Spanning trees with bounded degrees and leaves

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Abstract

Rivera-Campo provided a degree sum condition for a graph to have a spanning tree with bounded degrees and leaves. In this paper, we give an independence number condition for a graph to have a spanning tree with bounded degrees and leaves, which also partially solves the conjecture made by Enomoto and Ozeki (*J. Graph Theory*, 65, (2010) 173-184).

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the *order* of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v and by $N_G(v)$ the neighborhood of v. Thus $\deg_G(v) = |N_G(v)|$. An edge joining two vertices u and v is denoted by uv or vu. The *independence number* and the *connectivity*

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of G are denoted by $\alpha(G)$ and $\kappa(G)$, respectively. Let T be a tree. A vertex of T with degree one is often called a *leaf*, and the set of leaves of T is denoted by $Leaf(T)$. For a set X, the cardinality of X is denoted by $|X|$ or $\#X$.

Chvátal and Erdös [2] showed that if $\alpha(G) \leq \kappa(G) + 1$, then G has a hamiltonian path. This result was generalized to a spanning k-tree as follows, where a *k-tree* is a tree with maximum degree at most k.

Theorem 1 (Neumann-Lara and Rivera-Campo [6]). *Let $k \geq 2$ be an integer and G be a connected graph. If $\alpha(G) \leq (k-1)\kappa(G)+1$, then G has a spanning k-tree.*

On the other hand, a hamiltonian path is a spanning tree having exactly two leaves. From this point of view, Win provided a sufficient condition for a graph to have a spanning tree having a small number of leaves.

Theorem 2 (Win [9]). *Let $k \geq 2$ be an integer and G be a connected graph. If $\alpha(G) \leq \kappa(G) + k - 1$, then G has a spanning tree having at most k leaves.*

Recently, Rivera-Campo obtained a degree sum condition for a graph to have a spanning tree with bounded degree as well as with a small number of leaves.

Theorem 3 (Rivera-Campo [7]). *Let p, n and $d_1, d_2, \ldots, d_p$ be integers such that $1 \leq n \leq p-1$ and $2 \leq d_1 \leq d_2 \leq \cdots \leq d_p \leq p-1$. Let G be an n-connected graph of order p with vertex set $V(G) = \{v_1, v_2, \ldots, v_p\}$. If*

$$\deg_G(x) + \deg_G(y) \geq p - 1 - \sum_{j=1}^{n}(d_j - 2)$$

for any non-adjacent vertices x and y of G, then G has a spanning tree T that has at most $\sum_{j=1}^{n}(d_j - 2) + 2$ leaves and satisfies $\deg_T(v_i) \leq d_i$ for all $i = 1, 2, \ldots, p$.

For a function $f : V(G) \to \{1, 2, 3, \ldots\}$, a spanning tree T of a graph G is called a *spanning f-tree* if $\deg_T(v) \leq f(v)$ for all vertices v of G. Here, we give a sufficient condition using independence number for a graph to have a spanning f-tree with a small number of leaves. The following is our result.

Theorem 4. *Let $n \geq 1$ be an integer. Let G be an n-connected graph and $f : V(G) \to \{2, 3, 4, \ldots\}$ be a function. If*

$$\alpha(G) \leq \min_X \{\sum_{x \in X}(f(x) - 1) : X \subseteq V(G) \text{ and } |X| = n\} + 1, \qquad (1)$$

then G *has a spanning* f*-tree that has at most* $\min_X\{\sum_{x\in X}(f(x)-2) : X \subseteq V(G)$ *and* $|X|=n\}+2$ *leaves.*

We now give some other results and a conjecture related to our theorem, and explain the relation between our theorem and them. The following theorem gives a sufficient condition for a graph to have a spanning tree that contains specified vertices as leaves.

