m-dominating k-trees of graphs

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Abstract

Let $k \geq 2$, $l \geq 2$, $m \geq 0$ and $n \geq 1$ be integers, and let G be a connected graph. If there exists a subgraph H of G such that for every vertex v of G, the distance between v and H is at most m, then we say that H m-dominates G. A tree whose maximum degree is at most k is called a k-tree. Define $\alpha^l(G) = \max\{\ |S| : S \subseteq V(G),$ $d_G(x,y) \geq l$ for all distinct $x, y \in S\}$, where $d_G(x,y)$ denotes the distance between x and y in G. We prove the following theorem and

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show that the condition is sharp. If an n-connected graph G satisfies $\alpha^{2(m+1)}(G) \leq (k-1)n+1$, then G has a k-tree that m-dominates G. This theorem is a generalization of both a theorem of Neumann-Lara and Rivera-Campo on a spanning k-tree in an n-connected graph and a theorem of Broersma on an m-dominating path in an n-connected graph.

Keywords: k-tree, dominating tree, n-connected graph

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For two vertices u and v of G, let $d_G(u,v)$ denote the *distance* between u and v in G, which is the length of a shortest path of G connecting u and v. For a subgraph X or a vertex set X of G and a vertex v of G, the *distance* between v and X is defined to be the minimum value of $d_G(v,x)$ for all $x \in V(X)$ or $x \in X$, and denoted by $d_G(v,X)$. Thus $d_G(v,X) = 0$ if and only if v is contained in X.

Let $m \geq 0$ be an integer and X be a subgraph or a vertex set of G. Then the *m-th dominating set* of X, denoted by $Domi^m(X)$, is defined to be the following vertex set of G.

$$Domi^m(X) = \{\, v \in V(G) \ : \ d_G(v,X) \leq m \,\}.$$

If all the vertices of a subgraph Y or a vertex set Y of G are included in $Domi^m(X)$, then we say that X *m-dominates* Y. Thus a subgraph H of G 0-dominates G if and only if H is a spanning subgraph of G.

For an integer $l \geq 2$, the invariant $\alpha^l(G)$ of a graph G is defined as follows:

$$\alpha^l(G) = \max\{\, |S| : S \subseteq V(G),\ d_G(x,y) \geq l \text{ for all distinct } x,y \in S\}.$$

Thus the *independence number* $\alpha(G)$ of G is equal to $\alpha^2(G)$. A tree whose maximum degree is at most k is called a *k-tree*. So a hamiltonian path is a spanning 2-tree. The following theorem is well known.

Theorem 1 (Chvátal and Erdős [3]) *Let $n \geq 1$ be an integer, and let G be an n-connected graph. If $\alpha(G) \leq n+1$, then G has a hamiltonian path.*

The following theorem shows a k-tree version of Theorem 1.

Theorem 2 (Neumann-Lara and Rivera-Campo [5]) *Let $k \geq 2$ and $n \geq 1$ be integers, and let G be an n-connected graph. If $\alpha(G) \leq (k-1)n+1$, then G has a spanning k-tree.*

On the other hand, Broersma obtained the following result which is another generalization of Theorem 1.

Theorem 3 (Broersma [2]) *Let $m \geq 0$ and $n \geq 1$ be integers, and let G be an n-connected graph. If $\alpha^{2(m+1)}(G) \leq n+1$, then G has a path that m-dominates G.*

In this paper, we prove the following theorem, which is a generalization of both Theorems 2 and 3.

Theorem 4 *Let $k \geq 2$, $m \geq 0$ and $n \geq 1$ be integers, and let G be an n-connected graph. If $\alpha^{2(m+1)}(G) \leq (k-1)n+1$, then G has a k-tree that m-dominates G.*

We first show that the condition of Theorem 4 is sharp in the sense that there is a family of graphs G which satisfies $\alpha^{2(m+1)}(G) = (k-1)n+2$ but has no k-tree that m-dominates G. We construct such a graph G as follows (see Figure 1). Let $k \geq 2$, $m \geq 1$ and $n \geq 1$ be integers. Let $D_{i,1}, D_{i,2}, \ldots, D_{i,m}$ be disjoint copies of the complete graph of order n, where $1 \leq i \leq (k-1)n+2$. For each $1 \leq i \leq (k-1)n+2$ and $1 \leq j \leq m-1$, join all the vertices of $D_{i,j}$ to all the vertices of $D_{i,j+1}$ by edges. For each $1 \leq i \leq (k-1)n+2$, let v_i be a new vertex not contained in $D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,m}$, and join v_i to all the vertices of $D_{i,m}$ by edges. Let H be a graph of order n. For every $1 \leq i \leq (k-1)n+2$, join all the vertices of H to all the vertices of $D_{i,1}$ by edges. Then we obtain the desired graph G. It is easy to see that G is an n-connected graph, and has no k-tree that m-dominates G since the complete bipartite graph $K_{n,(k-1)n+2}$ whose vertex set is $V(H)$ and

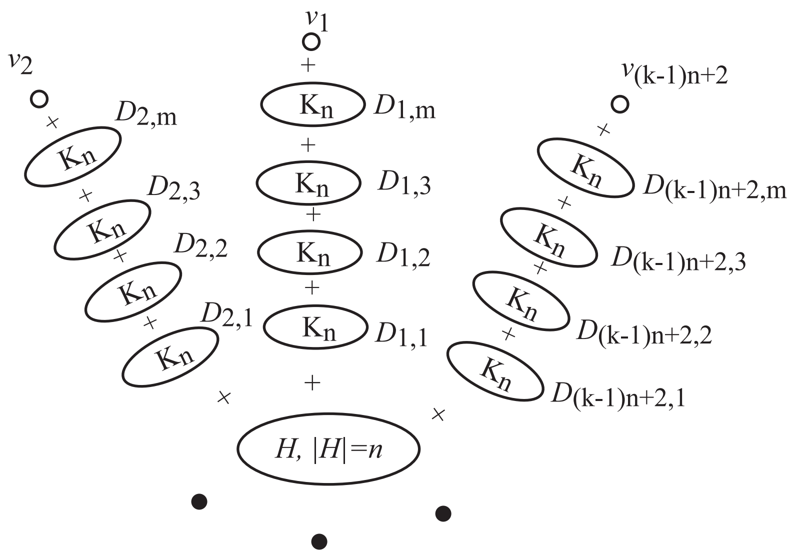

Figure 1: The graph G with $m = 4$, where $+$ denotes the join of two graphs.

