equitable 3-cutting since two open wedge $wedge\{r_1, r_2\}$ and $wedge\{r_3, r_4\}$ contains no blue point. If q has a label 2=WB-BW, then the weight of the half-plane above l_3 is less than $n_2 \cdot \omega$, which contradicts $sign(n_2) = +$ (see Fig. 3).

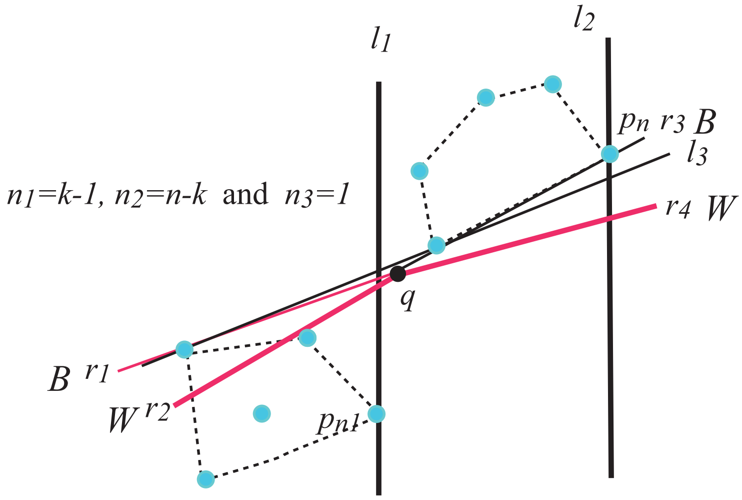

Figure 3: If q has a label 2=WB-BW, then this contradicts $sign(n_2) = +$.

Lemma 15 (Bespamyatnikh et al. [1]) *Let $\mathcal{Q}$ be any closed region in the plane. Let $\mathcal{B}$ be a finite arrangement of curves in $\mathcal{Q}$, in which each face of $\mathcal{B}$ is labeled with a 1, 2, 3, or 4. Suppose that there are two faces F_x and F_y, labeled 1 and 3, on the boundary $\partial(\mathcal{Q})$ such that one component of $\partial(\mathcal{Q}) - \{F_x, F_y\}$ uses labels 1,2 3 and the other component uses 1,3,4. Then some point of $\mathcal{Q}$ lies on the boundary between two faces whose labels differ by two.*

Lemma 16 *Let $k \geq 2$ be an integer and $n_1 = k-1$, $n_2 = n-k$, and $n_3 = 1$. Assume that $sign(n_1) = sign(n_2) = sign(n_3) = +$. Let p be a point of $\mathcal{A}$ on the boundary between two faces whose labels differs by two. Then one of the faces around p contains the apex of a weight-equitable 3-cutting.*

Proof. Throughout the proof, denote by p the point in the boundary of two faces F and F' of the arrangement $\mathcal{A}$ whose labels differ by 2. Let q be a point of F and let b_1, b_2, w_1, w_2 be the points that determine the canonical 3-cuttings with apex at q. Similarly, let q' be a point of F' and let b'_1, b'_2, w'_1, w'_2 be the points that determine the canonical 3-cuttings at q'. First, we prove the following claim.

Claim 2 *If $b_1 \neq w_1$, then b_1, w_1, p have to be collinear. A similar statement can be said about b_2, w_2, and p.*

We prove this by contradiction. Assume that $b_1 \neq w_1$ and b_1, w_1, and p are not collinear. Consider first the situation when q moves to a point q' of another face by crossing a non-vertical edge of $\mathcal{A}$ incident with p. Let l be the non-vertical line containing the non-vertical edge containing p. Then l is determined by two points, say $x, y \in R \cup B$. By definition of $\mathcal{A}$, no line except l intersects the line segment joining q and q'. We often use this fact without mention.

