0-Sum and 1-Sum Flows in Regular Graphs *†

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Abstract

Let G be a graph. Assume that l and k are two natural numbers. An l-sum flow on a graph G is an assignment of non-zero real numbers to the edges of G such that for every vertex v of G the sum of values of all edges incident with v equals l. An l-sum k-flow is an l-sum flow with values from the set $\{\pm 1, \ldots, \pm(k-1)\}$. Recently, it was proved that for every $r, r \geq 3$, $r \neq 5$, every r-regular graph admits a 0-sum 5-flow. In this paper we settle a conjecture by showing that every 5-regular graph admits a 0-sum 5-flow. Moreover, we prove that every r-regular graph of even order admits a 1-sum 5-flow.

1. Introduction

Throughout this paper a graph means a finite undirected graph without loop or multiple edges. Let G be a multigraph with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices and the number of edges of G are called the *order* and the *size* of G, respectively. A *k-regular* graph is a graph where each vertex is of degree k. The degree of vertex v in G is denoted by $d_G(v)$ and $N_G(v)$ denotes the set of all vertices adjacent to v. A graph G is called *k-edge connected* if

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the minimum number of edges whose removal would disconnect the graph is at least k. A *pendant edge* is an edge incident with a vertex of degree 1.

For a set $\{a_1, \ldots, a_r\}$ of non-negative integers an $\{a_1, \ldots, a_r\}$-*graph* is a graph each of whose vertices has degree from the set $\{a_1, \ldots, a_r\}$. For integers a and b, $1 \leq a \leq b$, an $[a, b]$-*graph* is defined to be a graph G such that for every $v \in V(G)$, $a \leq d_G(v) \leq b$. An $[a, b]$-*factor* is a spanning subgraph of G in which the degree of each vertex is in the interval $[a, b]$. When $a = b$, we call it an a-*factor.*

Assume that l and k are two natural numbers. An l-*sum flow* on a graph G is an assignment of non-zero real numbers to each edge of G such that for every vertex v in $V(G)$ the sum of values of all edges incident with v equals l and call it l-*sum rule.* An l-*sum* k-*flow* is an l-sum flow with values from the set $\{\pm 1, \ldots, \pm(k-1)\}$.

Let G be a graph. A k-*flow* of G is an assignment of integers with maximum value at most $k-1$ to each edge of G together with its orientation (or direction) such that for each vertex of G, the sum of the labels of incoming edges is equal to that of the labels of outgoing edges. A *nowhere-zero* k-*flow* is a k-flow with no zeros.

Tutte proposed the following interesting conjecture.

Conjecture A.[9](**Tutte's 5-flow Conjecture**) If G is 2-edge connected, then it has a nowhere-zero 5-flow.

In [2], it was proved that Tutte's 5-flow Conjecture is equivalent to show that every 2-edge connected bipartite graph admits a 0-sum 5-flow.

In 2009, an analagous version of Tutte's Conjecture proposed for undirected graphs.

Conjecture B.[2] **(0-Sum Conjecture (ZSC)).** *If a graph G admits a 0-sum flow, then G admits a 0-sum 6-flow.*

For r-regular graphs it was conjectured that 6 can be reduced to 5.

Conjecture C.[1] Every r-regular graphs ($r \geq 3$) admits a 0-sum 5-flow.

Conjecture C has been settled for cubic graphs in [2] and for every positive integer r, $r \neq 5$ in [3]. In [11], the authors proved that every r-regular graph ($r \geq 3$) admits a 0-sum 7-flow. Also in [10], for some r, k, l, the existence of l-sum k-flow for r-regular graphs has been studied.

In the present manuscript using strong tools in factorization of graphs, we show that Conjecture

C holds in general. Also, we prove that every r-regular graph of even order admits a 1-sum 5-flow.

1 0-sum 5-flow for 5-regular graphs

The main goal of this section is showing that Conjecture C is true.

We would like to prove the next result which settles Conjecture C.

Theorem 1. *Every 5-regular graph admits a 0-sum 5-flow.*

Proof. First let us state five lemmas.

