Properly colored geometric matchings and 3-trees without crossings on multicolored points in the plane

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Abstract. Let X be a set of multicolored points in the plane such that no three points are collinear and each color appears on at most $\lceil |X|/2 \rceil$ points. We show the existence of a non-crossing properly colored geometric perfect matching on X (if $|X|$ is even), and the existence of a non-crossing properly colored geometric spanning tree with maximum degree at most 3 on X. Moreover, we show the existence of a non-crossing properly colored geometric perfect matching in the plane lattice. In order to prove these our results, we propose an useful lemma that gives a good partition of a sequence of multicolored points.

Keyword(s): red and blue points, multicolored points, alternating matching, alternating tree, properly colored geometric graph, sequence of points.

MSC2010: 52C35, 05C70, 05C05[*1].

1 Introduction

Various topics on a set of red and blue points in the plane have been studied [3]. In this paper, we consider some problems for more colors. Given a set X of multicolored points in the plane, we want to draw a graph in the plane so that the vertex set is X and each edge is a straight-line segment whose two end-vertices have distinct colors. We call such a graph a *properly colored geometric graph* on X, which is also called an *alternating geometric graph* if X is a 2-colored point set. For alternating geometric perfect matchings on a 2-colored point set, the next theorem is well-known.

Theorem 1.1 ([5]). *Let R and B be sets of red and blue points in the plane, respectively. Assume that no three points of $R \cup B$ are collinear. If $|R| = |B|$, then there exists a non-crossing alternating geometric perfect matching on $R \cup B$.*

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In this paper, we first generalize this Theorem 1.1 for a 3-colored point set, stated as Theorem 1.2 (Fig. 1). Note that Theorem 1.1 is a special case of Theorem 1.2 with $G = \emptyset$.

Theorem 1.2. *Let R, B, G be sets of red, blue, and green points in the plane, respectively. Assume that no three points of $X = R \cup B \cup G$ are collinear. If $|X|$ is even and each color appears on at most $|X|/2$ points, then there exists a non-crossing properly colored geometric perfect matching on X.*

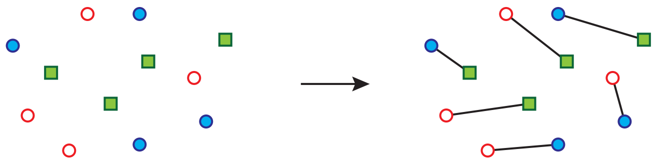

Fig. 1. A non-crossing properly colored geometric perfect matching.

Next, we consider a tree of maximum degree at most 3, which is called a *3-tree*. Kaneko [2] proved the following theorem.

Theorem 1.3 (Kaneko [2]). *Let R and B be sets of red and blue points in the plane, respectively. Assume that no three points of $R \cup B$ are collinear. If $|R| = |B|$, then there exists a non-crossing alternating geometric spanning 3-tree on $R \cup B$.*

Our second result is a generalization of this Theorem 1.3 for a 3-colored point set, stated as Theorem 1.4 (Fig. 2).

Theorem 1.4. *Let R, B, G be sets of red, blue, and green points in the plane, respectively. Assume that no three points of $X = R \cup B \cup G$ are collinear. If each color appears on at most $\lceil |X|/2 \rceil$ points, then there exists a non-crossing properly colored geometric spanning 3-tree on X.*

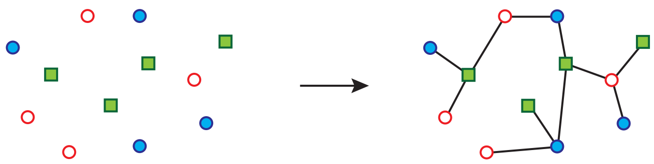

Fig. 2. A non-crossing properly colored geometric spanning 3-tree.

If $|X|$ is even, then we can obtain this Theorem 1.4 as a corollary from our Theorem 1.2 and the following theorem by Hoffmann and Tóth [1].

Theorem 1.5 (Hoffmann and Tóth [1]). *Every disconnected properly colored geometric graph with no isolated vertices can be augmented (by adding edges) into a connected properly colored geometric graph so that the degree of every vertex increases by at most two.*

By our Theorem 1.2, there exists a non-crossing properly colored geometric perfect matching M on X. By applying Theorem 1.5 to M, we can augment M into a non-crossing properly colored geometric spanning 3-tree on X. Note that if $|X|$ is odd, then a maximum matching M on X is not a perfect matching (one isolated vertex remains), so we cannot apply Theorem 1.5 to M. In Section 4, we present another proof of Theorem 1.4 for both even and odd $|X|$.

We can also consider problems on red and blue points in the plane lattice by using L-line segments instead of line segments, where an *L-line segment* in the plane lattice consists of a vertical line segment and a horizontal line segment having a common endpoint. Kano et al.[4] proved the following theorem.

Theorem 1.6 (Kano, Suzuki [4]). *Let R and B be sets of red and blue points in the plane lattice, respectively. Assume that every vertical line and horizontal line passes through at most one point of the points. If $|R| = |B|$, then there exists a non-crossing alternating geometric perfect matching on $R \cup B$ such that each edge is an L-line segment.*

Our third result is a generalization of this Theorem 1.6 for a 3-colored point set, stated as Theorem 1.7 (Fig. 3).

