$(1, f)$-factors of graphs with odd property

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Abstract

Let G be a graph and $f : V(G) \to \{1, 2, 3, 4, \dots\}$ be a function. We denote by $odd(G)$ the number of odd components of G. We prove that if $odd(G - X) \le \sum_{x \in X} f(x)$ for all $X \subset V(G)$, then G has a $(1, f)$-factor F such that, for every vertex v of G, if $f(v)$ is even, then $\deg_F(v) \in \{1, 3, \dots, f(v) - 1, f(v)\}$, and otherwise $\deg_F(v) \in \{1, 3, \dots, f(v)\}$. This theorem is a generalization of both the $(1, f)$-odd factor theorem and a recent result on $\{1, 3, \dots, 2n-1, 2n\}$-factors by Lu and Wang. We actually prove a result stronger than the above theorem.

Keywords: factor of graph, (1,f)-odd factor, odd components

1 Introduction

We consider finite graphs which has neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G . For a subset $S \subset V(G)$, we write $G - S$ for the subgraph of G induced by $V(G) - S$. A component of a graph is called an *odd component* if it is of odd order. Let $odd(G)$ and $Odd(G)$ denote the number of odd components and the set of odd components of G, respectively. Thus $odd(G) = |Odd(G)|$.

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For an odd integer-valued function $f : V(G) \to \{1, 3, 5, \ldots\}$, a spanning subgraph F of G is called a *$(1, f)$-odd factor* if $\deg_F(x) \in \{1, 3, 5, \ldots, f(x)\}$ for all $x \in V(G)$, while for an integer-valued function $g : V(G) \to \{1, 2, 3, \ldots\}$, a spanning subgraph H such that $1 \le \deg_H(x) \le g(x)$ for all $x \in V(G)$ is called a *$(1, g)$-factor*. A criterion for a graph to have a $(1, f)$-odd factor is given in the following theorem.

Theorem 1 (Cui and Kano [2]) *Let G be a graph and $f : V(G) \to \{1, 3, 5, \ldots\}$. Then G has a $(1, f)$-odd factor if and only if*

$$odd(G - X) \le \sum_{x \in X} f(x) \qquad \text{for all} \quad X \subset V(G).$$

Recently the following theorem, which settles a long-standing open problem given in [2], has been obtained by H.L. Lu and David G.L. Wang.

Theorem 2 (Lu and Wang [3]) *Let G be a graph and $n \ge 2$ be an even integer. If*

$$odd(G - X) \le n|X| \qquad \text{for all} \quad X \subset V(G)$$

then G has a factor F such that $\deg_F(x) \in \{1, 3, 5, \ldots, n-1, n\}$ for all $x \in V(G)$.

In this paper, we prove a theorem which is a generalization of both Theorems 1 and 2. Moreover, the proof in [3] uses a deep structure theorem for $\mathcal{H}$-factors obtained by Lovász, but our proof uses a standard proof technique in factor theory, called a β-method, and a class of special graphs and functions.

For an integer $n \ge 1$, we define a class $\mathcal{B}_{2n-1}$ of pairs (G, f) of a connected graph G of order $2n-1$ and a function $f : V(G) \to \{1, 2, 3, \ldots\}$ inductively as follows. Set $\mathcal{B}_1 = \emptyset$. Let $n \ge 2$, and assume that $\mathcal{B}_1, \mathcal{B}_3, \ldots, \mathcal{B}_{2n-3}$ have been defined. For a pair (H, f) of a graph H with $|H| \le 2n-2$ and a function $f : V(H) \to \{1, 2, 3, \ldots\}$, let

$$Odd_f(H) = \{C \in Odd(H) \;:\; (C, f) \notin \bigcup_{1 \le k \le n-1} \mathcal{B}_{2k-1}\}.$$

Set

$$\mathcal{B}_{2n-1} = \Big\{(G, f) \ : \ G \text{ is a connected graph of order } 2n-1,$$
$$f : V(G) \to \{1, 2, 3, \dots\},$$
$$|Odd_f(G - X)| \le \sum_{x \in X} f(x) \ \text{ for all } \ \emptyset \ne X \subset V(G),$$
$$\text{and there exists } \emptyset \ne S \subset V(G) \text{ such that}$$
$$|Odd_f(G - S)| = \sum_{x \in S} f(x)\Big\}.$$