Lemma 1.[5 and 6, p.91 and p.203] *Let G be an n-edge connected multigraph ($n \geq 1$), θ be a real number such that $0 < \theta < 1$ and $f : V(G) \to \{0, 1, 2, \ldots\}$. If (i), (ii) and one of $(iiia), (iiib)$ hold, then G has an f-factor.*

(i) $\sum_{x \in V(G)} f(x)$ *is even.*
(ii) $\sum_{x \in V(G)} |f(x) - \theta d_G(x)| < 2$.
(iiia) $n\theta \geq 1$ *and* $n(1 - \theta) \geq 1$.
(iiib) The set $\{f(x)\}$ *consists of even numbers and* $m(1-\theta) \geq 1$, *where* $m \in \{n, n+1\}$ *and* $m \equiv 1 \pmod 2$.

Now, we prove the following lemma.

Lemma 2. *Let G be a 2-edge connected $[2, 5]$-multigraph. If*

$$3|\{x \in V(G) : d_G(x) = 2\}| + 2|\{x \in V(G) : d_G(x) = 3\}| + |\{x \in V(G) : d_G(x) = 4\}| \leq 4,$$

then G has a 2-factor.

Proof. Define a function f on $V(G)$ as $f(x) = 2$, for all $x \in V(G)$, and let $\theta = \frac{2}{5}$. Then

$$\begin{aligned} & \sum_{x \in V(G)} |f(x) - \theta d_G(x)| \\ = \; & \frac{6}{5}|\{x : d_G(x) = 2\}| + \frac{4}{5}|\{x : d_G(x) = 3\}| + \frac{2}{5}|\{x : d_G(x) = 4\}| \\ \leq \; & \frac{8}{5} < 2. \end{aligned}$$

Hence Parts $(i), (ii)$ and $(iiib)$ of Lemma 1 are satisfied with $m = 3$, and thus G has a 2-factor. $\square$

In [2] the following result was proved.

Lemma 3. *If G is a connected $\{1,3\}$-graph and the subgraph of G induced by vertices of degree 3 is 2-edge connected, then there is a function f on $E(G)$ with $f(e) \in \{-2, 1, 4\}$ so that the 0-sum rule holds for each vertex of degree 3, and each pendant edge e has $f(e) \in \{-2, 4\}$. Moreover, one pendant edge e may have its value pre-assigned.*

The next lemma shows that Lemma 3 can be generalized to every $\{1,3\}$-graph.

Lemma 4. *Let G be a connected $\{1,3\}$-graph and let h be a pendant edge of G. For any arbitrary $\alpha \in \{-2, 4\}$, there exists a function $f : E(G) \to \{-2, 1, 4\}$ such that $f(h) = \alpha$ and 0-sum rule holds in each vertex of degree 3 and the value of any pendant edge is in the set $\{-2, 4\}$.*

Proof. Consider a rooted tree T obtained from G such that every maximal 2-edge connected subgraph of G is considered as a vertex of T and $E(T)$ consists of all cut edges of G, where the root is the maximal 2-edge connected subgraph one of whose vertices incident with the given pendant edge h, and a subgraph with one vertex is considered as a 2-edge connected subgraph. Now, we start by a root of T. If the root consists of one vertex, then we can easily assign the desired values to the three edges. So, we may assume that the maximal 2-edge connected subgraph, say H, of G corresponding to the root of T has order at least 2. Thus the subgraph of G obtained from H by adding all cut edges of G incident with H is a graph that satisfies the conditions of Lemma 3. Then apply Lemma 3 to obtain an edge assignment f for the root with values form $\{-2, 1, 4\}$ in which the pendant edges have even value and $f(h) = \alpha$. Consider a maximal 2-edge connected subgraph K of G corresponding to a child of the root of T and apply again Lemma 3, where the edge joining K to the root corresponds the given pendant edge in Lemma 3. By continuing this procedure we can find the desired function on the edge set of G. $\square$

Lemma 5. *If G is a connected $\{1,5\}$-graph, $\{e_1, \ldots, e_s\} \subseteq E(G)$ is the set of all pendant edges of G and $G - \{e_1, \ldots, e_s\}$ is 2-edge connected, then there is a function f on $E(G)$ with $f(e) \in \{\pm 1, \pm 2, \pm 3, \pm 4\}$ so that 0-sum rule holds for each vertex of degree 5 and for $i = 1, \ldots, s$, $f(e_i) \in \{-2, 2, 4\}$. Moreover, one pendant edge e_l may have value pre-assigned.*