Theorem 1.7. *Let R, B, G be sets of red, blue, and green points in the plane lattice, respectively. Assume that every vertical line and horizontal line passes through at most one point of $X = R \cup B \cup G$. If $|X|$ is even and each color appears on at most $|X|/2$ points, then there exists a non-crossing properly colored geometric perfect matching on X such that each edge is an L-line segment.*

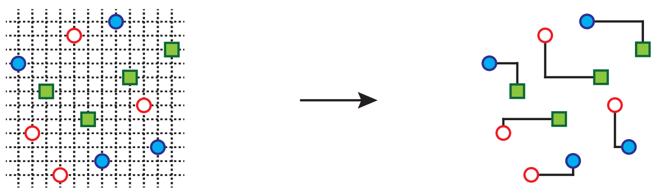

Fig. 3. A non-crossing properly colored geometric perfect matching with L-line segments.

In order to prove our results, we propose the following lemma (Fig. 4).

Lemma 1.8. *Let $(x_1, x_2, \ldots, x_n)$ be a sequence of $n \geq 3$ points colored with 3 colors, say red, blue, and green. Let R, B, G be sets of red, blue, and green points* *in the sequence, respectively. Assume that the both ends x_1 and x_n have the same color. If each color appears on at most $\lceil n/2 \rceil$ points, then there exists an even number p $(2 \le p \le n-1)$ such that x_p and x_1 have distinct colors and for every $C \in \{R, B, G\}$,*

$$|C \cap \{x_1, \ldots, x_p\}| \le \frac{p}{2},$$
$$|C \cap \{x_{p+1}, \ldots, x_n\}| \le \left\lceil \frac{n-p}{2} \right\rceil.$$

Fig. 4. Example of Lemma 1.8.

This lemma gives a *balanced* partition of a 3-colored sequence, in the sense that every color appears on at most half of the points in each part of the partition. In our inductive proofs, this lemma is useful in some cases where some "ends" have the same color. We expect applications of the Lemma to problems where each color appears on at most half of points.

We can more generalize above our results for a multicolored point set with 2, 3 or more colors, by using the following lemma.

Lemma 1.9. *Let $N_X = \{n_1, n_2, \ldots, n_r\}$ $(r \ge 4)$ be a set of positive integers. Set $n = n_1 + n_2 + \cdots + n_r$. If each integer n_i is at most $\lceil n/2 \rceil$ then there exists a tripartition $N_X = N_R \cup N_B \cup N_G$ such that*

$$\sum_{k \in N_R} k \le \left\lceil \frac{n}{2} \right\rceil, \quad \sum_{k \in N_B} k \le \left\lceil \frac{n}{2} \right\rceil, \quad \sum_{k \in N_G} k \le \left\lceil \frac{n}{2} \right\rceil.$$

Proof. We may assume that $n_1 \le n_2 \le \cdots \le n_r$. Then $n_1 \le \lfloor n/r \rfloor \le \lfloor n/4 \rfloor < \lfloor n/2 \rfloor$ since $r \ge 4$ and $n \ge 4$. Thus, for some integer n_j, $n_1 + n_2 + \cdots + n_j < \lfloor n/2 \rfloor$ and $n_1 + n_2 + \cdots + n_j + n_{j+1} \ge \lfloor n/2 \rfloor$. Then, $n_{j+2} + n_{j+3} + \cdots + n_r = n - (n_1 + n_2 + \cdots + n_j + n_{j+1}) \le n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$. Note that $1 \le j \le r-2$ because if $j = r-1$ then $n_r = n - (n_1 + \cdots + n_j) > n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$. Hence, we have the desired tripartition $N_R = \{n_1, \ldots, n_j\}$, $N_B = \{n_{j+1}\}$, and $N_G = \{n_{j+2}, \ldots, n_r\}$. □

By using this Lemma 1.9 where let n_i be the number of points of color i, we can obtain the following results for multicolored points from Theorem 1.2, 1.4, 1.7, and Lemma 1.8.

Corollary 1.10. *Let X be a set of multicolored points in the plane such that no three points are collinear. If $|X|$ is even and each color appears on at most $|X|/2$ points, then there exists a non-crossing properly colored geometric perfect matching on X.*

Corollary 1.11. *Let X be a set of multicolored points in the plane such that no three points are collinear. If each color appears on at most $\lceil |X|/2 \rceil$ points, then there exists a non-crossing properly colored geometric spanning 3-trees on X.*

Corollary 1.12. *Let X be a set of multicolored points in the plane lattice. Assume that every vertical line and horizontal line passes through at most one point of X. If $|X|$ is even and each color appears on at most $|X|/2$ points, then there exists a non-crossing properly colored geometric perfect matching on X such that each edge is an L-line segment.*

Corollary 1.13. *Let $(x_1, x_2, \dots, x_n)$ be a sequence of multicolored $n \geq 3$ points. For each color j, let C_j be a set of points colored with j in the sequence. Assume that the both ends x_1 and x_n have the same color. If $|C_j| \leq \lceil n/2 \rceil$ for every color j, then there exists an even number p $(2 \leq p \leq n-1)$ such that x_p and x_1 have distinct colors and for every color j,*

$$|C_j \cap \{x_1, \dots, x_p\}| \leq \frac{p}{2},$$
$$|C_j \cap \{x_{p+1}, \dots, x_n\}| \leq \left\lceil \frac{n-p}{2} \right\rceil.$$

In this paper, we will prove Lemma 1.8, Theorem 1.2, Theorem 1.4, and Theorem 1.7 in Section 2, 3, 4, and 5, respectively.

Throughout this paper, we will use the following definitions, notations, and a fact. For two points x and y in the plane, xy denotes the line segment joining x and y. For a set X of points in the plane, we denote by $conv(X)$ the boundary of the convex hull of X. We call a point in $X \cap conv(X)$ a *vertex* on $conv(X)$. For a graph G and its vertex v, we denote by $\deg_G(v)$ the degree of v in G. For positive integers n, a, and b such that $n = a + b$, we know the fact that $a \leq \lceil n/2 \rceil$ and $b \leq \lceil n/2 \rceil$ if and only if $|a - b| \leq 1$. We often use this fact without mentioning.