Theorem 5 (Matsuda and Matsumura [5]). *Let* n, k *and* s *be integers such that* $k \geq 2$, $0 \leq s \leq k$ *and* $s \leq n-1$, *and let let* G *be an* n*-connected graph. If* $\alpha(G) \leq (n-s)(k-1)+1$, *then for any* s *vertices of* G, G *has a spanning* k*-tree that includes the* s *specified vertices as leaves.*

Enomoto and Ozeki made the following conjecture on a spanning f-tree from the above theorem, and partially solved it (Theorem 7).

Conjecture 6 (Enomoto and Ozeki [3]). *Let* $n \geq 1$ *be an integer,* G *be an* n*-connected graph and* $f : V(G) \to \{1,2,3,\ldots\}$ *be a function. If* $\sum_{x\in V(G)} f(x) \geq 2(|G|-1)$ *and*

$$\alpha(G) \leq \min_X\{\sum_{x\in X}(f(x)-1) : X \subseteq V(G) \text{ and } |X|=n\} \ +1,$$

then G *has a spanning* f*-tree.*

Theorem 7 (Enomoto and Ozeki [3]). *Let* $n \geq 1$ *be an integer,* G *be an* n*-connected graph and and* $f : V(G) \to \{1,2,3,\ldots\}$ *be a function. If* $\#\{v \in V(G) : f(v) = 1 \text{ or } 2\} \leq n+1$, $\sum_{x\in V(G)} f(x) \geq 2(|G|-1)$ *and*

$$\alpha(G) \leq \min_X\{\sum_{x\in X}(f(x)-1) : X \subseteq V(G) \text{ and } |X|=n\} \ +1,$$

then G *has a spanning* f*-tree.*

Note that Conjecture 6 is a generalization of Theorem 5 since by setting $f(u) = 1$ for s specified verteices u and $f(v) = k$ for the other vertices v, Conjecture 6 implies Theorem 5. Moreover, another conjecture related to Conjecture 6 was proposed by Ozeki and Yamashita [8] using a new notation $Cut(G, f)$ instead of the connectivity. Our Theorem 4 solves the conjecture under the assumption that $f(v) \geq 2$ for all vertices v, and the proof techniques in this paper are different from those in Enomoto and Ozeki [3].

Some other results on spanning trees related to our theorem are given in [1], and many current results on spanning trees can be found in [8].

2 Proof of Theorem 4

In order to prove Theorem 4, we need the following Lemmas.

Lemma 1 (Kouider [4]). *Let H be a subgraph of an n-connected graph G, where $n \geq 2$. Then either $V(H)$ is covered by a cycle of G, or there is a cycle C in G such that $\alpha(H - V(C)) \leq \alpha(H) - n$. In particular, if $\alpha(H) \leq n$, then $V(H)$ is covered by a cycle of G.*

By Lemma 1, we can easily obtained following lemma.

Lemma 2. *Let H be a subgraph of an n-connected graph G, where $n \geq 1$. Then either $V(H)$ is covered by a path of G, or there is a path P in G such that $\alpha(H - V(P)) \leq \alpha(H) - (n+1)$.*

Proof. Let w be a new vertex not contained in G, and let $G + w$ denote the graph obtained from G by joining w to every vertex of G. Then $G + w$ is $(n+1)$-connected and H is its subgraph. By Lemma 1, H is covered by a cycle D of $G+w$ or there exists a cycle C in $G+w$ such that $\alpha(H-V(C)) \leq \alpha(H) - (n+1)$.

If D passes through w, then H is covered by the path $D - w$ of G; otherwise, H is covered by a path obtained from D by removing an edge. If C passes through w, then the path $C - w$ of G satisfies the condition since $H - V(C - w) = H - V(C)$; otherwise, the path $C - e$ obtained from C by removing an edge e of C satisfies the condition since $H - V(C - e) = H - V(C)$. Hence the lemma holds. $\square$

The next lemma is well-known.

Lemma 3. *Let T be a tree and $W = \{v \in V(T) : \deg_T(v) \geq 2\}$. Then the number of leaves of T is $\sum_{x \in W}(\deg_T(x) - 2) + 2$.*

We are ready to prove Theorem 4.