It is easy to see that if b_1 is not on line l, then $b_1' = b_1$. Moreover, if $b_1 \neq b_1'$ then $\{b_1, b_1'\} = \{x, y\}$. A similar statement also holds for w_1 and w_1'. Thus, if neither b_1 nor w_1 lies in l the labels of the left rays of the canonical blue and canonical weight 3-cuttings with apex at q and q' are the same. If b_1 is in l then, by assumption, w_1 is not in l. Also, if $b_1' \neq b_1$, that is both b_1 and b_1' are in l, then $w_1 = w_1'$. This means that the canonical weight 3-cutting in this particular instance remains unchanged whether the apex is at q or q'. Therefore, the labels of the left rays of the canonical 3-cuttings with apex at q and q' are the same. Namely, if their labels are different, then the line containing w_1 and b_1 or w_1' and b_1' passes through a face containing q or q', which contradicts the construction of the arrangement $\mathcal{A}$. This shows that if p lies in a non-vertical, the labels of q and q' do not differ by 2 (a contradiction).

Next we consider the case where p is on a vertical boundary. This means that the vertical line passing through p, denoted by l, contains a data point. Consequently, F and F' are on opposite sides of l. By symmetry, assume that $F \in \mathit{left}(l)$. If there is no data point below p, then the labels of the left rays of the canonical 3-cuttings with apex at q and q' are again the same. Suppose a data point, say t, is below p. Note here that t is below the line passing through q and q'. If t is blue then $b_2 = t$ (since $n_3 = 1$), which makes the upper wedge of the canonical blue 3-cutting with apex at q non-convex. This is a contradiction.

Suppose t is red. Note that the inclusion or exclusion of t to either the left or right wedge does not affect the clockwise order of blue points. Hence, $b_1 = b_1'$. If the label of q is WB-xx then w_1' is below w_1, and so the label of q' is still WB-xx (a contradiction). If the label of q is BW-xx, consider the weight of left wedge of the canonical weight 3-cutting at q.

$$n_1\omega = n_1 + 2r_1 + \mathrm{wt}(w_1) \Longrightarrow n_1(\omega - 1) = 2r_1 + \mathrm{wt}(w_1), \qquad (5)$$

where r_1 denotes the number of red points on the left wedge the canonical weight 3-cutting. Thus, $\mathrm{wt}(w_1)$ must be even, and so w_1 is a red circle, and whole w_1 is included in the left wedge of the canonical weight 3-cutting. Therefore, if we move q to q', the weight 2 of t is now counted in the left wedge of the canonical weight 3-cutting; hence, w_1' stays above b_1 or is equal to b_1. Hence, the label of q' is still BW-xx (a contradiction). This proves the claim.

Assume therefore that b_1, w_1, p and b_2, w_2, p are collinear and p lies in a vertical line in the arrangement. Note that p cannot be a data point. Moreover, a blue point cannot be found below p; otherwise, the upper wedge of the canonical blue 3-cutting becomes non-convex. Denote by r_1 and r_2 the number of red points on the left and right wedge, respectively, of the canonical weight 3-cutting. Consider the following cases:

Case 1 The label of q is BW-WB.

By (5), $\mathrm{wt}(w_1)$ must be even, and so w_1 is a red circle, and whole w_1 is included in the left wedge of the canonical weight 3-cutting. Similarly, by counting the weight of the right wedge of the canonical weight 3-cutting with apex at q, we have

$$n_3\omega = n_3 + 2r_2 + \mathrm{wt}(w_3) \Longrightarrow n_3(\omega - 1) = 2r_2 + \mathrm{wt}(w_3) \qquad (6)$$

Thus, $\mathrm{wt}(w_3)$ must be even, and so w_3 is a red circle and whole w_3 is included in the right wedge of the canonical weight 3-cutting. Observe that the desired weight-equitable 3-cutting is obtained by slightly rotating $\vec{qw_1}$ clockwise and $\vec{qw_1}$ counter-clockwise (see Figure **??**). Note that the said construction works whether or not there is a red point below p.

Figure 8

Case 2 The label of q is BW-BW.

