Spanning trees whose stems are spiders

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Abstract

Let T be a tree. The set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ is called the stem of T. A spider is a tree having at most one vertex with degree greater than two. In [2], it is shown that if a connected graph G satisfies $\delta(G) \geq (|G|-1)/3$, then G has a spanning spider. In this paper, we prove that if $\sigma_4^4(G) \geq |G| - 5$, then G has a spanning tree whose stem is a spider, where $\sigma_4^4(G)$ denotes the minimum degree sum of four vertices of G such that the distance between any two of them are at least four. Moreover, we show that this condition is sharp.

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. The minimum degree of G is denoted by $\delta(G)$. For two vertices u and v of G, the *distance* between u and v in G is denoted by $d_G(u, v)$. For an integer $l \geq 2$, let $\alpha^l(G)$ denote the number defined by

$$\alpha^l(G) = \max\{|S| : S \subset V(G), d_G(x, y) \geq l \text{ for all distinct } x, y \in S\}.$$

For an integer $k \geq 2$, we define

$$\sigma_k^l(G) = \min\{\sum_{x \in S} \deg_G(x) : S \subset V(G),\ |S| = k,\ d_G(x, y) \geq l \text{ for all distinct } x, y \in S\}.$$

For convenience, we define $\sigma_k^l(G) = \infty$ if $\alpha^l(G) < k$. Note that $\sigma_k^2(G)$ is often written $\sigma_k(G)$, which is the minimum degree sum of k independent vertices. For a tree T, a vertex with degree at least three is called a *branch*, and a tree having at most one branch is called a *spider*. Namely, a spider is a path or a tree having exactly one vertex with degree at least three.

In [2], Gargano, Hammar, Hell, Stacho and Vaccaro gave a sufficient condition for a graph to have a spanning spider as follows.

Theorem 1 (Gargano et al. [2]) *Let G be a connected graph. If $\delta(G) \geq (|G| - 1)/3$, then G has a spanning spider.*

However, it is not known whether the condition of Theorem 1 is sharp or not. In particular, Yamashita [9] raised the following problem.

Problem 1 *Is it true that a connected graph G with $\delta(G) \geq (|G| - 1)/4$ has a spanning spider? If it is true, then the condition $\delta(G) \geq (|G| - 1)/4$ is sharp. If it is not true, find a minimum value of $\delta(G)$ that guarantees the existence of a spanning spider in G.*

In [5], Kyaw obtained the following theorem which shows that the above problem is true for $K_{1,4}$-free graph.

Theorem 2 (Kyaw [5]) *Let G be a connected $K_{1,4}$-free graph. If $\sigma_4(G) \geq |G| - 1$, then G has a spanning spider with at most 3 leaves.*

A sharp sufficient condition for a claw-free graph to have a spanning spider is given in [6] as follows.

Theorem 3 (Matsuda, Ozeki and Yamashita [6]) *Let G be a connected claw-free graph. If $\sigma_5(G) \geq |G| - 2$, then G has a spanning spider.*

Let T be a tree. A vertex of T with degree one is often called a *leaf* of T, and the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the *stem* of T and denoted by $Stem(T)$. A spanning tree with specified stem was first considered in [3], and it is proved that if $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has maximum degree at most k. Sufficient conditions for a graph to have spanning tree whose stem has at most k leaves are obtained in [8] and [4]. In this paper, we give a sufficient condition for a graph to have a spanning tree whose stem is a spider. The following is our main theorem.

Theorem 4 *Let G be a connected graph. If*

$$\sigma_4^4(G) \geq |G| - 5, \tag{1}$$

then G has a spanning tree whose stem is a spider.

Before proving Theorem 4, we show that the condition in Theorem 4 is sharp. Let $m \geq 1$ be integer, and let D_1, D_2, D_3, D_4 be disjoint copies of the complete graph K_m of order m. Let $u, w, v_1, \ldots, v_4$ be six vertices not contained in $D_1 \cup D_2 \cup D_3 \cup D_4$. Join u to all the vertices of $D_1 \cup D_2$, and join w to all the vertices $D_3 \cup D_4$ by edges. Next join v_i to all vertices of D_i for $1 \leq i \leq 4$ by edges, and join u and w by an edge. Let G be the resulting graph. Then G satisfies $\sigma_4^4(G) = |G| - 6$, but G has no spanning tree whose stem is a spider. Therefore the condition (1) is sharp.