2 Proof of Lemma 1.8

By the symmetry of the colors, we may assume that x_1 and x_n are red. First, we claim the lemma holds when $B = \emptyset$ or $G = \emptyset$, say $G = \emptyset$.

Claim 1. *If $G = \emptyset$ then there exists an even number p $(2 \leq p \leq n-1)$ such that x_p is blue and*

$$|R \cap \{x_1, \dots, x_p\}| = |B \cap \{x_1, \dots, x_p\}| = \frac{p}{2},$$
$$|R \cap \{x_{p+1}, \dots, x_n\}| \leq \left\lceil \frac{n-p}{2} \right\rceil, |B \cap \{x_{p+1}, \dots, x_n\}| \leq \left\lceil \frac{n-p}{2} \right\rceil.$$

Proof. Define a function f from $\{1, 2, \ldots, n\}$ to the set of integers as

$$f(i) = |R \cap \{x_1, \ldots, x_i\}| - |B \cap \{x_1, \ldots, x_i\}|.$$

Then $f(i)$ increases or decreases by one, and $f(1) = |\{x_1\}| - |\emptyset| = 1$ and

$$\begin{aligned} f(n-1) &= |R \cap \{x_1, \ldots, x_{n-1}\}| - |B \cap \{x_1, \ldots, x_{n-1}\}| = |R \setminus \{x_n\}| - |B \setminus \{x_n\}| \\ &= (|R| - 1) - |B| = |R| - 1 - (n - |R|) = 2|R| - 1 - n \\ &\leq 2\left\lceil \frac{n}{2} \right\rceil - 1 - n \leq (n+1) - 1 - n = 0. \end{aligned}$$

Hence there exists the smallest number p $(2 \leq p \leq n-1)$ such that $f(p) = 0$. Then, $f(p-1) = 1$. Thus, x_p is a blue point since $f(i)$ decreases when x_i is a blue point. Since $f(p) = 0$, by the definition of f, we have

$$|R \cap \{x_1, \ldots, x_p\}| = |B \cap \{x_1, \ldots, x_p\}| = \frac{p}{2}.$$

Then, p is even and for each $C \in \{R, B\}$,

$$|C \cap \{x_{p+1}, \ldots, x_n\}| = |C| - |C \cap \{x_1, \ldots, x_p\}| \leq \left\lceil \frac{n}{2} \right\rceil - \frac{p}{2} = \left\lceil \frac{n-p}{2} \right\rceil.$$

□

Next, by using Claim 1, we will prove the lemma. We use induction on n. If $n = 3$ or $n = 4$ then x_i $(2 \leq i \leq n-1)$ are not red since $x_1, x_n \in R$ and $|R| \leq \lceil n/2 \rceil = 2$. Thus, $x_p = x_2$ is the desired point. For $n \geq 5$, we suppose that the lemma holds for a sequence of $n-2$ points.

Case 1. $|C| = \lceil n/2 \rceil$ *for some* $C \in \{R, B, G\}$.

Set $W = C$ and $K = (R \cup B \cup G) \setminus C$. We recolor all the points of W with white, and all the points of K with black[*2]. Then, we have

$$|W| = \left\lceil \frac{n}{2} \right\rceil, |K| = n - |W| = n - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor \leq \left\lceil \frac{n}{2} \right\rceil.$$

Since $x_1, x_n \in W$ or $x_1, x_n \in K$, by Claim 1, there exists an even number p $(2 \leq p \leq n-1)$ such that x_p and x_1 have distinct colors and

$$|W \cap \{x_1, \ldots, x_p\}| = |K \cap \{x_1, \ldots, x_p\}| = \frac{p}{2},$$
$$|W \cap \{x_{p+1}, \ldots, x_n\}| \leq \left\lceil \frac{n-p}{2} \right\rceil, |K \cap \{x_{p+1}, \ldots, x_n\}| \leq \left\lceil \frac{n-p}{2} \right\rceil.$$

Hence, since each of R, B and G is a subset of W or K, the point x_p is the desired point.

[*2] We denote a set of black points by K not by B, because B means a set of blue points in this paper.

Case 2. $|C| \le \lceil n/2 \rceil - 1$ *for every* $C \in \{R, B, G\}$.

If x_2 is not red, then the point $x_p = x_2$ is the desired point because for every $C \in \{R, B, G\}$,

$$|C \cap \{x_3, \ldots, x_n\}| \le |C| \le \left\lceil \frac{n}{2} \right\rceil - 1 = \left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n-p}{2} \right\rceil.$$

Hence we may assume that x_2 is red. Then there exists a blue or green point x_t $(t \ge 3)$, such that $x_1, \ldots, x_{t-1}$ are all red. We now consider a sequence

$$Y = (y_1, y_2, \ldots, y_{n-2}) = (x_2, \ldots, x_{t-1}, x_{t+1}, \ldots, x_n),$$

which is obtained from the original sequence by removing one red point x_1 and one blue or green point x_t. Note that the points $y_1(= x_2)$ and $y_{n-2}(= x_n)$ have the same color, namely red. For every $C \in \{R, B, G\}$, $|C \cap Y| \le |C| \le \lceil n/2 \rceil - 1 = \lceil (n-2)/2 \rceil$. Thus, by applying the inductive hypothesis to Y, there exists an even number q $(2 \le q \le n-3)$ such that y_q is a blue or green point and for every $C \in \{R, B, G\}$,

$$|C \cap \{y_1, \ldots, y_q\}| \le \frac{q}{2},$$
$$|C \cap \{y_{q+1}, \ldots, y_{n-2}\}| \le \left\lceil \frac{n-2-q}{2} \right\rceil.$$