Now let

$$\mathcal{B} = \bigcup_{n \ge 1} \mathcal{B}_{2n-1}.$$

Some elements of $\mathcal{B}$ are shown in Figure 1, where numbers in the figure indicate values of f. Note that for each example of $f : V(G_1) \to \{1, 2, \dots\}$, we have $|Odd_f(G_1 - r) = 2 = f(r)$. Similarly, for each example of $f : V(G_2) \to \{1, 2, \dots\}$, we have $|Odd_f(G - t)| = f(t)$. Pairs $(G, f) \in \mathcal{B}$ can be regarded as *benign pairs* because we do not count them in the inequality stated in Theorem 3.

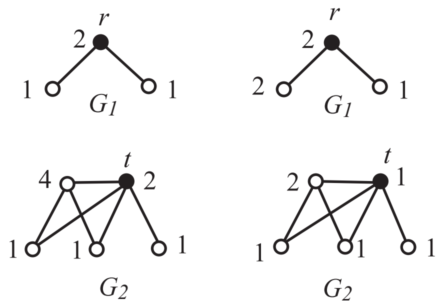

Figure 1: Some elements $(G_1, f) \in \mathcal{B}_3$ and $(G_2, f) \in \mathcal{B}_5$.

For a pair (G, f) of a graph G and a function $f : V(G) \to \{1, 2, 3, \dots\}$, let

$$Odd_f(G) = \{C \in Odd(G) \ : \ (C, f) \notin \mathcal{B}\}, \quad \text{and}$$
$$odd_f(G) = |Odd_f(G)|.$$

Note that this definition is consistent with the notation $Odd_f(H)$ introduced in the preceding paragraph in the course of the definition of $\mathcal{B}_{2n-1}$.

Theorem 3 *Let G be a connected graph of order at least two and $f : V(G) \to \{1, 2, 3, \dots\}$. Suppose that either $(G, f) \in \mathcal{B}$, or G is of even order and satisfies*

$$odd_f(G - X) \leq \sum_{x \in X} f(x) \qquad \text{for all} \ \ \emptyset \neq X \subset V(G). \tag{1}$$

Then G has a $(1, f)$-factor F such that, for every vertex v of G, if $f(v)$ is even, then $\deg_F(v) \in \{1, 3, \dots, f(v) - 1, f(v)\}$, and otherwise $\deg_F(v) \in \{1, 3, \dots, f(v)\}$.

The following theorem is an immediate consequence of the above theorem since $odd_f(G - X) \leq odd(G - X)$ for all $\emptyset \neq X \subset V(G)$ and $odd(G) = 0$ implies that every component of G is of even order. Moreover, the following theorem is a generalization of both Theorems 1 and 2.

Theorem 4 *Let G be a graph and $f : V(G) \to \{1, 2, 3, \dots\}$. If*

$$odd(G - X) \leq \sum_{x \in X} f(x) \qquad \text{for all} \quad X \subset V(G),$$

then G has a $(1, f)$-factor F such that, for every vertex v of G, if $f(v)$ is even, then $\deg_F(v) \in \{1, 3, \dots, f(v) - 1, f(v)\}$, and otherwise $\deg_F(v) \in \{1, 3, \dots, f(v)\}$.

2 Proof of Theorem 3

For two sets X and Y, $X \subset Y$ means that X is a proper subset of Y. Let G be a graph. For two vertices x and y of G, we write xy or yx for an edge joining x to y. For a vertex v of G, the neighborhood of v is denote by $N_G(v)$, and for a subset S of $V(G)$, we define $N_G(S) = \cup_{x \in S} N_G(x)$.

In order to prove Theorem 3, we need the following basic results.