Proof. Let $s = 5p + q$, where $p \geq 0$ and $0 \leq q \leq 4$ are integers. We divide $5p$ pendant edges of G into p groups each of which contains 5 edges, and identify the end points of every group to obtain the new p vertices of degree 5, called $v_1, \ldots, v_p$. Remove q remaining pendant edges from G and call the resultant multigraph by H. Since H is obtained from G by removing q pendant edges, H has the following property:

$$3|\{x : d_H(x) = 2\}| + 2|\{x : d_H(x) = 3\}| + |\{x : d_H(x) = 4\}| \leq 4.$$

Thus by Lemma 2, H has a 2-factor F. Now, we define a function $f : E(G) \to \{-2, 2, 3, 4\}$ for G so that the 0-sum rule holds for each vertex of degree 5. Assign value 3 to all edges of F, and assign value -2 to all remaining edges of H. Also assign value -2 to q removed pendant edges of G. Now, if a cycle C of F contains at least one vertex in $\{v_1, \ldots, v_p\}$, choose one vertex, say v_t, and change the values of edges of this cycle alternatively by 2 and 4 starting at an edge incident with v_t and ending at the other edge incident with v_t. Note that if a cycle C contains no vertex in $\{v_1, \ldots, v_p\}$, do not change the values of edges of C. Then we split $5p$ edges incident with $\{v_1, \ldots, v_p\}$ of H into $5p$ pendant edges of G. Thus we obtain a function f with the desired property.

For the last part of lemma we consider 3 cases:

(i) $f(e_\ell) = -2$. Consider the graph H as before. We know that H contains a 2-factor F. If e_ℓ is not contained in F, then the previous assignment works. If e_ℓ is contained in F and e_ℓ is incident with v_r, assign the value -3 to each edge of F, and assign 2 to all other edges. Then change the values of edges of the cycle C alternatively by -2 and -4, starting at e_ℓ. Note that we do the same procedure for every cycle of F containing a vertex in $\{v_1, \ldots, v_p\}$.

(ii) $f(e_\ell) = 2$. If e_ℓ is contained in a 2-factor F of H, then the previous assignment works. If e_ℓ is not contained in F, then assign -3 to all edges of F and assign 2 to all remaining edges, and do the same procedure for every cycle F containing at least one vertex in $\{v_1, \ldots, v_p\}$.

(iii) $f(e_\ell) = 4$. Consider the first assignment of edges of H. If e_ℓ is contained in F, then we are done. If e_ℓ is not contained in 2-factor F of H, then by removing all edges of F from H, we obtain a $[0, 3]$-graph which is not necessary connected. We have two possibilities: e_ℓ is an edge of $H - F$ or e_ℓ is not in H, i.e., e_ℓ is a removed pendant edge when H is obtained. In the first case suppose that v_t is a vertex of degree 3 in $H - F$ incident with e_ℓ. Now, for every $v_i \in \{v_1, \ldots, v_p\}$, we split 3 edges of $H - F$ incident with v_i to make 3 pendant edges. Add q removed pendant edges of G to $H - F$. Then the resultant graph is a $\{1, 3\}$-graph, say K, in which e_ℓ is a pendant edge.
By Lemma 4, we have a function $g : E(K) \to \{-2, 1, 4\}$ such that $g(e_\ell) = -2$ and the values of every pendant edge is in the set $\{-2, 4\}$ and moreover the 0-sum rule holds in each vertex of degree

3. Now, subtract 2 from all values of $E(K)$ and then multiply -1 to the values of all edges of K. Then assign -3 to all edges of F, and change the values of all edges of every cycle of F containing a vertex in $\{v_1, \ldots, v_p\}$ alternatively by -2 and -4. Clearly, the value of e_ℓ is 4 and 0-sum rule holds for each vertex of degree 5, as desired.