Then $y_q = x_{q+2}$ and $t+1 \le q+2$ since $x_1, \ldots, x_{t-1}$ are red, $x_t \notin Y$, and y_q is not red. Hence, since x_1 and x_t have distinct colors, for every $C \in \{R, B, G\}$,

$$|C \cap \{x_1, \ldots, x_{q+2}\}| = |C \cap \{x_1, x_t, y_1, \ldots, y_q\}| \le \frac{q}{2} + 1 = \frac{q+2}{2},$$
$$|C \cap \{x_{q+3}, \ldots, x_n\}| = |C \cap \{y_{q+1}, \ldots, y_{n-2}\}| \le \left\lceil \frac{n-2-q}{2} \right\rceil = \left\lceil \frac{n-(q+2)}{2} \right\rceil.$$

Therefore, since q is even, namely $q+2$ is even, the point $x_p = x_{q+2}$ is the desired point.

3 Proof of Theorem 1.2 by using Lemma 1.8

We briefly call a non-crossing properly colored geometric perfect matching *a Perfect Matching*. Set $2n = |X|$. We prove the theorem by induction on n. If $n = 1$ then the theorem is true. For $n \ge 2$, we suppose that the theorem holds for $2(n-1)$ points.

Suppose that $|C| = n$ for some $C \in \{R, B, G\}$. Set $W = C$ and $K = (R \cup B \cup G) \setminus C$. We recolor all the points of W with white, and all the points of K with black. Then there exists the desired Perfect Matching by applying Theorem 1.1 to $W \cup K$.

Hence, we may assume that

$$|C| \leq n-1 \qquad \text{for every } C \in \{R, B, G\}.$$

Suppose that some two adjacent vertices u and v on $conv(X)$ have distinct colors. By our assumption, we have

$$|C \cap (X - \{u, v\})| \leq |C| \leq n-1 \qquad \text{for every } C \in \{R, B, G\}.$$

Thus, since $|X - \{u, v\}| = 2(n-1)$, we can apply the inductive hypothesis to $X - \{u, v\}$ and there exists a Perfect Matching on $X - \{u, v\}$. By adding an edge uv to this matching, we can obtain the desired Perfect Matching.

Therefore, we may assume that all the vertices on $conv(X)$ have the same color. Let v be a vertex on $conv(X)$. By a suitable rotation of the plane, we may assume that v is the highest vertex on $conv(X)$, and a and b are the left and the right vertices on $conv(X)$ adjacent to v, respectively.

We sort all the points of X with respect to their counterclockwise angle from the ray emanating from v and passing through a, and denote the sorted sequence by $(x_1, x_2, \ldots, x_{2n})$ so that $x_1 = v$, $x_2 = a$, and $x_{2n} = b$. Since the two end-points x_1 and x_{2n} have the same color, by Lemma 1.8, there exists an even number p $(2 \leq p \leq 2n-1)$ such that for every $C \in \{R, B, G\}$,

$$|C \cap \{x_1, \ldots, x_p\}| \leq \frac{p}{2}, \quad |C \cap \{x_{p+1}, \ldots, x_{2n}\}| \leq \left\lceil \frac{2n-p}{2} \right\rceil.$$

This implies that the line passing through v and x_p partitions $X \setminus \{v, x_p\}$ into $Left = \{x_2(= a), x_3, \ldots, x_{p-1}\}$ and $Right = \{x_{p+1}, \ldots, x_{2n-1}, x_{2n}(= b)\}$ as shown in Fig. 5 so that $a \in Left$, $b \in Right$, $|Left \cup \{v, x_p\}| = p$, $|Right| = 2n-p$, and for every $C \in \{R, B, G\}$,

$$|C \cap (Left \cup \{v, x_p\})| \leq \frac{p}{2}, \quad |C \cap Right| \leq \left\lceil \frac{2n-p}{2} \right\rceil.$$

Since p is even, by applying the inductive hypothesis to each of $Left \cup \{v, x_p\}$ and $Right$, we can obtain the desired Perfect Matching.

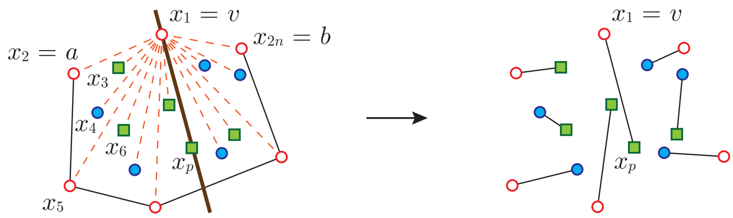

Fig. 5. A balanced partition and the desired Perfect Matching.

4 Proof of Theorem 1.4 by using Lemma 1.8

We first prove the following proposition, which is a stronger version of Theorem 1.3. Our proof of this proposition is also another proof of Theorem 1.3.

Proposition 4.1. *Let R and B be sets of red and blue points in the plane, respectively. Assume that no three points of $X = R \cup B$ are collinear. Let v be a vertex on $conv(X)$. If either of the following conditions (i), (ii), or (iii) holds:*

(i) $|B| = 1$, $1 \leq |R| \leq 3$, *and* $v \in R$,
(ii) $2 \leq |B|$, $|R| = |B| + 2$, *and* $v \in R$,
(iii) $2 \leq |B| \leq |R| \leq |B| + 1$,

then there exists a non-crossing alternating geometric spanning 3*-tree T on X such that* $\deg_T(v) = 1$.