By (5), whole w_1 is included in the left wedge of the canonical weight 3-cutting with apex q. Now, consider the weight of the right wedge of the canonical weight 3-cutting with apex at q.

$$n_3\omega = n_3 - 1 + 2r_2 + \mathrm{wt}(w_2) \Longrightarrow n_3(\omega - 1) + 1 = 2r_2 + \mathrm{wt}(w_3) \qquad (7)$$

The last equation implies that $\mathrm{wt}(w_2)$ is odd, and so w_2 is either a blue point or a red circle that contributes a weight 1 to the right wedge. If w_2 is blue then $b_2 = w_2$, since $n_3 = 1$. Thus, the desired weight-equitable 3-cutting is constructed as follows:

Figure 9

The same construction holds whether or not p has a red point below it. Suppose now that w_2 is a red circle that only contributes 1 to the weight of the right wedge. Then the construction of the weight-equitable 3-cutting depends on whether or not a red point below p exists.

Figure 10

Case 3 The label of q is WB-BW.

If there is a red point t below p, observe that the label of q' will be WB-xx, which means that the labels of q and q' do not differ by 2 from q (a contradiction). Assume, therefore, that that no red point lies below p. Consider the left wedge of the canonical weight 3-cutting at q.

$$n_1\omega = n_1 - 1 + 2r_1 + \mathrm{wt}(w_1) \Longrightarrow n_1(\omega - 1) + 1 = 2r_1 + \mathrm{wt}(w_1) \qquad (8)$$

The last equation implies that w_1 is either a blue point or a w_1 is a red circle that contributes a weight 1 to the left wedge. If w_1 is a blue point, then w_1 is the $(n_1 - 1)$th blue point in the clockwise order of blue points from q. Therefore, the left wedge of the canonical weight 3-cutting with apex q gives us

$$n_1\omega = n_1 - 1 + 2r_1 \Longrightarrow n_1(\omega - 1) + 1 = 2r_1 \qquad (9)$$

The last equation is a contradiction. Assume therefore that w_1 is a red circle in which $\mathrm{wt}(w_1) = 1$. By (7), w_2 is also either a blue point or a red circle that contributes a weight 1. For both situations, a weight-equitable 3-cutting can be constructed.

Figure 11

Case 4 The label of q is WB-WB.

By (8) and (9), both w_1 and w_2 are red circles such that $\mathrm{wt}(w_1) = \mathrm{wt}(w_2) = 1$. On the other hand, by (6), whole w_2 is contained in the right wedge of the canonical weight 3-cutting with apex at q. If there is a red point below p, observe that the label of q' will still be WB-xx (a contradiction). Assume, therefore, that there is no data point below p. Then the desired weight-equitable 3-cutting is constructed as follows:

Figure 12

The proof is complete. □

Finally, here is the proof of Theorem 6.

Proof of Theorem 6. Theorem 6 follows from Theorem 14, Lemma 11, and by induction on n. □

6 Computational Complexity

In this section, we show that there is a polynomial time algorithm for finding a weigh-equitable subdivision. This algorithm is directly obtained from the proof, and not so fast. It will be a future work to propose a fast algorithm. Let $|R| + |B| = m$ and $\mathrm{wt}(R \cap B) = 2|R| + |B| = n\omega$. For any two points x and y of $R \cup B$, let l be the line passing through x and y. Then we obtain the four lines l_1, l_2, l_3 and l_4 from l, whose left half planes contain $\mathit{left}(l)$ and satisfy (i) $\mathit{left}(l_1)$ contains both x and y, (ii) $\mathit{left}(l_2)$ contains x but not y, (iii) $\mathit{left}(l_3)$ contains y but not x, and (iv) l_4 contains neither x nor y. We first assume that ω is even. Then by Theorem 5, there exists a line l such that $\mathrm{wt}(\mathit{left}(l)) = \omega$. Since m red and blue points determine $(m^2 - m)/2$ lines, by checking all $2(m^2 - m)$ lines given above, we can find a line l with $\mathrm{wt}(\mathit{left}(l)) = \omega$. Since it takes m time to calculate the weight of the left half plane determined by a line, if we write $T(n)$ for the computation time of this procedure, we have $T(n) = O(m^3) + T(n-1)$, and thus $T(n) \leq O(nm^3)$.