Some results on spanning tree with given properties can be found in [1], and many current results on spanning trees can be found in [7].

2 Proof of Theorem 4

We begin with a new definition on spiders. If a spider is not a path, then it contains the unique vertex of degree at least three, which is called the *trunk*. We prove Theorem 4.

Proof of Theorem 4. Assume that G satisfies the condition (1) in Theorem 4 but does not have a spanning tree whose stem is a spider. In G, choose a tree T whose stem is a spider so that

(T1) $|T|$ is as large as possible.

(T2) The degree of the trunk of $Stem(T)$ in $Stem(T)$ is as small as possible subject to (T1).

(T3) $|Stem(T)|$ is as small as possible subject to (T1) and (T2).

Let w be the trunk of $Stem(T)$, and let $k = \deg_{stem(T)}(w)$, the degree of w in $Stem(T)$. Then it is obvious that $k \geq 3$. Let $x_1, x_2, \ldots, x_k$ be the leaves of $Stem(T)$. Since T is not a spanning tree of G, there exist two vertices $v_1 \in V(G) - V(T)$ and $v_2 \in Leaf(T)$ which are adjacent in G. Note that v_1 can not be adjacent to any vertex of the $Stem(T)$. Moreover, v_2 is adjacent to a vertex v_3 of $Stem(T)$ in T.

By the choice (T1), we can obtain the following Claim 1.

Claim 1. For every $v \in V(G) - V(T)$, $N_G(v) \subseteq Leaf(T) \cup (V(G) - V(T))$.

Claim 2. For every x_i, $1 \leq i \leq k$, there exists a leaf y_i of T that is adjacent to x_i in T and satisfies $N_G(y_i) \subseteq leaf(T) \cup \{x_i\}$.

Suppose that there exists a leaf x_a, $1 \leq a \leq k$, of $Stem(T)$ for which $N_G(y) \cap (Stem(T) - \{x_a\}) \neq \emptyset$ for every leaf y of T adjacent to x_a in T. For every leaf y adjacent to x_a in T, remove the edge yx_a from T and add an edge yz of G, where z is a vertex of $Stem(T) - \{x_a\}$. Denote the resulting tree by T_1. Then T_1 is a tree whose stem is a spider, $|T_1| = |T|$ and $Stem(T_1) = Stem(T) - \{x_a\}$, which contradicts the condition (T3). Therefore, the claim holds.

Claim 3. For any two distinct vertices $y, z \in \{v_1, y_1, y_2, \ldots, y_k\}$, $d_G(y, z) \geq 4$.

First, we show that $d_G(v_1, y_i) \geq 4$ for every $1 \leq i \leq k$. Let $P(v_1, y_a)$ be a shortest path connecting v_1 and y_a in G, where $1 \leq a \leq k$. If $P(v_1, y_a)$ does not pass through $Stem(T)$, then add $P(v_1, y_a)$ to T and remove the edges of T joining $P(v_1, y_a) \cap Leaf(T)$ to $Stem(T)$ except the edge $y_a x_a$. Then resulting tree of G is a tree whose stem is a spider and whose order is greater than $|T|$, which contradicts the condition (T1). Hence $P(v_1, y_a)$ passes through a vertex s in $Stem(T)$. Let $P(v_1, s)$ denote the sub-path of $P(v_1, y_a)$ connecting v_1 and s. We can choose s so that $P(v_1, s)$ does not passes through any other vertex of $Stem(T)$.

If $s = x_c$ for some $1 \leq c \leq k$, then add $P(v_1, x_c)$ to T and remove the edges of T joining $P(v_1, x_c) \cap Leaf(T)$ to $Stem(T)$ except $x_c y_c$. Then resulting tree is a tree whose stem is a spider and whose order is greater than $|T|$, which is a contradiction. Thus, $s \in Stem(T) - \{x_1, \ldots, x_k\}$. By Claims 1 and 2, $d_G(v_1, s) \geq 2$ and $d_G(s, y_a) \geq 2$, which implies $d_G(v_1, y_a) = d_G(v_1, s) + d_G(s, y_a) \geq 4$.