If e_ℓ is not in H, we add q removed pendant edges of G including e_ℓ to $H - F$ to obtain a $\{1, 3\}$-graph, say K. Now, a similar method given above completes the proof. $\square$

Now, we are in a position to prove Theorem 1.

If G is 2-edge connected, then by Lemma 2, G has a 2-factor F. Then assign value 3 to all the edges of F, and assign value -2 to all remaining edges of G, which is the desired 0-sum 5-flow. Hence we may assume that G is not 2-edge connected. Consider a rooted tree T obtained from G such that every maximal 2-edge connected subgraph of G is considered as a vertex of T and $E(T)$ consists of all cut edges of G, where a subgraph consisting of one vertex is considered as a 2-edge connected subgraph. Now, we start by a root of T whose induced subgraph on the vertices of degree 5 is 2-edge connected. Let H be the maximal 2-edge connected subgraph of G corresponding to the root of T. Apply Lemma 5 to the subgraph of G obtained from H by adding all the cut edges of G incident with H to obtain an edge assignment for the root with values from the set $\{\pm1, \pm2, \pm3, \pm4\}$ in which every pendant edge has a value from the set $\{-2, 2, 4\}$. Consider a maximal 2-edge connected subgraph K of G corresponding to a child of the root of T and apply again Lemma 5 to the subgraph of G obtained from K by adding all cut edges of G incident with K to obtain an assignment on the root and K. By continuing this procedure we find a 0-sum 5-flow for G and the proof is complete.

2 1-sum flows in regular graphs

As we mentioned that before every r-regular graph $r \geq 3$, admits a 0-sum 5-flow. In this section we prove that every r-regular graph of even order $r \geq 3$, admits a 1-sum 5-flow. Before establishing our results we need some theorems.

Remark 1. We note that if a graph G admits a 1-sum k-flow, then G has even order. To see this assume that f is a 1-sum k-flow for G. We have

$$|V(G)| = \sum_{v \in V(G)} \sum_{u \in N_G(v)} f(uv) = 2 \sum_{e \in E(G)} f(e).$$

Thus $|V(G)|$ should be even.

In the sequel we need the following result.

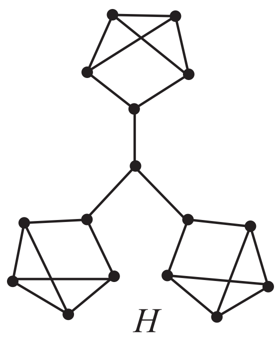
Theorem 2.[6 and 7, p. 184-190] *Let $r \geq 3$ be an odd integer and let k be an integer such that $1 \leq k \leq \frac{2r}{3}$. Then every r-regular graph has a $[k-1, k]$-factor each component of which is regular.*

Also, we need the following theorem due to Petersen.

Theorem 3.[8] *Every $2k$-regular multigraph admits a 2-factorization.*

The following remark shows that there are some regular graphs with no 1-sum 3-flow.

Remark 2. It is not hard to see that following 3-regular graph does not admit a 1-sum 3-flow.



H

Now, we are ready to show that every r-regular graph of even order admits a 1-sum 5-flow.

Theorem 4. *Let G be an r-regular connected graph of even order. Then the following hold:*

(i) If r is an odd integer or $r = 4k+2$, for some integer $k \geq 0$, then G admits a 1-sum 4-flow.

(ii) If $r = 4k$, for some integer $k \geq 1$, then G admits a 1-sum 5-flow.

Proof. Assume that $V(G) = \{1, \ldots, n\}$. First suppose that r is an odd integer. We define a bipartite graph from G, called B, with two parts $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ and $x_i y_j \in E(B)$ if and only if $ij \in E(G)$ for every i and j, $1 \leq i, j \leq n$. So, B is an r-regular graph and by Theorem [4, p.79], B has a 1-factorization $F_1, \ldots, F_r$. Now, for every $e \in E(F_i)$, $1 \leq i \leq r$, define a function $g : E(B) \to \{\pm\frac{1}{2}, \pm\frac{3}{2}\}$ as follows.