$((k-1)n+2)K_n$ has no spanning k-tree. On the other hand, it follows that $\alpha^{2(m+1)}(G) = |\{ v_i : 1 \le i \le (k-1)n+2 \}| = (k-1)n+2$.

We conclude this section with a similar result on k-ended tree instead of k-tree, where a *k-ended tree* is a tree that contains at most k leaves.

Theorem 5 (Kano, Tsugaki and Yan [4]) *Let $k \ge 2$ and $m \ge 0$ be integers, and let G be a connected graph. If $\alpha^{2(m+1)}(G) \le k$, then G has a k-ended tree that m-dominates G.*

Notice that Theorem 5 has not yet been extended to n-connected graphs, and this extension might be an interesting problem. For other related results on spanning trees, the reader is referred to the book [1] and the survey [6].

2 Proof of Theorem 4

We begin with some notations. An edge joining a vertex x to a vertex y is denoted by xy or yx. Let G be a graph and H be a subgraph of G. For a vertex v of H, we denote by $N_H(v)$ the *neighborhood* of v in H. Thus $\deg_H(v) = |N_H(v)|$.

An *inner vertex* of a path is a vertex not being its end-vertex. For two vertices x and y of G, a path connecting x and y is called an (x, y)-path. Let X and Y be disjoint vertex sets of G. If a path P connects a vertex of X and a vertex of Y, and all the inner vertices of P are contained in $V(G) - (X \cup Y)$, then we call P an *(X, Y)-path* of G. For a vertex $z \notin X$, we abbreviate a $(\{ z \}, X)$-path as a (z, X)-path.

Let T be a tree. An end-vertex of T, which has degree one, is often called a *leaf* of T. We denote the set of leaves of T by $Leaf(T)$. For two vertices u and v of T, there exists a unique path connecting u and v in T, and it is denoted by $P_T(u, v)$. Let T be a *rooted tree with root* r, and let v be a nonroot vertex of T. Then the vertex adjacent to v and lying on the path $P_T(v, r)$ is called the *parent* of v and denoted by v^-. A vertex whose parent is v is called a *child* of v. In particular, there are $\deg_T(v) - 1$ children of v and $\deg_T(r)$ children of r. The set of children of v is denoted by $Child(v)$.

For simplicity, we often identify a tree T with its vertex set $V(T)$. For example, we write $G - T$ for $G - V(T)$.

A vertex set X of a subgraph H of a graph G is called an *independent set* of H if no two vertices of X are joined by an edge of H. The following three lemmas are useful in our proof. Lemma 2.1 is well-known.

Lemma 2.1 *Let T be a tree, and let X be an independent set of T. Then*

(i) *The number of leaves of T is $\sum_{v \in W}(\deg_T(v) - 2) + 2$, where $W = \{ v \in V(T) : \deg_T(v) \geq 3 \}$.*

(ii) *The number of components of $T - X$ is $\sum_{x \in X}(\deg_T(x) - 1) + 1$.*

Lemma 2.2 *Let $m \geq 1$ be an integer, and let G be a connected graph and H a subgraph of G. Let y_1 and y_2 be two distinct vertices of $Domi^m(H) - H$. Assume that there exist two disjoint vertex sets $S(y_1), S(y_2) \subseteq V(H)$ such that*

(i) *$d_G(y_i, S(y_i)) = m$ and $d_G(y_i, H - S(y_i)) \geq m + 1$ for $i = 1, 2$; and*

(ii) *there exists no $(S(y_1), S(y_2))$-path in G whose inner vertices are contained in $G - H$, in particular, no edge of G connects $S(y_1)$ and $S(y_2)$.*

Then $d_G(y_1, y_2) \geq 2(m+1)$.

Proof. Suppose that $d_G(y_1, y_2) \leq 2m+1$. Let $P(y_1, y_2)$ be a shortest path in G connecting y_1 and y_2, and let $P(y_i, S(y_i))$ be a shortest path in G connecting y_i and $S(y_i)$ for $i = 1, 2$. Then, by (i), the inner vertices of $P(y_i, S(y_i))$ are contained in $G - H$. Moreover, $P(y_1, y_2)$ passes through H by (ii) since otherwise $P(y_1, y_2) \cup P(y_1, S(y_1)) \cup P(y_2, S(y_2))$ contains an $(S(y_1), S(y_2))$-path whose inner vertices are contained in $G - H$. Proceeding along $P(y_1, y_2)$ from y_1 to y_2, let z_1 be the first vertex of $P(y_1, y_2)$ that lies in H and let z_2 be the last vertex of $P(y_1, y_2)$ that lies in H. Then $d_G(y_i, z_i) \geq m$ by (i) for $i = 1, 2$.

First, suppose that $z_1 \neq z_2$. Then $d_G(y_1, y_2) = d_G(y_1, z_1) + d_G(z_1, z_2) + d_G(z_2, y_2) \geq 2m + 1$. Since $d_G(y_1, y_2) \leq 2m + 1$, equality holds in the above inequality. Therefore, for $i = 1, 2$, $d_G(y_i, z_i) = m$, and so $z_i \in S(y_i)$ by (i). Moreover we obtain $d_G(z_1, z_2) = 1$, which implies that $z_1 \in S(y_1)$ and $z_2 \in S(y_2)$ are adjacent in G. This contradicts (ii).