Proof. We briefly call an alternating geometric spanning 3-tree *a Spanning* 3*-Tree*. If Condition (i) holds then the star $K_{1,|R|}$ whose center is blue is the desired Spanning 3-Tree.

Hence, we may assume that (ii) or (iii) holds. Set $n = |X|$. We prove the proposition by induction on n. By the assumption of the proposition, $n \geq 4$. If $n = 4$ then $|R| = |B| = 2$. Thus, there exists a non-crossing alternating geometric matching $M = \{va, bc\}$ where a and b have distinct colors. Then, the path $vabc$ is the desired Spanning 3-Tree.

For $n \geq 5$, we suppose that the proposition holds for at most $n-1$ points. The outline of the proof is that we will find a Spanning 3-Tree on $X - v$ and connect v and a point with degree at most 2 in the tree. We consider the following two cases depending on the colors of the two neighbors of v on $conv(X)$.

Case 1. *v and some neighbor vertex u of v on $conv(X)$ have distinct colors.*

Subcase 1.1. *Condition (ii) holds.*

Since $v \in R$ and $|R| = |B| + 2$, $X - v = (R - v) \cup B$ and $|R - v| = |B| + 1$. Since $2 \leq |B|$, we have $2 \leq |B| \leq |R - v| \leq |B| + 1$. Thus, $R - v$ and B satisfy Condition (iii). Hence, since u is a vertex on $conv(X - v)$, we can apply the inductive hypothesis to $R - v$, B, and u. Then there exists a Spanning 3-Tree T_1 on $(R - v) \cup B$ such that $\deg_{T_1}(u) = 1$. Therefore, $T = T_1 + vu$ is the desired Spanning 3-Tree on X.

Subcase 1.2. *Condition (iii) holds and $v \in R$.*

$3 \leq |R|$ since $n \geq 5$. Thus, $2 \leq |R| - 1 \leq |R - v|$. By Condition (iii), $|R - v| = |R| - 1 \leq |B| \leq |R| = |R - v| + 1$. Hence, we have $2 \leq |R - v| \leq |B| \leq |R - v| + 1$. Thus, $R - v$ and B satisfy Condition (iii). Hence, since u is a vertex on $conv(X - v)$, we can apply the inductive hypothesis to $R - v$, B, and u. Then there exists a Spanning 3-Tree T_1 on $(R - v) \cup B$ such that $\deg_{T_1}(u) = 1$. Therefore, $T = T_1 + vu$ is the desired Spanning 3-Tree on X.

Subcase 1.3. *Condition (iii) holds and $v \in B$.*

If $|B| = 2$ then $2 \le |R| \le 3$ and $|B - v| = 1$. Thus, since $u \in R$, R and $B - v$ satisfy Condition (i). If $3 \le |B|$ then $2 \le |B - v|$. By Condition (iii), we have $2 \le |B - v| \le |B| \le |R| \le |B| + 1 = |B - v| + 2$. Thus, since $u \in R$, R and $B - v$ satisfy Condition (ii) or (iii). Hence, since u is a vertex on $conv(X - v)$, we can apply the inductive hypothesis to R, $B - v$, and u. Then there exists a Spanning 3-Tree T_1 on $R \cup (B - v)$ such that $\deg_{T_1}(u) = 1$. Therefore, $T = T_1 + vu$ is the desired Spanning 3-Tree on X.

Case 2. *v and its two neighbor vertices on $conv(X)$ have the same color.*

By a suitable rotation of the plane, we may assume that v is the highest vertex on $conv(X)$, and a and b are the left and the right vertices on $conv(X)$ adjacent to v, respectively.

Subcase 2.1. *$v, a, b \in R$.*

We sort all the points of $X - v$ with respect to their counterclockwise angle from the ray emanating from v and passing through a, and denote the sorted sequence by $(x_1, x_2, \ldots, x_{n-1})$ so that $x_1 = a$ and $x_{n-1} = b$. Since $2 \le |B| \le |R| \le |B| + 2$, we have $1 \le |B| - 1 \le |R - v| \le |B| + 1$, which implies $||R - v| - |B|| \le 1$. Thus, in the sequence, each color appears on at most $\lceil (n-1)/2 \rceil$ points. Since the two end-points x_1 and x_{n-1} have the same color, namely red, by Lemma 1.8, there exists an even number p $(2 \le p \le n - 2)$ such that $x_p \in B$ and for every $C \in \{R, B\}$,

$$|C \cap \{x_1, \ldots, x_p\}| \le \frac{p}{2}, \quad |C \cap \{x_{p+1}, \ldots, x_{n-1}\}| \le \left\lceil \frac{n-1-p}{2} \right\rceil.$$

This implies that the line passing through v and x_p partitions $X \setminus \{v, x_p\}$ into $Left = \{x_1(= a), x_2, \ldots, x_{p-1}\}$ and $Right = \{x_{p+1}, \ldots, x_{n-2}, x_{n-1}(= b)\}$ as shown in Fig. 6 so that $a \in Left$, $b \in Right$, $|Left \cup \{x_p\}| = p$, $|Right| = n-1-p$, and for every $C \in \{R, B\}$,

$$|C \cap (Left \cup \{x_p\})| \le \frac{p}{2}, \quad |C \cap Right| \le \left\lceil \frac{n-1-p}{2} \right\rceil. \tag{1}$$

Here, we will find two Spanning 3-Trees T_1 and T_2 on $Left \cup \{x_p\}$ and $Right \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$ and $\deg_{T_2}(x_p) = 1$, respectively, and connect the red point v to the blue point x_p.