For $r = 4k+1$ define:

$$g(e) = \begin{cases} \frac{-3}{2}, & 1 \leq i \leq k; \\ \frac{1}{2}, & k < i \leq r. \end{cases}$$

Also, for $r = 4k+3$ define:

$$g(e) = \begin{cases} \frac{3}{2}, & 1 \leq i \leq k+1; \\ \frac{-1}{2}, & k+1 < i \leq r. \end{cases}$$

Clearly, for each $u \in V(B)$, $\sum_{v \in N_B(u)} g(uv) = \frac{1}{2}$. Now, define a function $f : E(G) \to \{\pm 1, \pm 3\}$

such that for every $ij \in E(G)$, $f(ij) = g(x_i y_j) + g(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, suppose that r is an even integer. If G is a 2-regular graph, then by assigning the integers $-1, 2$ to the edges of G alternatively, we are done.

Let $r = 4k$. Double all edges of G to obtain an $8k$-regular multigraph G'. Since G' contains two edge disjoint spanning subgraphs H_1 and H_2 isomorphic to G and H_1 is decomposed into 2-factors $F_1, \ldots, F_{2k}$, we can obtain a $(4k+2)$-regular multigraph $G'' = G' \setminus E(F_1) \cup \cdots \cup E(F_{2k-1})$, which contains a $4k$-regular graph H_2. Since G'' is 2-edge connected, then by Parts (i), (ii) and $(iiia)$ of Lemma 1, if we define $f(i) = 2k+1$, for all $i \in V(G'')$ and $\theta = \frac{1}{2}$, G'' is decomposed into two $(2k+1)$-factors G''_1 and G''_2. Now, for every $e \in E(G')$, we define a function $g : E(G') \rightarrow \{-2, 1, 3\}$ as follows:

$$g(e) = \begin{cases} -2, & e \in E(F_1) \cup \cdots \cup E(F_{k-2}) \cup E(G''_1); \\ 1, & e \in E(F_{k-1}) \cup E(F_k) \cup E(F_{k+1}) \cup E(G''_2); \\ 3, & e \in E(F_{k+2}) \cup \cdots \cup E(F_{2k-1}). \end{cases}$$

Clearly, for each $i \in V(G')$, $\sum_{j \in N_{G'}(i)} g(ij) = 1$. Now, define a function $f : E(G) \rightarrow \{-4, -1, 1, 2, 4\}$ such that for every $e \in E(G)$, $f(e) = g(e) + g(e')$, where e' is the copy of e in duplicating of this edge in G'. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, assume that $r = 4k+2$ and $r \neq 6, 10, 14, 22$. First note that every integer of the form $4k+2$ can be written as $12k+2$, $12k+6$ or $12k+10$, for some integer $k \geq 0$.

Let $r = 12k+2$. Since G is 2-edge connected, then by Parts (i), (ii) and $(iiia)$ of Lemma 1, if we define $f(i) = 6k+1$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(6k+1)$-factors H_1 and H_2. On the other hand, by Theorem 2, H_2 has a $[4k-1, 4k]$-factor, say T whose components are regular. Let T_1 be the union of the $(4k-1)$-regular components of T and let T_2 be the union of $4k$-regular components of T. Note that by Theorem 3, T_2 has a 2-factorization with 2-factors $F_1, \ldots, F_{2k}$. Now, we define a function $g : E(G) \setminus E(T_1) \rightarrow \{-3, -2, -1, 2\}$ as follows:

$$g(e) = \begin{cases} -3, & e \in E(H_2) \setminus E(T); \\ -2, & e \in E(F_i),\ 1 \leq i \leq k-1\ ; \\ -1, & e \in E(F_i),\ k \leq i \leq 2k\ ; \\ 2, & e \in E(H_1). \end{cases}$$

Now, we want to assign some labels to the edges of T_1. With no loss of generality one can assume that $V(T_1) = \{1, \ldots, q\}$. We define a bipartite graph, call L, with two parts $X = \{x_1, \ldots, x_q\}$ and $Y = \{y_1, \ldots, y_q\}$ and $x_i y_j \in E(L)$ if and only if $ij \in E(T_1)$ for every i and j, $1 \le i, j \le q$. So, L is a $(4k-1)$-regular graph and by Theorem [4, p.79], L has a 1-factorization $F'_1, \ldots, F'_{4k-1}$. Now, for every $e \in E(F'_i)$, $1 \le i \le 4k-1$, define a function $g' : E(L) \to \{-\frac{1}{2}, -\frac{3}{2}\}$ as follows:

$$g'(e) = \begin{cases} -\frac{3}{2}, & 1 \le i \le k-2; \\ -\frac{1}{2}, & k-1 \le i \le 4k-1. \end{cases}$$

