

Generalizations of Marriage Theorem for Degree Factors

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Abstract

Let G be a bipartite graph with bipartition (A, B) . We give new criteria for a bipartite graph to have an f -factor, a (g, f) -factor and other factors together with some applications of these criteria. These criteria can be considered as direct generalizations of Hall's marriage theorem. Among some results, we prove that for a function $h : A \cup B \rightarrow \{0, 1, 2, \dots\}$, G has a factor F such that $\deg_F(x) = h(x)$ for $x \in A$ and $\deg_H(y) \leq h(y)$ for $y \in B$ if and only if $h(X) \leq \sum_{x \in N_G(X)} \min\{h(x), e_G(x, X)\}$ for all $X \subseteq A$.

Keywords: Marriage theorem, degree factor, bipartite graph, f -factor, (g, f) -factor

1 Introduction

We mainly consider a finite graph which may have multiple edges but has no loops. Such a graph is called a *multigraph*. A graph having neither multiple edges nor loops is called a *simple graph*. For a graph G , let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G , respectively. If $V(G)$ is partitioned into two sets A and B so that every edge of G joins a vertex of

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A to a vertex of B , then G is called a *bipartite graph with bipartition* (A, B) . For a vertex v of G , the *degree* of v in G is denoted by $\deg_G(v)$, and the *neighborhood* of v is denoted by $N_G(v)$. For a vertex set S of G , let $N_G(S)$ denote $\cup_{x \in S} N_G(x)$. Thus if G is a simple graph, then $\deg_G(v) = |N_G(v)|$. For two disjoint subsets S and T of $V(G)$, we write $e_G(S, T)$ for the number of edges of G joining S to T . For a subgraph H of G , $\deg_H(v)$ and $e_H(S, T)$ are analogously defined.

Let $f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ be a function defined on $V(G)$. Then a spanning subgraph F of G is called an *f -factor* if $\deg_F(v) = f(v)$ for all $v \in V(G)$. On the other hand, for two functions $g, f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ such that $g(v) \leq f(v)$ for all $v \in V(G)$, a spanning subgraph H is called a *(g, f) -factor* if $g(v) \leq \deg_H(v) \leq f(v)$ for all $v \in V(G)$. For a function f defined on $V(G)$ and a subset $S \subseteq V(G)$, $f(S)$ denotes $\sum_{x \in S} f(x)$.

In this paper, we give new criteria for a bipartite graph with bipartition (A, B) to have an f -factor, a (g, f) -factor and other factors, in which one variable $X \subseteq A$ is used. In known criteria, usually two variables $X \subseteq A$ and $Y \subseteq B$ are used. Moreover, these new criteria can be considered as direct generalizations of Hall's marriage theorem since this is a direct corollary of the criteria. Then we obtain some results on factors of bipartite graphs by applying these criteria.

2 f -factors of bipartite graphs

A known criterion for a bipartite graph to have an f -factor is the following.

Theorem 1 (Ore [3], Folkman and Fulkerson [2], Theorem 3.38 of [1])

Let G be a bipartite multigraph with bipartition (A, B) , and let $f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ be a function such that $f(A) = f(B)$. Then G has an f -factor if and only if

$$\delta(T, S) = f(T) + \sum_{x \in S} (\deg_G(x) - f(x)) - e_G(T, S) \geq 0 \quad (1)$$

for all $S \subseteq A$ and $T \subseteq B$, where the function $\delta(T, S)$ is defined by (1).

It is known that two conditions $f(A) = f(B)$ and (1) implies $\delta(S, T) \geq 0$ (see Theorem 3.38 of [1]). Moreover, (1) implies $f(x) \leq \deg_G(x)$ for $x \in A$ since $\delta(\emptyset, x) = \deg_G(x) - f(x) \geq 0$. The next our theorem gives a new criterion for a bipartite graph to have an f -factor, which uses only one variable $X \subseteq A$. A similar result was obtained by Ore [4] though his proof is different from our proof.

Theorem 2 Let G be a bipartite multigraph with bipartition (A, B) , and let $f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ be a function such that $f(A) = f(B)$. Then G has an f -factor if and only if

$$f(X) \leq \sum_{y \in N_G(X)} \min\{f(y), e_G(y, X)\} \quad \text{for all } X \subseteq A. \quad (2)$$

Proof. First assume that G has an f -factor F . Then $f(A) = e_F(A, B) = f(B)$. For every subset $\emptyset \neq X \subseteq A$, it follows that

$$\begin{aligned} f(X) &= \sum_{x \in X} \deg_F(x) = e_F(X, N_F(X)) \leq \sum_{y \in N_G(X)} e_F(y, X) \\ &\leq \sum_{y \in N_G(X)} \min\{f(y), e_G(y, X)\}. \quad (\text{by } e_F(y, X) \leq f(y)) \end{aligned}$$

Hence the necessity is proved.