First, set $W = R \cap Left$ and $K = (B \cap Left) \cup \{x_p\}$. Since p is even, $|K| = |W|$. Hence, since $x_p \in K$ is a vertex on $conv(K \cup W)$, we can apply the inductive hypothesis to K, W, and x_p. Then there exists a Spanning 3-Tree T_1 on $Left \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$.

Next, set $W = R \cap Right$ and $K = (B \cap Right) \cup \{x_p\}$. By the inequality (1), $||W| - (|K| - 1)| = ||R \cap Right| - |B \cap Right|| \le 1$. Thus, we have $-1 \le |W| - (|K| - 1) \le 1$, that is, $0 \le |K| - |W| \le 2$. Then, either of the following

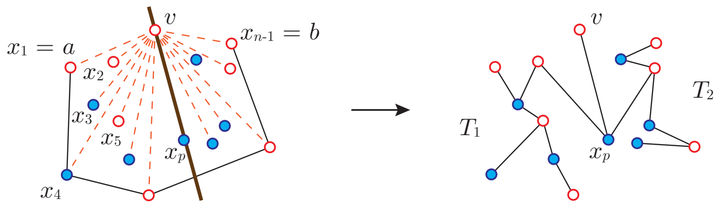

Fig. 6. A balanced partition and the desired Spanning 3-tree.

conditions (i), (ii), or (iii) holds: (i) $|W| = 1$, $1 \leq |K| \leq 3$, and $x_p \in K$, (ii) $2 \leq |W|$, $|K| = |W| + 2$, and $x_p \in K$, (iii) $2 \leq |W| \leq |K| \leq |W| + 1$. Hence, since $x_p \in K$ is a vertex on $conv(K \cup W)$, we can apply the inductive hypothesis to K, W, and x_p. Then there exists a Spanning 3-Tree T_2 on $Right \cup \{x_p\}$ such that $\deg_{T_2}(x_p) = 1$.

Consequently, $T = T_1 + T_2 + vx_p$ is the desired Spanning 3-tree on X.

Subcase 2.2. $v, a, b \in B$.

We sort all the points of X with respect to their counterclockwise angle from the ray emanating from v and passing through a, and denote the sorted sequence by $(x_1, x_2, \ldots, x_n)$ so that $x_1 = v$, $x_2 = a$, and $x_n = b$. Note that in this subcase R and B satisfy Condition (iii) since $v \in B$, that is, $2 \leq |B| \leq |R| \leq |B| + 1$. Thus, in the sequence, each color appears on at most $\lceil n/2 \rceil$ points. Since the two end-points x_1 and x_n have the same color, namely blue, by Lemma 1.8, there exists an even number p $(2 \leq p \leq n-1)$ such that $x_p \in R$ and for every $C \in \{R, B\}$,

$$|C \cap \{x_1, \ldots, x_p\}| \leq \frac{p}{2}, \quad |C \cap \{x_{p+1}, \ldots, x_n\}| \leq \left\lceil \frac{n-p}{2} \right\rceil.$$

This implies that the line passing through v and x_p partitions $X \setminus \{v, x_p\}$ into $Left = \{x_2(= a), x_3, \ldots, x_{p-1}\}$ and $Right = \{x_{p+1}, \ldots, x_{n-1}, x_n(= b)\}$ as shown in Fig. 7 so that $a \in Left$, $b \in Right$, $|Left \cup \{x_1(= v), x_p\}| = p$, $|Right| = n - p$, and for every $C \in \{R, B\}$,

$$|C \cap (Left \cup \{v, x_p\})| \leq \frac{p}{2}, \quad |C \cap Right| \leq \left\lceil \frac{n-p}{2} \right\rceil. \tag{2}$$

Here, we will find two Spanning 3-Trees T_1 and T_2 on $Left \cup \{x_p\}$ and $Right \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$ and $\deg_{T_2}(x_p) = 1$, respectively, and connect the blue point v to the red point x_p.

First, set $W = (R \cap Left) \cup \{x_p\}$ and $K = B \cap Left$. Since $Left \cup \{x_1(= v), x_p\}$ has even points and $x_1(= v)$ is blue, $|W| = |K| + 1$. Hence, since $x_p \in W$ is a vertex on $conv(W \cup K)$, we can apply the inductive hypothesis to W, K, and x_p. Then there exists a Spanning 3-Tree T_1 on $Left \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$.

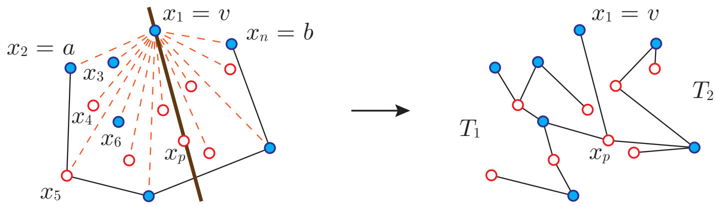

Fig. 7. A balanced partition and the desired Spanning 3-tree.

Next, set $W = (R \cap \mathit{Right}) \cup \{x_p\}$ and $K = B \cap \mathit{Right}$. By the inequality (2), $|(|W| - 1) - |K|| = ||R \cap \mathit{Right}| - |B \cap \mathit{Right}|| \leq 1$. Thus, we have $-1 \leq (|W| - 1) - |K| \leq 1$, that is, $0 \leq |W| - |K| \leq 2$. Then, either of the following conditions (i), (ii), or (iii) holds: (i) $|K| = 1$, $1 \leq |W| \leq 3$, and $x_p \in W$, (ii) $2 \leq |K|$, $|W| = |K| + 2$, and $x_p \in W$, (iii) $2 \leq |K| \leq |W| \leq |K| + 1$. Hence, since $x_p \in W$ is a vertex on $conv(W \cup K)$, we can apply the inductive hypothesis to W, K, and x_p. Then there exists a Spanning 3-Tree T_2 on $\mathit{Right} \cup \{x_p\}$ such that $\deg_{T_2}(x_p) = 1$.