Lemma 5 ([1] Lemma 2.26) *Let G be a graph of even order and $S \subseteq V(G)$. Then*

$$odd(G - S) \equiv |S| \pmod 2.$$

Lemma 6 (Generalized Marriage Theorem, [1] Theorem 2.9) *Let G be a bipartite graph with bipartition (X, Y), and let $f : X \to \{1, 2, 3, \dots\}$ be a function. Then G has a spanning subgraph F such that*

$$\deg_F(x) = f(x) \ \ \text{for all } x \in X, \quad \text{and} \quad \deg_F(y) = 1 \ \ \text{for all} \ \ y \in Y$$

if and only if

$$|N_G(S)| \geq \sum_{x\in S} f(x) \qquad \textit{for all } \ \emptyset \neq S \subseteq X, \quad \textit{and} \quad |Y| = \sum_{x\in X} f(x).$$

Note that if such a subgraph F exists, every component of F is a star.

Lemma 7 *If $(G, f) \in \mathcal{B}$, then G has at least one vertex u such that $f(u)$ is even.*

Proof. By the definition of $\mathcal{B}$, G is a connected graph of odd order. We prove the lemma by induction on $|G|$. If $|G| = 3$, then G has a vertex u such that $odd_f(G-u) = f(u) \geq 1$. Then $odd_f(G-u) = odd(G-u) = 2 = f(u)$.

Assume that $|G| \geq 5$. Suppose, to the contrary, that $f(x)$ is odd for every vertex x. Let S be a non-empty vertex set of G such that $odd_f(G-S) = \sum_{x\in S} f(x)$. Since $f(x)$ is odd for all $x \in S$, $\sum_{x\in S} f(x) \equiv |S| \pmod 2$. Since $|G|$ is odd, $odd(G-S) \not\equiv |S| \pmod 2$. Thus $odd_f(G-S) \neq odd(G-S)$, and hence there exists an odd component C of $G-S$ such that $(C, f) \in \mathcal{B}$. By the induction hypothesis, C has at least one vertex u such that $f(u)$ is even, which contradicts the assumption that $f(x)$ is odd for every vertex x. Hence G has a desired vertex u. $\Box$

Proof of Theorem 3. We denote the number of components of a graph H by $\omega(H)$. For brevity, we refer to a $(1, f)$-factor satisfying the property required in Theorem 3 as a *$(1, f)$-factor with odd property.*

We prove Theorem 3 by induction on $\sum_{x\in V(G)} f(x)$. If G is of even order and $f(x)$ is odd for all $x \in V(G)$, then for each $\emptyset \neq X \subset V(G)$, we have $odd(G-X) = odd_f(G-X)$ by the definition of $odd_f(G-X)$ and by Lemma 7, and hence Theorem 3 follows from Theorem 1. By this observation and by Lemma 7, we may assume that G has a vertex w such that $f(w)$ is even. Throughout the proof, w always denotes this special vertex.

Let us define the number β by

$$\beta = \min\{\sum_{x\in X} f(x) - odd_f(G-X) \ : \ \emptyset \neq X \subset V(G)\}.$$

Then $\beta \geq 0$ by (1) and the definition of $\mathcal{B}$, and

$$odd_f(G-Y) \leq \sum_{x\in Y} f(x) - \beta \qquad \text{for all} \quad \emptyset \neq Y \subset V(G). \tag{2}$$

Choose a vertex set S of G so that

(S1) S is a maximal set with $\beta = \sum_{x\in S} f(x) - odd_f(G-S)$, and

(S2) $\omega(G-S)$ is as large as possible subject to (S1).

By the maximality of S, for every $X \subset V(G)$ with $|S| < |X|$, we have

$$\beta + 1 \le \sum_{x\in X} f(x) - odd_f(G-X). \tag{3}$$

Claim 1. *If $\beta \ge 1$, then G has a desired $(1, f)$-factor with odd property.*

Proof. Assume $\beta \ge 1$. Then $(G, f) \notin \mathcal{B}$ by (2) and the definition of $\mathcal{B}$, and so G is of even order. Define $f^* : V(G) \to \{1, 2, 3, \ldots\}$ by

$$f^*(x) = \begin{cases} f(x) - 1 & \text{if } x = w; \\ f(x) & \text{otherwise.} \end{cases}$$

Let $\emptyset \ne X \subset V(G)$. If $w \notin X$, then $\sum_{x\in X} f^*(x) = \sum_{x\in X} f(x)$ and $odd_{f^*}(G-X) \le odd_f(G-X)+1$; if $w \in X$, then $\sum_{x\in X} f^*(x) = \sum_{x\in X} f(x) - 1$ and $odd_{f^*}(G-X) = odd_f(G-X)$. In either case,

$$\sum_{x\in X} f^*(x) - odd_{f^*}(G-X) \ge \sum_{x\in X} f(x) - odd_f(G-X) - 1.$$