We next prove the sufficiency by showing (1). Assume that (2) holds. For every $x \in A$, it follows that $f(x) \leq \sum_{y \in N_G(x)} \min\{f(y), e_G(y, x)\} \leq \sum_{y \in N_G(x)} e_G(y, x) = \deg_G(x)$, and thus $f(x) \leq \deg_G(x)$. It is immediate that $\delta(T, \emptyset) = f(T) \geq 0$ and $\delta(\emptyset, S) = \sum_{x \in S} (\deg_G(x) - f(x)) \geq 0$ for all $T \subseteq B$ and $S \subseteq A$. Let $\emptyset \neq S \subseteq A$ and $\emptyset \neq T \subseteq B$, and let

$$\begin{aligned} T_1 &= \{y \in T : e_G(y, S) = 0\}, \quad \text{and} \\ T_2 &= \{y \in T : e_G(y, S) \geq 1\} = N_G(S) \cap T. \end{aligned}$$

Moreover, let

$$\begin{aligned} N_1 &= \{y \in N_G(S) - T_2 : e_G(y, S) \geq f(y)\}, \quad \text{and} \\ N_2 &= \{y \in N_G(S) - T_2 : e_G(y, S) < f(y)\}. \end{aligned}$$

Then $N_G(S) = T_2 \cup N_1 \cup N_2$ (disjoint union) and $\sum_{x \in S} \deg_G(x) = e_G(N_G(S), S)$. Also it is clear that $\delta(T_1, S) = f(T_1) + \sum_{x \in S} (\deg_G(x) - f(x)) \geq 0$. Thus $\delta(T, S) \geq \delta(T_2, S)$. Hence we have

$$\begin{aligned} \delta(T, S) &\geq \delta(T_2, S) \\ &= f(T_2) + \sum_{x \in S} (\deg_G(x) - f(x)) - e_G(T_2, S) \\ &\geq f(T_2) + f(N_1) - e_G(N_1, S) + \sum_{x \in S} \deg_G(x) - f(S) - e_G(T_2, S) \\ &= f(T_2 \cup N_1) + e_G(N_2, S) - f(S) \\ &\geq \sum_{y \in N_G(S)} \min\{f(y), e_G(y, S)\} - f(S) \geq 0. \quad (\text{by (2)}) \end{aligned}$$

Therefore G has an f -factor by Theorem 1. \square

Corollary 3 *Let G be a bipartite multigraph with bipartition (A, B) , and let s, t, m, n be positive integers such that $s \leq m$ and $t \leq n$. Assume that $\deg_G(x) = m$ for all $x \in A$ and $\deg_G(y) = n$ for all $y \in B$. If*

$$\frac{s}{m} = \frac{t}{n} \quad (3)$$

then G has a factor F such that $\deg_F(x) = s$ for all $x \in A$ and $\deg_F(y) = t$ for all $y \in B$.

Proof. Define a function f by $f(x) = s$ for all $x \in A$ and $f(y) = t$ for all $y \in B$. Since $m|A| = e_G(A, B) = n|B|$, it follows from (3) that

$$f(A) = s|A| = \frac{tm}{n} \times \frac{n|B|}{m} = t|B| = f(B).$$

Let $\emptyset \neq X \subseteq A$. Then $(t/n)e_G(y, X) \leq t$ and $(t/n)e_G(y, X) \leq e_G(y, X)$ for $y \in N_G(X)$, and thus

$$f(X) = s|X| = \frac{tm|X|}{n} = \frac{t}{n} e_G(N_G(X), X) \leq \sum_{y \in N_G(X)} \min\{t, e_G(y, X)\}.$$

Hence G has the desired factor by Theorem 2. \square

Notice that if $m = n$ and $s = t = 1$ in Corollary 3, then Corollary 3 means that every regular bipartite multigraph has a 1-factor, which is well-known, and hence Corollary 3 can be considered as a generalization of this result.

3 Generalizations of marriage theorem for degree factors

A criterion for a bipartite graph to have a (g, f) -factor is given in the following theorem, and its elementary proof can be found in [1].

Theorem 4 (Folkman and Fulkerson [2], Theorem 4.3 of [1]) *Let G be a bipartite multigraph with bipartition (A, B) , and let $g, f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ be functions such that $g(v) \leq f(v)$ for all vertices v of G . Then G has a (g, f) -factor if and only if*

$$\gamma(S, T) = f(S) + \sum_{x \in T} (\deg_G(x) - g(x)) - e_G(S, T) \geq 0 \quad (4)$$

and

$$\gamma(T, S) = f(T) + \sum_{x \in S} (\deg_G(x) - g(x)) - e_G(T, S) \geq 0 \quad (5)$$

for all $S \subseteq A$ and $T \subseteq B$, where the function γ is defined by (4) and (5).