Clearly, for each $i \in V(L)$, $\sum_{j \in N_L(i)} g'(ij) = \frac{-6k+5}{2}$. Now, define a function $f : E(G) \to \{-3, -2, -1, 2\}$ such that for every $e \in E(G) \setminus E(T_1)$, $f(e) = g(e)$ and for every $e = ij \in E(T_1)$, $f(e) = g'(x_i y_j) + g'(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, suppose that $r = 12k+6$. Since G is 2-edge connected, then by Parts (i), (ii) and $(iiia)$ of Lemma 1, if we define $f(i) = 6k+3$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(6k+3)$-factors H_1 and H_2. On the other hand, by Theorem 2, H_2 has a $[4k+1, 4k+2]$-factor, say T whose components are regular. Let T_1 be the union of the $(4k+1)$-regular components of T and let T_2 be the union of $(4k+2)$-regular components of T. Note that by Theorem 3, T_2 has a 2-factorization with 2-factors $F_1, \ldots, F_{2k+1}$. Now, we define a function $g : E(G) \setminus E(T_1) \to \{-3, -2, -1, 2\}$ as follows:

$$g(e) = \begin{cases} -3, & e \in E(H_2) \setminus E(T); \\ -2, & e \in E(F_i),\ 1 \le i \le k\ ; \\ -1, & e \in E(F_i),\ k+1 \le i \le 2k+1\ ; \\ 2, & e \in E(H_1). \end{cases}$$

Now, we want to assign some labels to the edges of T_1. With no loss of generality one can assume that $V(T_1) = \{1, \ldots, q\}$. We define a bipartite graph, call L, with two parts $X = \{x_1, \ldots, x_q\}$ and $Y = \{y_1, \ldots, y_q\}$ and $x_i y_j \in E(L)$ if and only if $ij \in E(T_1)$ for every i and j, $1 \le i, j \le q$. So, L is a $(4k+1)$-regular graph and by Theorem [4, p.79], L has a 1-factorization $F'_1, \ldots, F'_{4k+1}$. Now, for every $e \in E(F'_i)$, $1 \le i \le 4k+1$, define a function $g' : E(L) \to \{-\frac{1}{2}, -\frac{3}{2}\}$ as follows:

$$g'(e) = \begin{cases} -\frac{3}{2}, & 1 \le i \le k-1; \\ -\frac{1}{2}, & k \le i \le 4k+1. \end{cases}$$

Clearly, for each $v \in V(L)$, $\sum_{v \in N_L(u)} g'(uv) = \frac{-6k+1}{2}$. Now, define a function $f : E(G) \to$

$\{-3,-2,-1,2\}$ such that for every $e \in E(G) \setminus E(T_1)$, $f(e) = g(e)$ and for every $e = ij \in E(T_1)$, $f(e) = g'(x_i y_j) + g'(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Next, assume that $r = 12k + 10$. Since G is 2-edge connected, then by Parts (i), (ii) and $(iiia)$ of Lemma 1, if we define $f(i) = 6k + 5$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(6k+5)$-factors H_1 and H_2. On the other hand, by Theorem 2, H_2 has a $[4k+1, 4k+2]$-factor, say T whose components are regular. Let T_1 be the union of the $(4k+1)$-regular components of T and let T_2 be the union of $(4k+2)$-regular components of T. Note that by Theorem 3, T_2 has a 2-factorization with 2-factors $F_1, \ldots, F_{2k+1}$. Now, we define a function $g : E(G) \setminus E(T_1) \to \{-3,-2,-1,2\}$ as follows:

$$g(e) = \begin{cases} -3, & e \in E(H_2) \setminus E(T); \\ -2, & e \in E(F_i),\ 1 \leq i \leq k-1\ ; \\ -1, & e \in E(F_i),\ k \leq i \leq 2k+1\ ; \\ 2, & e \in E(H_1). \end{cases}$$