We next assume that ω is odd. If there is a equitable 2-cutting, then we can find it in $O(nm^3)$ time as above. Otherwise, we can find $(k-1, n-k, 1)$ equitable 3-cutting, whose existence is guaranteed by Theorem6. The number of lines passing through two points of $R \cup B$ is $(m^2 - m)/2$, and the number of vertical line passing through a point of $R \cup B$ is m. Since k lines in the plane subdivide the plane into at most $(k^2 - k + 2)/2$ regions, the number of regions determined by these two types of lines is $O(m^4)$. For each region, we can check a point q in it determine a $(k-1, n-k, 1)$ equitable 3-cutting in $O(m \log m)$ time by making a list of all the red and blue points in clockwise order starting from the downward ray from q. Hence we obtain $T(n) = O(m^4 \log m) + T(k-1) + T(n-k)$, and thus $T(n) \leq O(m^4 \log m)n$. Therefore, we can show that there is a a polynomial time algorithm for finding a weigh-equitable subdivision.

7 Conclusion and Future Work

Section 2 proved that given any configuration of red and blue points in the plane in general position, where each red point has weight α and each blue point has weight β, the total weight of the plane can be bisected by a line if and only if $\alpha = 2\beta$ and $|B|$ is even. In fact, if one of these two conditions is violated, one can construct a configuration of red and blue points whose total weight cannot be bisected by a line. On the other hand, Sections 3, 4, and 5, focused on the essential case where $\alpha = 2$ and $\beta = 1$ and, further, aimed to partition the plane into weight-equitable convex regions. If the weight of the plane is $n\omega$, where ω is even then, according to Theorem 5, one can always construct n disjoint convex regions of weight ω using lines. If ω is odd, then a necessary and sufficient condition to carry out the subdivision is for the number of blue points in the plane to be at least n. The construction is carried out by induction. First, we trying to partition the plane into two weight-equitable half-planes by way of a 2-cutting. If such a 2-cutting does not occur then Theorem 14 guarantees that a weight-equitable 3-cutting of the plane exists.

A natural extension of subdivision problems in the plane is to check whether it also works for points in the plane lattice, which the authors have done quite recently [2]. One can also check what results would be obtained if the assigned weights of colored points are changed (e.g. $\alpha = 3$ and $\beta = 1$), or if it is possible to produce a weight-equitable subdivision of the plane containing more than two colors.

References

[1] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink, Generalizing ham sandwich cuts to equitable subdivisions, *Discrete and Computational Geometry*, **24** (2000), 605-622.

[2] J. Buot and M. Kano, Weight-Equitable Orthogonal Subdivision of Red and Blue Points in the Plane Lattice, submitted.

[3] H. Ito, H. Uehara, and M. Yokoyama, 2-dimensional ham-sandwich theorem for partitioning into three convex pieces, *Discrete Comput. Geom.* LNCS, **1763** (2000), 129-157.

[4] A. Kaneko and M. Kano, Balanced partitions of two sets of points in the plane, *Computational Geometry: Theory and Applications*, **13** (1999), 253-261.

[5] A. Kaneko and M. Kano, Discrete geometry on red and blue points in the plane – a survey, *Discrete and Computational Geometry Algorithms and Combininatorics*, **25** (2003), 551-570.

[6] M. Kano and M. Uno, General balanced subdivision of two sets of points in the plane, *Discrete Geometry, Combinatorics and Graph Theory* (Lecture Notes in Computer Science), **4381** (2007), 79-87.

[7] M. Kano and M. Uno, Balanced subdivisions with boundary condition of two sets of points in the plane, *International Journal of Computational Geometry & Applications*, **20** (2010), 527–541.

[8] T. Sakai, Balanced Convex Partitions of Measures in R^2, *Graphs and Combinatorics*, **18** (2002), 169–192.

[9] A.H. Stone and J.W. Tukey, Generalized "sandwich" theorems, Duke Math. J. 9 (1942) 356岾59