The next our theorem gives a new criterion for a bipartite graph to have a (g, f) -factor, which has one variable X with $X \subseteq A$ or $X \subseteq B$.

Theorem 5 *Let G be a bipartite multigraph with bipartition (A, B) , and let $g, f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ be functions such that $g(v) \leq f(v)$ for all vertices v of G . Then G has a (g, f) -factor if and only if*

$$g(X) \leq \sum_{y \in N_G(X)} \min\{f(y), e_G(y, X)\} \quad (6)$$

for all $X \subseteq A$ and $X \subseteq B$.

Proof. First assume that G has a (g, f) -factor F . For any subset $\emptyset \neq X \subseteq A$ or $\emptyset \neq X \subseteq B$, it follows that

$$\begin{aligned} g(X) &\leq \sum_{x \in X} \deg_F(x) = e_F(X, N_F(X)) \leq \sum_{y \in N_G(X)} e_F(y, X) \\ &\leq \sum_{y \in N_G(X)} \min\{f(y), e_G(y, X)\}. \quad (\text{by } e_F(y, X) \leq f(y)) \end{aligned}$$

Hence the necessity is proved.

We next prove the sufficiency by using Theorem 4. We shall show that $\gamma(T, S) \geq 0$ under the assumption that (6) holds. First note that $g(v) \leq \deg_G(v)$ for all $v \in A \cup B$ since $g(v) \leq \sum_{y \in N_G(v)} \min\{f(y), e_G(y, v)\} \leq \sum_{y \in N_G(v)} e_G(y, v) = \deg_G(v)$ by (6). It is immediate that $\gamma(T, \emptyset) = f(T) \geq 0$ and $\gamma(\emptyset, S) = \sum_{x \in S} (\deg_G(x) - g(x)) \geq 0$ for all $T \subseteq B$ and $S \subseteq A$. Let $\emptyset \neq S \subseteq A$ and $\emptyset \neq T \subseteq B$, and let

$$\begin{aligned} T_1 &= \{y \in T : e_G(y, S) = 0\}, \quad \text{and} \\ T_2 &= \{y \in T : e_G(y, S) \geq 1\} = N_G(S) \cap T. \end{aligned}$$

Moreover, let

$$\begin{aligned} N_1 &= \{y \in N_G(S) - T_2 : e_G(y, S) \geq f(y)\}, \quad \text{and} \\ N_2 &= \{y \in N_G(S) - T_2 : e_G(y, S) < f(y)\}. \end{aligned}$$

Then $N_G(S) = T_2 \cup N_1 \cup N_2$ (disjoint union). Also it is clear that $\gamma(T_1, S) = f(T_1) + \sum_{x \in S} (\deg_G(x) - g(x)) \geq 0$. Thus $\gamma(T, S) \geq \gamma(T_2, S)$. Hence we have

$$\begin{aligned}
\gamma(T, S) &\geq \gamma(T_2, S) \\
&= f(T_2) + \sum_{x \in S} (\deg_G(x) - g(x)) - e_G(T_2, S) \\
&\geq f(T_2) + f(N_1) - e_G(N_1, S) + \sum_{x \in S} \deg_G(x) - g(S) - e_G(T_2, S) \\
&= f(T_2 \cup N_1) + e_G(N_2, S) - g(S) \\
&\geq \sum_{x \in N_G(S)} \min\{f(x), e_G(x, S)\} - g(S) \geq 0. \quad (\text{by (6)})
\end{aligned}$$

By symmetry, we can similarly show that $\gamma(S, T) \geq 0$. Hence by Theorem 4, G has the desired (g, f) -factor. \square

Theorem 6 *Let G be a bipartite multigraph with bipartition (A, B) , and let $h : A \cup B \rightarrow \{0, 1, 2, 3, \dots\}$ be a function. Then G has a factor F such that*

$$\begin{aligned}
\deg_F(x) &= h(x) \quad \text{for all } x \in A, \text{ and} \\
\deg_F(y) &\leq h(y) \quad \text{for all } y \in B
\end{aligned}$$

if and only if

$$h(X) \leq \sum_{y \in N_G(X)} \min\{h(y), e_G(y, X)\} \quad (7)$$

for all $X \subseteq A$.