Now, we want to assign some labels to the edges of T_1. With no loss of generality one can assume that $V(T_1) = \{1, \ldots, q\}$. We define a bipartite graph, call L, with two parts $X = \{x_1, \ldots, x_q\}$ and $Y = \{y_1, \ldots, y_q\}$ and $x_i y_j \in E(L)$ if and only if $ij \in E(T_1)$ for every i and j, $1 \leq i, j \leq q$. So, L is a $(4k+1)$-regular graph and by Theorem [4, p.79], L has a 1-factorization $F'_1, \ldots, F'_{4k+1}$. Now, for every $e \in E(F'_i)$, $1 \leq i \leq 4k+1$, define a function $g' : E(L) \to \{-\frac{1}{2}, -\frac{3}{2}\}$ as follows:

$$g'(e) = \begin{cases} -\frac{3}{2}, & 1 \leq i \leq k-2; \\ -\frac{1}{2}, & k-1 \leq i \leq 4k+1. \end{cases}$$

Clearly, for each $v \in V(L)$, $\sum_{v \in N_L(u)} g'(uv) = \frac{-6k+3}{2}$. Now, define a function $f : E(G) \to \{-3,-2,-1,2\}$ such that for every $e \in E(G) \setminus E(T_1)$, $f(e) = g(e)$ and for every $e = ij \in E(T_1)$, $f(e) = g'(x_i y_j) + g'(x_j y_i)$. Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired.

Now, suppose that G is an r-regular graph such that $r \in \{6, 10, 14, 22\}$ and $r = 4k+2$. Since G is 2-edge connected then by Parts (i),(ii) and $(iiia)$ of Lemma 1, if we define $f(i) = 2k+1$, for all $i \in V(G)$ and $\theta = \frac{1}{2}$, then G has two $(2k+1)$-factors G_1 and G_2. Then by Theorem 2, G_2 has a $[t-1, t]$-factor T, for every t, $1 \leq t \leq \frac{2r}{3}$, whose components are regular. Let T_1 be the union of the $(t-1)$-regular components of T and let T_2 be the union of t-regular components of T.

If $r = 6$, then G_2 has a $[1,2]$-factor. Define a function $f : E(G) \to \{-2, 1, 2, 3\}$, where $f(e) = -2$ for $e \in E(G_1)$, $f(e) = 3$ for $e \in E(G_2) \setminus E(T)$, $f(e) = 1$ for $e \in E(T_1)$ and $f(e) = 2$ for $e \in E(T_2)$.

If $r = 10$, then G_2 has a $[1, 2]$-factor. Define a function $f : E(G) \to \{-2, -1, 1, 3\}$, where $f(e) = -2$ for $e \in E(G_1)$, $f(e) = 3$ for $e \in E(G_2) \setminus E(T)$, $f(e) = -1$ for $e \in E(T_1)$ and $f(e) = 1$ for $e \in E(T_2)$.

If $r = 14$, then G_2 has a $[3, 4]$-factor. Note that by Theorem 3, T_2 has two 2-factors, say T'_1 and T'_2. Now, define a function $f : E(G) \to \{-3, -1, -2, 2\}$, where $f(e) = 2$ for $e \in E(G_1)$, $f(e) = -1$ for $e \in E(G_2) \setminus E(T)$, $f(e) = -3$ for $e \in E(T_1)$, $f(e) = -2$ for $e \in E(T'_1)$ and $f(e) = -3$ for $e \in E(T'_2)$.

If $r = 22$, then G_2 has a $[2, 3]$-factor. Define a function $f : E(G) \to \{-3, 1, 2, 3\}$, where $f(e) = 2$ for $e \in E(G_1)$, $f(e) = -3$ for $e \in E(G_2) \setminus E(T)$, $f(e) = 3$ for $e \in E(T_1)$ and $f(e) = 1$ for $e \in E(T_2)$.

Then for every $i \in V(G)$, $\sum_{j \in N_G(i)} f(ij) = 1$, as desired. □

We close the paper with the following conjecture.

Conjecture 1. *Every connected $4k$-regular graph of even order admits a 1-sum 4-flow.*

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