Next, suppose that $z_1 = z_2$. If $z_1 = z_2 \in S(y_1)$, then $P(y_2, S(y_2)) \cup P(y_2, z_1)$ contains an $(S(y_2), S(y_1))$-path whose inner vertices are contained in $G - H$, which contradicts (ii). Thus $z_1 = z_2 \notin S(y_1)$. By symmetry, $z_1 = z_2 \notin S(y_1) \cup S(y_2)$. Hence by (i), $d_G(y_1, y_2) = d_G(y_1, z_1) + d_G(z_1, y_2) \geq 2(m+1)$, which is again a contradiction. Therefore Lemma 2.2 holds. □

Lemma 2.3 *Let* $m \geq 1$ *be an integer, and let* G *be a connected graph and* H *a subgraph of* G. *Let* $y \in Domi^m(H) - H$ *and* $w \in G - Domi^m(H)$ *be two vertices. Assume that there exists a vertex set* $S(y) \subseteq V(H)$ *such that*

(i) $d_G(y, S(y)) = m$ *and* $d_G(y, H - S(y)) \geq m + 1$; *and*

(ii) *there exists no* $(w, S(y))$*-path whose inner vertices are contained in* $G - H$.

Then $d_G(w, y) \geq 2(m+1)$.

Proof. Let $P(w, y)$ be a shortest path connecting w and y. By (i), there exists a path $P(y, S(y))$ of length m which connects y and $S(y)$ and whose inner vertices are contained in $G - H$. By (ii), $P(w, y)$ passes through H

since otherwise $P(w, y) \cup P(y, S(y))$ contains a $(w, S(y))$-path whose inner vertices are contained in $G - H$. Proceeding along $P(w, y)$ from w to y, let z_1 be the first vertex of $P(w, y)$ that lies in H and let z_2 be the last vertex of $P(w, y)$ that lies in H.

If $z_1 \neq z_2$, then $d_G(w, y) = d_G(w, z_1) + d_G(z_1, z_2) + d_G(z_2, y) \geq 2(m + 1)$. Hence we may assume $z_1 = z_2$. By (ii), we obtain $z_1 = z_2 \notin S(y)$. Thus by (i), $d_G(w, y) = d_G(w, z_1) + d_G(z_1, y) \geq 2(m + 1)$. Hence Lemma 2.3 holds. $\square$

We are ready to prove Theorem 4.

Proof of Theorem 4. If $m = 0$, then Theorem 4 follows from Theorem 2. Thus we may assume that $m \geq 1$. If $k = 2$, then Theorem 4 follows from Theorem 3. Thus we may assume that $k \geq 3$.

Let G be an n-connected graph that satisfies the condition in Theorem 4. Suppose that G has no k-tree that m-dominates G. Let T be a k-tree of G with $|T| \geq n$. Notice that the minimum degree of G is at least n and so G has a path order at least n. Since T does not m-dominate G, there exists a vertex w in $G - Domi^m(T)$. Since G is an n-connected graph, there exist n distinct (w, T)-paths $Q_1, Q_2, \ldots, Q_n$ such that each Q_i connects w and a vertex v_i of T, $Q_i \cap Q_j = \{ w \}$ for all $i \neq j$, and $Q_i \cap T = \{ v_i \}$ for all i. Let $V^* = \{ v_1, v_2, \ldots, v_n \}$. Let $D_1, D_2, \ldots, D_l$ be the components of $T - V^*$, and let $\mathcal{D} = \{ D_1, D_2, \ldots, D_l \}$.

Define $\partial_T(D_i) = \{ v \in V^* : v \in N_T(D_i) \}$ for each $D_i \in \mathcal{D}$. Thus $\partial_T(D_i)$ consists of the vertices of V^* which are adjacent to D_i in T. Let

$$\begin{aligned}
\mathcal{D}_1^T &= \mathcal{D}_1 = \{ D \in \mathcal{D} : |\partial_T(D)| = 1 \}, \\
\mathcal{D}_2^T &= \mathcal{D}_2 = \{ D \in \mathcal{D} : |\partial_T(D)| = 2 \}, \quad \text{and} \\
\mathcal{D}_{\geq 3}^T &= \mathcal{D}_{\geq 3} = \{ D \in \mathcal{D} : |\partial_T(D)| \geq 3 \}.
\end{aligned}$$

Notice that if there is no confusion, we often abbreviate $\mathcal{D}_*^T$ as $\mathcal{D}_*$. Choose a k-tree T, a vertex w and n paths $Q_1, Q_2, \ldots, Q_n$ so that

(T1) $|Domi^m(T)|$ is as large as possible,

(T2) $|\mathcal{D}_1^T \cup \mathcal{D}_{\geq 3}^T|$ is as small as possible, subject to (T1), (T3) $|Leaf(T)|$ is as small as possible, subject to (T2) and

(T4) $|T|$ is as small as possible, subject to (T3).

By the choice of T for (T1), we can obtain Claim 1.

Claim 1 $\deg_T(v) = k$ *for every* $v \in V^*$.

Claim 2 (i) *No two vertices of* V^* *are adjacent in* T.
(ii) $|\mathcal{D}| = (k-1)n + 1$ *and* $|\mathcal{D}_1^T| = (k-2)n + \sum_{D \in \mathcal{D}_{\geq 3}^T}(|\partial_T(D)| - 2) + 2$.

Proof. Assume that two vertices v_a and v_b of V^* are adjacent in T. Choose two paths Q_a and Q_b that connect w to v_a and v_b, respectively. Then $T' = T + Q_a + Q_b - v_a v_b$ is a k-tree and satisfies $Domi^m(T') \supseteq Domi^m(T) \cup \{ w \}$, which contradicts (T1). Hence (i) holds.