Consequently, $T = T_1 + T_2 + vx_p$ is the desired Spanning 3-tree on X. □

Now, we will prove Theorem 1.4. If $|X| \leq 3$ then the theorem is true. Thus, we may assume that $|X| \geq 4$. Instead of Theorem 1.4 with $|X| \geq 4$, we prove the following stronger Proposition 4.2 by using Lemma 1.8 and Proposition 4.1.

Proposition 4.2. *Let R, B, G be sets of red, blue, and green points in the plane, respectively. Assume that no three points of $X = R \cup B \cup G$ are collinear. $|X| \geq 4$. Let v be a vertex on $conv(X)$. If each color appears on at most $\lceil |X|/2 \rceil$ points, then there exists a non-crossing properly colored geometric spanning 3-tree T on X such that $\deg_T(v) = 1$.*

Proof. Set $n = |X|$. We briefly call a properly colored geometric spanning 3-tree *a Spanning 3-Tree*. If there are exactly two colors then the proposition holds by Proposition 4.1. Thus, we may assume that $R \neq \emptyset$, $B \neq \emptyset$, $G \neq \emptyset$.

Suppose that the number of points colored with some color is exactly $\lceil n/2 \rceil$, say $|R| = \lceil n/2 \rceil$. Then $|B \cup G| = n - |R| = \lfloor n/2 \rfloor \leq \lceil n/2 \rceil$. Set $W = R$ and $K = B \cup G$. Then, $2 \leq |K| \leq |W| \leq |K| + 1$ since $n \geq 4$. Thus, we can apply Proposition 4.1 to W, K, and v. Then there exists a Spanning 3-Tree T with $\deg_T(v) = 1$ on $X = W \cup K$, which is the desired tree. Therefore, we have the following claim.

Claim 1. *We may assume that each color appears on at most $\lceil n/2 \rceil - 1$ points.*

We prove Proposition 4.2 by induction on n. If $n = 4$ then we may assume that $|R| = 2$, $|B| = 1$, and $|G| = 1$ by the symmetry of the colors. Set $W = R$ and $K = B \cup G$. Then, $2 \leq |W| = |K|$. Thus, we can apply Proposition 4.1 to

W, K, and v. Then there exists a Spanning 3-Tree T on $X = W \cup K$ such that $\deg_T(v) = 1$, which is the desired tree.

For $n \geq 5$, we suppose that the proposition holds for at most $n-1$ points. The outline of the proof is that we will find a Spanning 3-Tree on $X - v$ and connect v and a point with degree at most 2 in the tree. We consider the following two cases depending on the colors of the two neighbors of v on $conv(X)$. By the symmetry of the colors, we may assume that $v \in R$.

Case 1. *v and some neighbor vertex u of v on $conv(X)$ have distinct colors, namely, $u \notin R$.*

$|X-v| \geq 4$. By Claim 1, for every $C \in \{R, B, G\}$, $|C-v| \leq |C| \leq \lceil n/2 \rceil - 1 \leq \lceil |X-v|/2 \rceil$ points. Hence, since u is a vertex on $conv(X-v)$, we can apply the inductive hypothesis to $X - v$ and u. Then there exists a Spanning 3-Tree T_1 on $X - v$ such that $\deg_{T_1}(u) = 1$. Therefore, $T = T_1 + vu$ is the desired Spanning 3-Tree on X.

Case 2. *v and its two neighbor vertices on $conv(X)$ have the same color.*

By a suitable rotation of the plane, we may assume that v is the highest vertex on $conv(X)$, and a and b are the left and the right vertices on $conv(X)$ adjacent to v, respectively. We sort all the points of $X - v$ with respect to their counterclockwise angle from the ray emanating from v and passing through a, and denote the sorted sequence by $(x_1, x_2, \ldots, x_{n-1})$ so that $x_1 = a$ and $x_{n-1} = b$.

By Claim 1, for every $C \in \{R, B, G\}$, $|C-v| \leq |C| \leq \lceil n/2 \rceil - 1 \leq \lceil (n-1)/2 \rceil$ points. Thus, in the sequence, each color appears on at most $\lceil |X-v|/2 \rceil$ points. The two end-points x_1 and x_{n-1} have the same color, namely red. Hence, by Lemma 1.8, there exists an even number p $(2 \leq p \leq n-2)$ such that $x_p \notin R$ and for every $C \in \{R, B, G\}$,

$$|C \cap \{x_1, \ldots, x_p\}| \leq \frac{p}{2}, \quad |C \cap \{x_{p+1}, \ldots, x_{n-1}\}| \leq \left\lceil \frac{n-1-p}{2} \right\rceil.$$

This implies that the line passing through v and x_p partitions $X \setminus \{v, x_p\}$ into $Left = \{x_1(=a), x_2, \ldots, x_{p-1}\}$ and $Right = \{x_{p+1}, \ldots, x_{n-2}, x_{n-1}(=b)\}$ as shown in Fig. 8 so that $a \in Left$, $b \in Right$, $|Left \cup \{x_p\}| = p$, $|Right| = n-1-p$, and for every $C \in \{R, B, G\}$,

$$|C \cap (Left \cup \{x_p\})| \leq \frac{p}{2}, \quad |C \cap Right| \leq \left\lceil \frac{n-1-p}{2} \right\rceil. \tag{3}$$

Here, we will find two Spanning 3-Trees T_1 and T_2 on $Left \cup \{x_p\}$ and $Right \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$ and $\deg_{T_2}(x_p) = 1$, respectively, and connect the red point v to the non-red point x_p. By the symmetry of B and G, we may assume that $x_p \in B$.