Proof of Theorem 4. If G has a hamiltonian path, the theorem holds. So we may assume that G does not have a hamiltonian path. By applying Lemma 2 with $H = G$, we can take a path P in G such that

(P1) $\alpha(G - V(P)) \leq \alpha(G) - (n+1)$.

(P2) P is as long as possible subject to (P1).

By the assumption, P is not a hamiltonian path of G. Let x_1 and x_2 be the endvertices of P. Then, by the choice (P2), we obtain the following claim.

Claim 1. x_1 and x_2 are not adjacent in G and $N_G(x_1) \cup N_G(x_2) \subseteq V(P)$.

Next, we choose a spanning tree T of G so that

(T1) T includes P.
(T2) $\sum_{x \in V_T^*} \deg_T(x)$ is as small as possible subject to (T1), where

$$V_T^* = \{x \in V(G) : \deg_T(x) > f(x)\}.$$

(T3) $|Leaf(T)|$ is as small as possible subject to (T2).

By the choice (T1), (T3) and Claim 1, we can obtain the following claim.

Claim 2. x_1 and x_2 are leaves of T, and $Leaf(T)$ is an independent set of G.

Claim 3. $|Leaf(T)| \leq \min_X \{\sum_{x \in X} (f(x) - 2) : X \subseteq V(G),\ |X| = n\} \ + 2.$

Since $Leaf(T) - \{x_1, x_2\} \subseteq V(G) - V(P)$, by Claim 2, (P1) and by (1), we have

$$\begin{aligned} |Leaf(T)| - 2 &\leq \alpha(G - V(P)) \\ &\leq \alpha(G) - (n+1) \\ &\leq \min_X \{\sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n\}. \end{aligned}$$

Hence, $|Leaf(T)| \leq \min_X \{\sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n\} + 2.$

Claim 4. $V_T^* = \emptyset$.

Assume that there exists a vertex $w \in V(G)$ such that $\deg_T(w) > f(w) \geq 2$. We regard T as a rooted tree with root x_1. Let u be a child of w in T such that $u \notin V(P)$. Since G is a n-connected graph, there exist n internal disjoint paths $P_1, P_2, \ldots, P_n$ connecting x_1 and u in G. By Claim 1, x_1 and u are not adjacent in G, and so $V(P_i) - \{x_1, u\} \neq \emptyset$ for every $1 \leq i \leq n$.

Assume that for every path P_i, $1 \leq i \leq n$, there exists a vertex $w_i \in V(P_i) - \{x_1, u\}$ such that $\deg_T(w_i) \geq f(w_i)$. Let $W = \{w_1, w_2, \ldots, w_n\} \cup \{w\}$. Then, by Lemma 3, we have

$$\begin{aligned} |Leaf(T)| &\geq \sum_{x \in W} (\deg_T(x) - 2) + 2 \geq \sum_{x \in W - \{w\}} (f(x) - 2) + 3 \\ &\geq \min_X \{\sum_{x \in X} (f(x) - 2) : X \subseteq V(G) \text{ and } |X| = n\} \ + 3. \end{aligned}$$

This contradicts Claim 3. Therefore there exists a path $P^* \in \{P_1, P_2, \dots, P_n\}$ such that $\deg_T(x) < f(x)$ for all $x \in V(P^*) - \{x_1, u\}$. Note that $\deg_T(x_1) = 1 < f(x_1)$.

Since x_1 and u are contained in the two distinct components of $T - wu$ and P^* connects x_1 and u, P^* contains an edge $e = u_1u_2$ such that u_1 and u_2 are contained in the two distinct components of $T - wu$. Obviously, $e \notin E(T)$. Let $T_1 = T - wu + e$. Then T_1 is a spanning tree of G that includes P and satisfies

$$\sum_{x \in V^*_{T_1}} \deg_{T_1}(x) < \sum_{x \in V^*_T} \deg_T(x).$$

This contradicts the choice (T2). Hence, Claim 4 holds.

Consequently, the theorem follows from Claim 3 and Claim 4. □

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