By the above statement (i), Lemma 2.1 and by Claim 1, we have $|\mathcal{D}| = (k-1)n + 1$. By contracting every component of $\mathcal{D}$ to a single vertex, we obtain a tree $T/\mathcal{D}$ from T. Then $V(T/\mathcal{D}) = V^* \cup \mathcal{D}_1^T \cup \mathcal{D}_2^T \cup \mathcal{D}_{\geq 3}^T$ and each component of $\mathcal{D}_1^T$ corresponds to a leaf of $T/\mathcal{D}$. The number of leaves of $T/\mathcal{D}$ is given by Lemma 2.1, and so the second equality holds. □

Claim 3 *For every leaf* x *of* T*, there exists a vertex* $y_x \in Domi^m(T)$ *such that* $d_G(y_x, x) = m$ *and* $d_G(y_x, T - x) \geq m + 1$.

Proof. Let x be a leaf of T. Let $W = \{ y \in V(G) : d_G(y, x) = m \}$. Suppose that either $W = \emptyset$ or $d_G(y, T - x) \leq m$ for every $y \in W$. Then $Domi^m(T) = Domi^m(T - x)$

It follows that $\{ x \}$ is not a component in $\mathcal{D}$ since otherwise $T - x + Q_a$ is a k-tree of G for some $1 \leq a \leq n$, and it m-dominates $Domi^m(T) \cup \{ w \}$, which contradicts (T1). We may assume $x \in D_a$, $1 \leq a \leq l$. Then $T - x$ is a k-tree, $w \notin Domi^m(T - x)$, $Q_1, Q_2, \ldots, Q_n$ are $(w, T - x)$-paths, and $\{ D_a - x \} \cup \{ D_i : 1 \leq i \leq n, i \neq a \}$ is the set of components of $(T - x) - V^*$. Thus $|\mathcal{D}_1^T \cup \mathcal{D}_{\geq 3}^T| = |\mathcal{D}_1^{T-x} \cup \mathcal{D}_{\geq 3}^{T-x}|$, $|Leaf(T)| \geq |Leaf(T - x)|$ and $|T| > |T - x|$. This contradicts (T3) or (T4). Hence there exists $y_x \in W$ such that $d_G(y_x, T - x) \geq m + 1$. Therefore Claim 3 holds. □

By Claim 3, we can obtain the following claim.

Claim 4 *$y_{x_1} \neq y_{x_2}$ for any distinct $x_1, x_2 \in Leaf(T)$.*

Let $Y_{Leaf} = \{ y_x : x \in Leaf(T) \}$, and for each $y \in Y_{Leaf}$, let $S(y) = \{ x \in Leaf(T) : y_x = y \}$. Then $S(y)$ consists of exactly one leaf of T by Claim 4.

By the choice of T for (T1), we can obtain the following claim.

Claim 5 *For every $y \in Y_{Leaf}$, there exists no $(w, S(y))$-path in G whose inner vertices are contained in $G - T$.*

Claim 6 *For any distinct $y_1, y_2 \in Y_{Leaf} \cup \{ w \}$, $d_G(y_1, y_2) \geq 2(m+1)$.*

Proof. If $y_1 = w$, then by Claim 5 there exists no $(w, S(y_2))$-path in G whose inner vertices are contained in $G - T$, and hence $d_G(w, y_2) \geq 2(m+1)$ by Claim 3 and Lemma 2.3. Therefore, we may assume that $y_1, y_2 \in Y_{Leaf}$. Let $S(y_i) = \{ x_i \}$ for $i = 1, 2$. Then $d_G(y_i, x_i) = m$ and $d_G(y_i, T - x_i) \geq m + 1$ by Claim 3.

We shall show that there exists no (x_1, x_2)-path in G whose inner vertices are contained in $G - T$. This fact implies $d_G(y_1, y_2) \geq 2(m+1)$ by Lemma 2.2. Suppose, to the contrary, that there exists a (x_1, x_2)-path P in G whose inner vertices are contained in $G - T$. By Claim 5, P intersects no Q_i for $1 \leq i \leq n$.

First, suppose that there exists $D \in \mathcal{D}$ such that $x_1, x_2 \in D$. Then $D + P$ contains a cycle C. Let T' be a tree obtained from $T + P$ by deleting one edge e of C which is adjacent to a vertex of degree at least 3 in T. Let $D' = D + P - e$. Since T' is a k-tree, $w \notin Domi^m(T')$ by (T1). Furthermore, $Q_1, Q_2, \ldots, Q_n$ are (w, T')-paths, and $\mathcal{D}' = (\mathcal{D} - \{ D \}) \cup \{ D' \}$ is the set of components of $T' - V^*$. Moreover, $|Domi^m(T)| \leq |Domi^m(T')|$, $|\mathcal{D}_1^T \cup \mathcal{D}_{\geq 3}^T| = |\mathcal{D}_1^{T'} \cup \mathcal{D}_{\geq 3}^{T'}|$ and $|Leaf(T)| > |Leaf(T')|$. This contradicts (T1) or (T3).