Next, we show that $d_G(y_i, y_j) \geq 4$ for all $1 \leq i < j \leq k$. For fixed a and b ($1 \leq a < b \leq k$), let $P(y_a, y_b)$ be a shortest path connecting y_a and y_b in G. Assume first that $P(y_a, y_b)$ does not pass through $Stem(T)$. Then add $P(y_a, y_b)$ to T and remove the edges of T joining $P(y_a, y_b) \cap Leaf(T)$ to $Stem(T)$ except the edges $y_a x_a$ and $y_b x_b$. Then the resulting subgraph of G includes a unique cycle, which contains an edge e_1 of $Stem(T)$ incident with the trunk w. By removing the edge e_1, we obtain a tree whose stem is a spider with trunk w of degree $k-1$ in its stem and whose order is $|T|$. This contradicts the condition (T2). Hence $P(y_a, y_b)$ passes through a vertex s in $Stem(T)$.

If $s \notin Stem(T) - \{x_a, x_b\}$, then $d_G(y_a, s) \geq 2$ and $d_G(s, y_b) \geq 2$ by Claim 2, and thus $d_G(y_a, y_b) = d_G(y_a, s) + d_G(s, y_b) \geq 4$. So we may assume that $s = x_a$ by symmetry.

If x_a and x_b are adjacent in G, then $T + x_a x_b$ contains a cycle passing through the trunk w. Let e_2 be an edge of the cycle adjacent to w. Then $T + x_a x_b - e_2$ is a tree whose stem is a spider with trunk w of degree $k-1$ in its stem and whose order is $|T|$. This contradicts the condition (T2). Hence x_a and x_b are not adjacent in G. If $P(y_a, y_b)$ passes through both x_a and x_b, then $d_G(y_a, y_b) = d_G(y_a, x_a) + d_G(x_a, x_b) + d_G(x_b, y_b) \geq 1 + 2 + 1 = 4$. Hence we may assume that $P(y_a, y_b)$ passes through x_a but not x_b.

Let $P(x_a, y_b)$ be the sub-path of $P(y_a, y_b)$ connecting x_a and y_b. Add $P(x_a, y_b)$ to T and remove the edges of T joining $P(x_a, y_b) \cap Leaf(T)$ to $Stem(T)$ except $y_b x_b$. Then the resulting subgraph of G includes a unique cycle, which contains an edge e_3 of $Stem(T)$ incident with the trunk w. By removing the edge e_3, we obtain a tree whose stem is a spider with trunk w of degree $k-1$ in its stem and whose order is $|T|$. This contradicts the condition (T2). Therefore the claim holds.

By Claim 3, we can obtain the following claim.

Claim 4. (i) $N_G(v_1) \cap N_G(y_i) = \emptyset$ for all $1 \leq i \leq k$.
(ii) $N_G(y_i) \cap N_G(y_j) = \emptyset$ for all $1 \leq i \neq j \leq k$.

By Claims 1-4 and $k \geq 3$, we have

$$N_G(v_1) \subseteq (V(G) - V(T) - \{v_1\}) \cup (Leaf(T) - Y)$$
$$\bigcup_{i=1}^{3} N_G(y_i) \subseteq (Leaf(T) - Y) \cup \{x_1, x_2, x_3\},$$

where $Y = \{y_1, \ldots, y_k\}$. Hence by letting $m = |N_G(v_1) \cap (Leaf(T) - Y)|$, we have by Claim 4 that

$$\begin{aligned}
\sigma_4^4(G) \leq & \deg_G(v_1) + \deg_G(y_1) + \deg_G(y_2) + \deg_G(y_3) \\
\leq & |G| - |T| - 1 + m + |Leaf(T)| - k - m + 3 \\
= & |G| - |T| + |Leaf(T)| - k + 2 \\
= & |G| - |Stem(T)| - k + 2.
\end{aligned}$$

Recall that a vertex v_3 of $Stem(T)$ is adjacent to v_2 in T and v_2 is adjacent to v_1 in G. If $v_3 = w$ or $v_3 = x_b$ for some $1 \leq b \leq k$, then $T + v_1 v_2$ is a tree of G whose stem is a spider and whose order is greater than that of T. Thus $v_3 \notin \{w, x_1, \ldots, x_k\}$. Hence $|Stem(T)| \geq k + 2$. Therefore we obtain

$$\begin{aligned}
& \deg_G(v_1) + \deg_G(y_1) + \deg_G(y_2) + \deg_G(y_3) \\
\leq & |G| - |Stem(T)| - k + 2 \leq |G| - 2k \leq |G| - 6.
\end{aligned}$$

This contradicts the assumption $\sigma_4^4 \geq |G| - 5$. Consequently, the proof is complete. □

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