Hence $\sum_{x\in X} f^*(x) - odd_{f^*}(G-X) \ge \beta - 1 \ge 0$ by (2). Since X is arbitrary, this implies that G has a $(1, f^*)$-factor F^* with odd property by the induction hypothesis. Since $f^*(w)$ is odd, we have $\deg_{F^*}(w) \in \{1, 3, \ldots, f^*(w) = f(w) - 1\}$. Therefore, F^* is a desired $(1, f)$-factor with odd property. □

Hereafter we assume that $\beta = 0$.

Claim 2. *If $\omega(G-S) = 1$, then G has a desired $(1, f)$-factor with odd property.*

Proof. Assume that $\omega(G-S) = 1$. Then $\omega(G-S) = 1 = odd_f(G-S) = \sum_{x\in S} f(x)$, and thus $S = \{s\}$ and $odd_f(G-s) = f(s) = 1$. Then $w \ne s$. Since $\omega(G-s) = odd_f(G-s) = 1$, $G-s$ is of odd order, and hence G is of even order. Define $f^* : V(G) \to \{1, 2, 3, \ldots\}$ by

$$f^*(x) = \begin{cases} f(x) - 1 & \text{if } x = w; \\ f(x) & \text{otherwise.} \end{cases}$$

Let $\emptyset \ne X \subset V(G)$. If $|X| = 1$ and $\omega(G-X) = 1$, then $odd_{f^*}(G-X) \le \omega(G-X) = 1 \le \sum_{x\in X} f^*(x)$; if $|X| \ge 2$ or $\omega(G-X) \ge 2$, then $\sum_{x\in X} f(x) - odd_f(G-X) \ge 1$ by (3) or condition (S2), and we therefore obtain $\sum_{x\in X} f^*(x) - odd_{f^*}(G-X) \ge \sum_{x\in X} f(x) - odd_f(G-X) - 1 \ge 0$ by arguing as in the proof of Claim 1. Thus $odd_{f^*}(G-X) \le \sum_{x\in X} f^*(x)$ for every $\emptyset \ne X \subset V(G)$. Hence G has a $(1,f^*)$-factor F^* with odd property by induction. Since $f^*(w)$ is odd, it follows that $\deg_{F^*}(w) \in \{1,3,\dots,f^*(w)=f(w)-1\}$. Therefore F^* is a desired $(1,f)$-factor with odd property. □

Hereafter we assume that $\omega(G-S) \ge 2$.

Claim 3. *Every even component of $G-S$ has a $(1,f)$-factor with odd property.*

Proof. Let D be an even component of $G-S$. Let $\emptyset \ne X \subset V(D)$. By (2), we have

$$odd_f(G-S) + odd_f(D-X) = odd_f(G-S\cup X)$$
$$\le \sum_{x\in S\cup X} f(x) = \sum_{x\in S} f(x) + \sum_{x\in X} f(x).$$

Hence $odd_f(D-X) \le \sum_{x\in X} f(x)$. Since X is arbitrary, this implies that D has a $(1,f)$-factor with odd property by the induction hypothesis.

Claim 4. *Every odd component of $G-S$ not contained in $Odd_f(G-S)$ has a $(1,f)$-factor with odd property.*

Proof. Let D be an odd component of $G-S$ not contained in $Odd_f(G-S)$. Then $(D,f) \in \mathcal{B}$ by the definition of $Odd_f(G-S)$. Since $|D| < |G|$, D has a $(1,f)$-factor with the odd property by induction. □

Claim 5. *If $C \in Odd_f(G-S)$, then $odd_f(G-X) < \sum_{x\in X} f(x)$ for every $\emptyset \ne X \subset V(C)$.*

Proof. If $|C|=1$, then there is nothing to be proved. Thus assume $|C| \ge 3$. For each $\emptyset \ne X \subset V(C)$, we obtain by (3)

$$odd_f(G-S) + odd_f(C-X) = odd_f(G-S\cup X) + 1$$
$$\le \sum_{x\in S} f(x) + \sum_{x\in X} f(x).$$

Hence $odd_f(C-X) \le \sum_{x\in X} f(x)$. Since $(C,f) \notin \mathcal{B}$, this together with the definition of $\mathcal{B}$ implies that we have $odd_f(G-X) < \sum_{x\in X} f(x)$ for every $\emptyset \ne X \subset V(C)$. □

We construct a bipartite graph B with bipartition $(S, Odd_f(G-S))$ in which two vertices $x \in S$ and $C \in Odd_f(G-S)$ are joined by an edge of B if and only if x is adjacent to C in G.