Next, suppose that there exist two distinct $D_1, D_2 \in \mathcal{D}$ such that $x_i \in D_i$ for $i = 1, 2$. Then $T + P$ contains a unique cycle C, which passes through a vertex v_a of V^*. Let e be an edge of C incident with v_a, and let $T' = T + P + Q_a - e$. Then T' is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$. This contradicts (T1). Hence Claim 6 holds. □

Claim 7 *There exists a component $D^* \in \mathcal{D}_1 = \mathcal{D}_1^T$ such that $\deg_T(x) \leq k-1$ for all $x \in V(D^*)$.*

Proof. Suppose that there exists no $D \in \mathcal{D}_1$ such that $\deg_T(x) \le k-1$ for all $x \in V(D)$. Then every component $D \in \mathcal{D}_1$ has a vertex of degree k in T, and so D has at least $k-1$ leaves of T. Hence, it follows from Claims 2, 4 and 6 and from $k \ge 3$ that

$$\begin{aligned}\alpha^{2(m+1)}(G) \ge |Y_{Leaf}| = |Leaf(T)| &\ge \sum_{D \in \mathcal{D}_1} |Leaf(T) \cap V(D)| \\ &\ge |\mathcal{D}_1|(k-1) \ge ((k-2)n+2)(k-1) \\ &\ge (n+2)(k-1) \ge (k-1)n+2.\end{aligned}$$

This contradicts the assumption on $\alpha^{2(m+1)}(G)$ in the theorem. Hence Claim 7 holds. $\square$

Without of loss of generality, we may assume that $D_1 = D^*$ and $\{ v_1 \} = \partial_T(D_1)$. We regard T as a *rooted tree with root* v_1. For each $D \in \mathcal{D}$, let r_D be the root of D, and let $v_D = {r_D}^- \in V^*$, where the root of D is the vertex that has no parent in D.

Since G is an n-connected graph, there exist n distinct $(D_1, T-D_1)$-paths $R_1, R_2, \ldots, R_n$ in G such that every R_i connects a vertex of D_1 and a vertex of $T - D_1$, the end-vertices of R_i and R_j in $T - D_1$ are distinct if $i \ne j$, and the inner vertices of every R_i are contained in $G - T$. In particular, $R_i \cap R_j \subseteq V(D_1)$ if $i \ne j$, and $|R_i \cap D_1| = 1$ for every i. It may happen that some R_c consists of an edge $r_{D_1}v_1$. Let U^* be the set of end-vertices of $R_i, 1 \le i \le n$, which are contained in $T - D_1$. Then $|U^*| = n$.

Claim 8 *There exists no (w, D_1)-path whose inner vertices are contained in $G - T$. Especially, $V(Q_i) \cap V(R_j) \subseteq \{ v_i \}$ for all $1 \le i, j \le n$.*

Proof. Suppose that there exists (w, D_1)-path Q whose inner vertices are contained in $G - T$. Let $T' = T + Q$. By Claim 7, T' is a k-tree and satisfies $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1). Hence Claim 8 holds. $\square$

Claim 9 $\deg_T(u) = k$ *for every $u \in U^*$.*

Proof. Suppose that $\deg_T(u) \le k-1$ for some $u \in U^*$. Let $R_a, 1 \le a \le n$, be the path connecting D_1 and u. Then $u \ne v_1$ by Claim 1. Let $T' = (T + Q_1 + R_a) - v_1 r_{D_1}$. Then it follows from Claims 7 and 8 that T' is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1). Hence Claim 9 holds. □

Claim 10 *For every $D \in \mathcal{D}_{\ge 3} = \mathcal{D}^T_{\ge 3}$, $U^* \cap \partial_T(D) \subseteq \{ v_D \}$.*

Proof. Suppose that there exists a vertex $u \in U^* \cap (\partial_T(D) - \{ v_D \})$ for some $D \in \mathcal{D}_{\ge 3}$. Let R_a be the path connecting D_1 and u. Let $T' = (T + R_a) - uu^-$ and $D_1{}' = D_1 + (R_a - \{ u \})$. Then T' is a k-tree. By the choice of T for (T1), $w \notin Domi^m(T')$. By Claim 8, $Q_1, Q_2, \dots, Q_n$ are (w, T')-paths, and $D_1{}', D_2, \dots, D_l$ are the components of $T' - V^*$. Moreover, $|\partial_T(D_1)| = 1$, $|\partial_{T'}(D_1{}')| = 2$, $|\partial_{T'}(D)| = |\partial_T(D)| - 1 \ge 2$ and $|\partial_{T'}(D_i)| = |\partial_T(D_i)|$ for every $D_i \in \mathcal{D} - \{ D_1, D \}$. Hence, $|Domi^m(T)| \le |Domi^m(T')|$ and $|\mathcal{D}^{T'}_1 \cup \mathcal{D}^{T'}_{\ge 3}| < |\mathcal{D}^T_1 \cup \mathcal{D}^T_{\ge 3}|$, which contradicts (T1) or (T2). Hence Claim 10 holds. □

For convenience, we introduce four notations $P(s,t)$, $P[s,t)$, $P(s,t]$ and $P[s,t]$ of a path in T connecting two veritces s and t. Namely, $P[s,t]$ contains both s and t, $P(s,t)$ contains neither s nor t, $P[s,t)$ contains s but not t, and $P(s,t]$ contains t but not s. From now on, we use these four different notations of a path in T.

For each $D \in \mathcal{D}_2 = \mathcal{D}^T_2$, let $\partial_T(D) = \{ v_D = r_D^-, s_D, \}$, where r_D is the root of D. So if D is a path and $r_D \ne s_D^-$, then r_D and s_D^- are the end-vertices of D. On the other hand, if D is a path of order at least two and $r_D = s_D^-$, then one end-vertex of D is a leaf of T and the other end-vertex $r_D = s_D^-$ has degree 3 in T.

If $D \in \mathcal{D}_2$ possesses one of the following three properties, then we call D a *pseudo-path component*.

(P1) $r_D = s_D^-$ and $D = \{r_D\}$.

(P2) D is a path and $r_D \ne s_D^-$.

(P3) There exists a vertex $z_D \in P[r_D, s_D^-]$ such that $z_D \in U^*$ and $\deg_T(z) = 2$ for every vertex $z \in P(z_D, s_D^-]$, where $P(z_D, s_D^-] = \emptyset$ if $z_D = s_D^-$.