Proof. Assume that G has a factor F that satisfies $\deg_F(x) = h(x)$ for all $x \in A$ and $\deg_F(y) \leq h(y)$ for all $y \in B$. Then for every $\emptyset \neq X \subseteq A$, it follows that

$$\begin{aligned}
h(X) &= \sum_{x \in X} \deg_F(x) = e_F(X, N_F(X)) \leq \sum_{y \in N_G(X)} e_F(y, X) \\
&\leq \sum_{y \in N_G(X)} \min\{f(y), e_G(y, X)\}. \quad (\text{by } e_F(y, X) \leq h(y))
\end{aligned}$$

Hence the necessity is proved.

We next prove the sufficiency using Theorem 5. We define two functions g and f on $A \cup B$ by using h as follows:

$$\begin{aligned}
g(x) &= h(x) \quad \text{for } x \in A, \quad \text{and} \quad g(y) = 0 \quad \text{for } y \in B, \\
f(x) &= h(x) \quad \text{for } x \in A, \quad \text{and} \quad f(y) = h(y) \quad \text{for } y \in B.
\end{aligned}$$

Then G has the desired factor if and only if G has a (g, f) -factor.

For every subset $\emptyset \neq X \subset A$, it follows from (7) that

$$g(X) \leq \sum_{y \in N_G(X)} \min\{f(y), e_G(y, X)\}.$$

For every subset $\emptyset \neq Y \subset B$, we have

$$g(Y) = 0 \leq \sum_{x \in N_G(Y)} \min\{f(x), e_G(x, Y)\}.$$

Hence by Theorem 5, G has a (g, f) -factor. \square

The following theorem with $h(x) = 1$ is called Hall's marriage theorem. We now show that the following theorem is a direct corollary of Theorem 6.

Theorem 7 (Generalized marriage theorem) *Let G be a bipartite simple graph with bipartition (A, B) , and let $h : A \rightarrow \{0, 1, 2, 3, \dots\}$ be a function such that $h(x) \leq \deg_G(x)$ for all $x \in A$. Then G has a factor H such that*

$$\begin{aligned} \deg_H(x) &= h(x) \quad \text{for all } x \in A, \text{ and} \\ \deg_H(y) &\in \{0, 1\} \quad \text{for all } y \in B \end{aligned}$$

if and only if

$$h(X) \leq |N_G(X)| \quad \text{for all } \emptyset \neq X \subseteq A. \quad (8)$$

Proof. We use Theorem 6. Extend h on B by $h(y) = 1$ for $y \in B$. Since G is a simple bipartite graph, it follows that

$$\sum_{y \in N_G(X)} \min\{h(y), e_G(y, X)\} = |N_G(X)|.$$

It is obvious that (7) follows from (8) and the above equality, and hence the theorem is proved. \square

By combining Theorems 5 and 6, we obtain the following corollary.

Corollary 8 *Let G be a bipartite multigraph with bipartition (A, B) , and let $g, f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ such that $g(v) \leq f(v)$ for all $v \in V(G)$. Then G has a (g, f) -factor if and only if G has two factors H and K that satisfy the following conditions.*

$$\begin{aligned} \deg_H(x) &= g(x) \quad \text{for all } x \in A, \text{ and} \quad \deg_H(y) \leq f(y) \quad \text{for all } y \in B, \\ \deg_K(x) &\leq f(x) \quad \text{for all } x \in A, \text{ and} \quad \deg_K(y) = g(y) \quad \text{for all } y \in B. \end{aligned}$$

Corollary 9 *Let G be a bipartite multigraph with bipartition (A, B) , and let s, t, m, n be positive integers such that $s \leq m$ and $t \leq n$. Assume that $\deg_G(x) = m$ for all $x \in A$ and $\deg_G(y) = n$ for all $y \in B$. If*

$$\frac{s}{m} \leq \frac{t}{n} \tag{9}$$

then G has a factor F such that $\deg_F(x) = s$ for all $x \in A$ and $\deg_F(y) \leq t$ for all $y \in B$.

Proof. Define a function h by $h(x) = s$ for $x \in A$ and $h(y) = t$ for $y \in B$. Let $\emptyset \neq X \subseteq A$. Since $(t/n)e_G(y, X) \leq t$ and $(t/n)e_G(y, X) \leq e_G(y, X)$ for $y \in N_G(X)$, we have

$$h(X) = s|X| \leq \frac{tm|X|}{n} = \frac{t}{n} e_G(N_G(X), X) \leq \sum_{x \in N_G(X)} \min\{t, e_G(y, X)\}.$$

Hence G has the desired factor by Theorem 6. \square

References

- [1] J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, LNM 2031, Springer (2011).
- [2] J. Folkman and D.R. Fulkerson, Flows in infinite graphs, *J. Combin. Theory* **8** (1970) 30–44.
- [3] O. Ore, Graphs and subgraphs, *Trans. Amer. Math. Soc.* **84** (1957) 109–136.
- [4] O. Ore, Studies on directed graphs I, *Annals of Mathematics* **63** (1956) 383–406.