Claim 6. *For every $\emptyset \ne Y \subset S$, we have $|N_B(Y)| \ge \sum_{x\in Y} f(x)$, and $|N_B(S)| = |Odd_f(G-S)| = \sum_{x\in S} f(x)$.*

Proof. Since G is connected, it follows that $N_B(S) = Odd_f(G - S)$. Let $\emptyset \neq Y \subset S$. Since $Odd_f(G - S) - N_B(Y) \subseteq Odd_f(G - (S - Y))$, it follows from (2) that

$$\begin{aligned} & \sum_{x \in S} f(x) - \sum_{x \in Y} f(x) = \sum_{x \in S - Y} f(x) \geq odd_f(G - (S - Y)) \\ \geq \;& odd_f(G - S) - |N_B(Y)| = \sum_{x \in S} f(x) - |N_B(Y)|. \end{aligned}$$

Hence $|N_B(Y)| \geq \sum_{x \in Y} f(x)$, and so Claim 6 holds. □

For a component $C \in Odd_f(G - S)$ and a vertex $s \in S$ with $sC \in E(B)$, take an edge e joining s and C in G, an let $C + e$ denote the subgraph of G obtained from C by adding the edge e together with its end-point s. Moreover a function $g : V(C + e) \to \{1, 2, 3 \ldots\}$ is defined by letting $g(x) = f(x)$ for all $x \in V(C)$ and $g(s) = 1$. Then the following claim holds.

Claim 7. *Assume that $C \in Odd_f(G - S)$ and $s \in S$ are are adjacent in B, and let e and g be as above. Then $C + e$ has a $(1, g)$-factor H_{sC} with odd property.*

Proof. Let $\emptyset \neq X \subset V(C + e) = V(C) \cup \{s\}$. We shall show that $odd_g(C + e - X) \leq \sum_{x \in X} g(x)$, which implies that $C + e$ has a desired $(1, g)$-factor with odd property by induction. Note that we have $|V(C) \cup \{s\}| < |G|$ because $\omega(G - S) \geq 2$, and hence $\sum_{x \in V(C) \cup \{s\}} g(x) < \sum_{x \in V(G)} f(x)$.

First assume $s \in X$. If $X = \{s\}$, then $odd_g(C + e - X) = 1 = g(s) = \sum_{x \in X} g(x)$; if $\{s\} \subset X$, then by Claim 5, $odd_g(C + e - X) = odd_f(C - (X - \{s\} < \sum_{x \in X - \{s\}} g(x)$, which implies $odd_g(C + e - X) \leq \sum_{x \in X} g(x)$. Next assume $s \notin X$. If $X = V(C)$, then $odd_g(C + e - X) = 1 \leq \sum_{x \in X} g(x)$. Thus we may assume $X \neq V(C)$. Then $odd_g(C + e - X) \leq odd_f(C - X) + 1$. Therefore by Claim 5, $odd_g(C + e - X) \leq \sum_{x \in X} f(x) - 1 + 1 = \sum_{x \in X} g(x)$, as desired. □.

By Claim 7 and Lemma 5, B has a spanning subgraph F such that

$$\begin{aligned} & \deg_F(C) = 1 \quad \text{for all} \;\; C \in Odd_f(G - S), \quad \text{and} \\ & \deg_F(s) = f(s) \quad \text{for all } s \in S. \end{aligned}$$

Consequently, we can obtain a desired $(1, f)$-factor of G with odd property by combining $(1, f)$-odd factors with odd property of all even components of $G - S$ and all odd components of $G - S$ not contained in $Odd_f(G - S)$, and $(1, g)$-odd factors H_{sC} with odd property given in Claim 7 for all edges sC of F. □

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