Let

$$\mathcal{D}_2^p = \{\, D \in \mathcal{D}_2 = \mathcal{D}_2^T \ :\ D \text{ is a pseudo-path component} \,\}.$$

Claim 11 *If $D \in \mathcal{D}_2^p$, then there exists a vertex $x_D \in P[r_D, s_D^-]$ that satisfies the following two properties, where $P(s_D^-, s_D^-] = \emptyset$.*

(i) $\deg_T(z) = 2$ *for every vertex $z \in P[x_D, s_D^-]$.*

(ii) $Domi^m(P[x_D, s_D^-]) \subseteq Domi^m(T - P(x_D, s_D^-])$.

Proof. If $\deg_T({s_D}^-) = 2$, then $x_D = {s_D}^-$ satisfies the properties (i) and (ii). Hence, we may assume that D satisfies the property (P3) and $z_D = {s_D}^-$. Choose paths Q_a, Q_b and R_c so that $v_D \in Q_a$, $s_D \in Q_b$ and $z_D \in R_c$. Let $T' = (T + Q_a + Q_b + R_c) - v_D r_D - s_D {s_D}^-$. By Claim 8, T' is a k-tree such that $Domi^m(T) \cup \{\, w \,\} \subseteq Domi^m(T')$, which contradicts (T1). Hence Claim 11 holds. □

For each $D \in \mathcal{D}_2^p$, choose a vertex $x_D \in P[r_D, s_D^-]$ that satisfies (i) and (ii) of Claim 11 so that the order of $P[x_D, s_D^-]$ is as large as possible.

Claim 12 *If $D \in \mathcal{D}_2^p$, then there exists a vertex y_D such that $d_G(y_D, P[x_D, s_D^-]) = m$ and $d_G(y_D, T - P[x_D, s_D^-]) \geq m + 1$.*

Proof. Let $W = \{\, y \in V(G) : d_G(y, P[x_D, s_D^-]) = m \,\}$. Suppose that either $W = \emptyset$ or $d_G(y, T - P[x_D, s_D^-]) \leq m$ for every $y \in W$. Choose paths Q_a and Q_b so that $v_D \in Q_a$ and $s_D \in Q_b$.

First, suppose that D satisfies the property (P1). Then $T' = T + Q_a + Q_b - D$ is a k-tree and satisfies $Domi^m(T) \cup \{\, w \,\} \subseteq Domi^m(T')$, which contradicts (T1).

Next assume that D satisfies (P2). If $x_D \neq r_D$, then ${x_D}^-$ satisfies the properties (i) and (ii) of Claim 11, which contradicts the choice of x_D. Hence $x_D = r_D$. Let $T' = (T + Q_a + Q_b) - P[r_D, s_D^-]$. Then T' is a k-tree such that $Domi^m(T) \cup \{\, w \,\} \subseteq Domi^m(T')$, which contradicts (T1).

Finally, suppose that D satisfies (P3). If $x_D \notin Child(z_D)$, then ${x_D}^-$ satisfies the properties (i) and (ii) of Claim 11, which contradicts the choice of x_D. Hence $x_D \in Child(z_D)$. Choose a path R_c such that $z_D \in R_c$. Let

$T' = (T+Q_a+Q_b+R_c) - P[x_D, s_D^-] - v_D r_D$. By Claim 8, T' is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1).

Therefore there exists a vertex $y_D \in W$ such that $d_G(y_D, T - P[x_D, s_D^-]) \geq m+1$. Therefore Claim 12 holds. $\square$

Let $Y_{Path} = \{ y_D : D \in \mathcal{D}_2^p \}$. For each $y \in Y_{Path}$, choose $D \in \mathcal{D}_2^p$ so that $y_D = y$, and let $S(y) = V(P[x_D, s_D^-])$. Note that $S(y) = D = \{r_D\}$ if D satisfies (P1). By Claims 3 and 12, we obtain the following claim.

Claim 13 *$Y_{Leaf} \cap Y_{Path} = \emptyset$, and $y_{D_1} \neq y_{D_2}$ for any distinct $D_1, D_2 \in \mathcal{D}_2^p$.*

Claim 14 *For every $y \in Y_{Path}$, the following two statements hold.*

(i) *There exists no $(w, S(y))$-path whose inner vertices are contained in $G - T$.*

(ii) *There exists no $(D_1, S(y))$-path whose inner vertices are contained in $G - T$.*

Proof. (i) Suppose that there exists a $(w, S(y))$-path Q whose inner vertices are contained in $G - T$. Choose $D \in \mathcal{D}_2^p$ and a path Q_b such that $y_D = y$ and $s_D \in Q_b$. Then D satisfies (P3) since otherwise $T + Q$ is a k-tree and $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T+Q)$. Let z be the end-vertex of Q in $S(y)$. Choose a vertex w_0 of $Q \cap Q_b$ such that w_0 is the closest vertex of $Q \cap Q_b$ to z in Q. Then by Claim 11 and the choice of w_0, $T' = (T+Q_b+Q[z, w_0]) - P(z, s_D^-]$ is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1). Hence (i) holds.

(ii) Suppose that there exists a $(D_1, S(y))$-path Q such that all the inner vertices of Q are contained in $G - T$. Choose $D \in \mathcal{D}_2^p$ and paths Q_a and Q_b such that $y_D = y$, $v_D \in Q_a$ and $s_D \in Q_b$. Note that Q intersects neither Q_a nor Q_b by Claim 8. Then $T' = (T + Q_a + Q_b + Q) - v_D r_D - s_D s_D^-$ is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1). Hence (ii) holds. $\square$

Claim 15 *For two distinct $y_1, y_2 \in Y_{Leaf} \cup Y_{Path} \cup \{ w \}$, $d_G(y_1, y_2) \geq 2(m+1)$.*

Proof. If $y_i \in Y_{Leaf} \cup Y_{Path}$, then $d_G(y_i, S(y_i)) = m$ and $d_G(y_i, T - S(y_i)) \geq m + 1$ by Claims 3 and 12.

Suppose that $y_1 = w$. By Claims 5 and 14, there exists no $(w, S(y_2))$-path whose inner vertices are contained in $G - T$. By Lemma 2.3, this implies that $d_G(w, y_2) \geq 2(m + 1)$.

Therefore, we may assume that $y_1, y_2 \in Y_{Leaf} \cup Y_{Path}$. By Claim 6, we may assume that either (i) $y_1 \in Y_{Leaf}$ and $y_2 \in Y_{Path}$, or (ii) $y_1, y_2 \in Y_{Path}$. By Lemma 2.2, it suffices to show that there exists no $(S(y_1), S(y_2))$-path whose inner vertices are contained in $G - T$. We shall prove this fact by considering the following two cases.

Case 1. $y_1 \in Y_{Leaf}$ and $y_2 \in Y_{Path}$

Choose a leaf $x \in Leaf(T)$ and a component $D \in \mathcal{D}_2^p$ such that $y_x = y_1$ and $y_D = y_2$. Then $S(y_1) = \{ x \}$ and $S(y_2) = V(P[x_D, s_D^-])$. Suppose that there exists a $(x, S(y_2))$-path Q whose inner vertices are contained in $G - T$. Let z be the end-vertex of Q in $S(y_2)$. Choose paths Q_a and Q_b such that $v_D \in Q_a$ and $s_D \in Q_b$. By Claim 8, Q intersects neither Q_a nor Q_b.

Suppose first that $x \in V(D)$. Then D satisfies the property (P3). Choose a path R_c such that $z_D \in R_c$. By Claim 8, R_c intersects neither Q_a nor Q_b. By Claim 14, R_c does not intersect Q also. Hence $T' = (T + Q_a + Q_b + R_c + Q) - P(z, s_D) - v_D r_D - z_D z^*$, where $z_D z^*$ is an edge contained in a path $P_T(z_D, x)$, is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1).

Next suppose that $x \notin V(D)$. Then $T' = (T + Q_a + Q_b + Q) - P(z, s_D) - v_D r_D$ is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1).

Case 2. $y_1, y_2 \in Y_{Path}$

Choose two components $D_a, D_b \in \mathcal{D}_2^p$ so that $y_{D_a} = y_1$ and $y_{D_b} = y_2$. Choose paths Q_i, Q_j and Q_h such that $s_{D_a} \in Q_i$, $s_{D_b} \in Q_j$ and $v_{D_a} \in Q_h$. Suppose that there exists an $(S(y_1), S(y_2))$-path Q whose inner vertices are contained in $G - T$. Let z_a and z_b be the end-vertices of Q contained in $S(y_a)$ and $S(y_b)$, respectively. If D_a satisfies (P1) or (P2), then $T' = T + Q_i + Q_j + Q_h + Q - P(z_b, s_{D_b}) - s_{D_a} s_{D_a}^- - v_{D_a} r_{D_a}$ is a k-tree and satisfies $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, a contradiction. Hence by symmetry, we may assume that both D_a and D_b satisfy (P3). Choose Q so that $|P(z_a, s_{D_a})|+|P(z_b, s_{D_b})|$ is as small as possible.

Suppose that there exists a vertex $u \in Domi^m(P(z_a, s_{D_a}) \cup P(z_b, s_{D_b}))$ such that $u \notin Domi^m(T - (P(z_a, s_{D_a}) \cup P(z_b, s_{D_b})))$. By Claim 11 (ii), there exist either a $(u, P(z_a, s_{D_a}))$-path whose inner vertices are contained in $(G-T) \cup P(z_b, s_{D_b})$ or a $(u, P(z_b, s_{D_b}))$-path whose inner vertices are contained in $(G-T) \cup P(z_a, s_{D_a})$. This implies that there exists a $(P(z_a, s_{D_a}), P(z_b, s_{D_b}))$-path whose inner vertices are contained in $G-T$. This contradicts the minimality of $|P(z_a, s_{D_a})|+|P(z_b, s_{D_b})|$. Hence $Domi^m\big(P(z_a, s_{D_a}) \cup P(z_b, s_{D_b})\big) \subseteq Domi^m\big(T - (P(z_a, s_{D_a}) \cup P(z_2, s_{D_b}))\big)$.

Let $T' = (T + Q_i + Q_j + Q_h + Q) - P(z_a, s_{D_a}) - P(z_b, s_{D_b}) - v_{D_a} r_{D_a}$. By Claim 8 and the above fact, T' is a k-tree such that $Domi^m(T) \cup \{ w \} \subseteq Domi^m(T')$, which contradicts (T1). Hence Claim 15 holds. $\square$

Claim 16 $|V^* \cup U^*| \geq |V^*| + \sum_{D \in \mathcal{D}_{\geq 3}} (\partial_T(D) - 1)$.

Proof. We first construct a new tree T^* from T as follows. Remove all the components of $\mathcal{D}_1$, replace every component D of $\mathcal{D}_2$ by an edge joining two vertices v_D and s_D, and contract every component of $\mathcal{D}_{\geq 3}$ to a single vertex. Then the vertex set of T^* is $V^* \cup \mathcal{D}_{\geq 3}$. We consider T^* as a rooted tree with root v_1. Then for every vertex $D \in \mathcal{D}_{\geq 3}$, there are $|\partial_T(D)| - 1$ children of D in T^*. By Claim 10, these children of D are contained in $V^* - U^* \cap V^*$. Since $|U^*| = |V^*| = n$, it follows that $|U^* - U^* \cap V^*| = |V^* - U^* \cap V^*| \geq \sum_{D \in \mathcal{D}_{\geq 3}} (|\partial_T(D)| - 1)$. Hence $|V^* \cup U^*| = |V^*| + |U^* - U^* \cap V^*| \geq |V^*| + \sum_{D \in \mathcal{D}_{\geq 3}} (|\partial_T(D)| - 1)$. $\square$

Claim 17 (i) *If $D \in \mathcal{D}_2^p$, then $|Leaf(T) \cap D| \geq (k-2)|U^* \cap D|$.*

(ii) *If $D \in \mathcal{D}_2 - \mathcal{D}_2^p$, then $|Leaf(T) \cap D| \geq (k-2)|U^* \cap D| + 1$.*

(iii) $|Leaf(T) \cap \bigcup_{D \in \mathcal{D}_1} D| \geq (k-2)|U^* \cap \bigcup_{D \in \mathcal{D}_1} D| + (k-2)|V^*|$
$+ \sum_{D \in \mathcal{D}_{\geq 3}} (|\partial_T(D)| - 2) + 2$.

(iv) *If $D \in \mathcal{D}_{\geq 3}$, then $|Leaf(T) \cap D| \geq (k-2)|U^* \cap D| - |\partial_T(D)| + 2$.*

Proof. (i) This follows immediately from Lemma 2.1 and Claim 9.

(ii) Let $D \in \mathcal{D}_2 - \mathcal{D}_2^p$. If D is a path with $r_D = s_D^-$, then $U^* \cap D = \emptyset$ or $\{ r_D \}$. If $U^* \cap D = \emptyset$, then (ii) holds since D contains a leaf of T. Thus we may assume $U^* \cap D = \{ r_D \}$. Then we derive a contradiction by considering $T + Q_a + Q_b + R_c - v_D r_D - s_D^- s_D$, where $v_D \in Q_a$, $s_D \in Q_b$ and $r_D \in R_c$. Hence there exists a vertex z in $P[r_D, s_D)$ such that $\deg_T(z) \geq 3$.

Choose such a vertex z so that z is closest to s_D in D. Then every vertex in $P(z, s_D)$ has degree 2 in T. If $z \in U^*$, then D satisfies (P3) and so $D \in \mathcal{D}_2^p$, which contradicts $D \in \mathcal{D}_2 - \mathcal{D}_2^p$. Hence $z \notin U^*$.

Thus it follows from Claim 9 that

$$\begin{aligned} |Leaf(T) \cap D| &\geq \sum_{v \in V(D)} \max\{ \deg_T(v) - 2, 0 \} \\ &\geq \sum_{v \in U^* \cap V(D)} (\deg_T(v) - 2) + \deg_T(z) - 2 \\ &\geq (k-2)|U^* \cap D| + 1. \end{aligned}$$

Hence (ii) holds.

(iii) We first construct a new tree $T/\mathcal{D}_3$ from T as follows. Replace every component D of $\mathcal{D}_2$ by an edge joining two vertices v_D and s_D, and contract every component of $\mathcal{D}_{\geq 3}$ into a single vertex. Then the number of leaves of T contained in $\bigcup_{D \in \mathcal{D}_1} D$ is equal to the number of leaves of $T/\mathcal{D}_3$. Since every vertex $D \in \mathcal{D}_{\geq 3}$ has degree $\partial_T(D)$ in $T/\mathcal{D}_3$, (iii) follows from Lemma 2.1 and Claim 1.

(iv) Let $D \in \mathcal{D}_{\geq 3}$. By adding the edges joining D to V^* together with their endvertices contained in V^* to D, we obtain a tree D'. The number of leaves of T contained in D is equal to the number of leaves of D' minus $\partial_T(D)$. Hence (iv) holds. □

Claim 18 $|Leaf(T)| \geq |\mathcal{D}| - |\mathcal{D}_2^p|$.

Proof. By Claims 16 and 17, we obtain

$$
\begin{aligned}
&|Leaf(T)| \\
\geq &(k-2)|U^* \cap \bigcup_{D \in \mathcal{D}_2} D| + |\mathcal{D}_2| - |\mathcal{D}_2^p| \\
&+ (k-2)|U^* \cap \bigcup_{D \in \mathcal{D}_1} D| + (k-2)|V^*| + \sum_{D \in \mathcal{D}_{\geq 3}} (|\partial_T(D)| - 2) + 2 \\
&+ (k-2)|U^* \cap \bigcup_{D \in \mathcal{D}_{\geq 3}} D| - \sum_{D \in \mathcal{D}_{\geq 3}} |\partial_T(D)| + 2|\mathcal{D}_{\geq 3}| \\
\geq &(k-2)|V^* \cup U^*| + |\mathcal{D}_2| - |\mathcal{D}_2^p| + 2 \\
\geq &(k-2)|V^*| + (k-2) \sum_{D \in \mathcal{D}_{\geq 3}} (|\partial_T(D)| - 1) + |\mathcal{D}_2| - |\mathcal{D}_2^p| + 2 \\
\geq &(k-2)n + \sum_{D \in \mathcal{D}_{\geq 3}} (|\partial_T(D)| - 2) + |\mathcal{D}_{\geq 3}| + |\mathcal{D}_2| - |\mathcal{D}_2^p| + 2 \\
= &|\mathcal{D}_1| + |\mathcal{D}_{\geq 3}| + |\mathcal{D}_2| - |\mathcal{D}_2^p| \qquad \text{(by Claim 2)} \\
= &|\mathcal{D}| - |\mathcal{D}_2^p|
\end{aligned}
$$

Hence Claim 18 holds. □

By Claims 2, 4, 13, 15 and 18, we have

$$
\begin{aligned}
\alpha^{2(m+1)}(G) &\geq |Y_{Leaf} \cup Y_{Path} \cup \{w\}| = |Y_{Leaf}| + |Y_{Path}| + 1 \\
&= |Leaf(T)| + |\mathcal{D}_2^p| + 1 \\
&\geq |\mathcal{D}| + 1 \\
&= (k-1)n + 2.
\end{aligned}
$$

This contradicts the condition in the theorem. Consequently Theorem 4 is proved. □

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