First, we will find a Spanning 3-Tree T_1 on $Left \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$. If $|Left \cup \{x_p\}| = 2$ then the path $x_p a$ is the desired Spanning 3-Tree T_1. Thus,

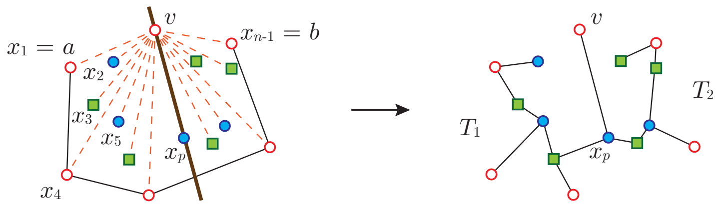

Fig. 8. A balanced partition and the desired Spanning 3-tree.

since p is even, we suppose that $|Left \cup \{x_p\}| \geq 4$. Then, since x_p is a vertex on $conv(Left \cup \{x_p\})$, we can apply the inductive hypothesis to $Left \cup \{x_p\}$ and x_p. Then there exists a Spanning 3-Tree T_1 on $Left \cup \{x_p\}$ such that $\deg_{T_1}(x_p) = 1$.

Next, we will find a Spanning 3-Tree T_2 on $Right \cup \{x_p\}$ such that $\deg_{T_2}(x_p) = 1$. If $|Right \cup \{x_p\}| = 2$ then the path $x_p b$ is the desired Spanning 3-Tree T_2. If $|Right \cup \{x_p\}| = 3$ then $n - 1 - p = |Right| = 2$. Thus, by the inequality (3), $|C \cap Right| \leq 1$. Thus implies that $Right \cup \{x_p\}$ has one red point b, one blue point x_p, and one blue or green point g. Hence, the path $x_p bg$ is the desired Spanning 3-Tree T_2.

Thus, we suppose that $|Right \cup \{x_p\}| \geq 4$. If for every $C \in \{R, B, G\}$, $|C \cap (Right \cup \{x_p\})| \leq \lceil (n-p)/2 \rceil$, then, since x_p is a vertex on $conv(Right \cup \{x_p\})$, we can apply the inductive hypothesis to $Right \cup \{x_p\}$ and x_p. Then there exists a Spanning 3-Tree T_2 on $Right \cup \{x_p\}$ such that $\deg_{T_2}(x_p) = 1$.

Hence, we suppose that for some $C \in \{R, B, G\}$, $|C \cap (Right \cup \{x_p\})| > \lceil (n-p)/2 \rceil$. Since x_p is blue, by the inequality (3), we have

$$\left\lceil \frac{n-p}{2} \right\rceil < |B \cap (Right \cup \{x_p\})| \leq \left\lceil \frac{n-1-p}{2} \right\rceil + 1 = \left\lceil \frac{n-p+1}{2} \right\rceil .$$

This implies that $n - p$ is even and $|B \cap (Right \cup \{x_p\})| = (n-p)/2 + 1$. Set $W = B \cap (Right \cup \{x_p\})$ and $K = (Right \cup \{x_p\}) \setminus W$. Then,

$$|K| = |Right \cup \{x_p\}| - |W| = (n - 1 - p + 1) - (\frac{n-p}{2} + 1) = \frac{n-p}{2} - 1$$

Thus, $|W| = |K| + 2$. Hence, since $x_p \in W$ is a vertex on $conv(W \cup K)$, we can apply Proposition 4.1 to W, K, and x_p. Then there exists a Spanning 3-Tree T_2 on $Right \cup \{x_p\}$ such that $\deg_{T_2}(x_p) = 1$.

Consequently, $T = T_1 + T_2 + vx_p$ is the desired Spanning 3-tree on X. □

5 Proof of Theorem 1.7 by using Lemma 1.8

We can prove Theorem 1.7 in the same way as the proof of Theorem 1.2 in Section 3. We briefly call a non-crossing properly colored geometric perfect matching (such that each edge is an L-line segment) *a Perfect Matching*. Set $2n = |X|$.

We prove the theorem by induction on n. If $n = 1$ then the theorem is true. For $n \geq 2$, we suppose that the theorem holds for $2(n-1)$ points.

Suppose that $|C| = n$ for some $C \in \{R, B, G\}$. Set $W = C$ and $K = (R \cup B \cup G) \setminus C$. We recolor all the points of W with white, and all the points of K with black. Then there exists the desired Perfect Matching by applying Theorem 1.6 to $W \cup K$.

Hence, we may assume that

$$|C| \leq n - 1 \qquad \text{for every } C \in \{R, B, G\}.$$

The *rectangular hull* of X is the smallest closed rectangular enclosing X. We denote by $rect(X)$ the boundary of the rectangular hull of X. We call a point in $X \cap rect(X)$ a *vertex* on $rect(X)$.

Suppose that some two adjacent vertices u and v on $rect(X)$ have distinct colors. By our assumption, we have

$$|C \cap (X - \{u, v\})| \leq |C| \leq n - 1 \qquad \text{for every } C \in \{R, B, G\}.$$

Thus, since $|X - \{u, v\}| = 2(n-1)$, we can apply the inductive hypothesis to $X - \{u, v\}$ and there exists a Perfect Matching on $X - \{u, v\}$. By adding an L-line segment uv on $rect(X)$ to this matching, we can obtain the desired Perfect Matching.

Therefore, we may assume that all the vertices on $rect(X)$ have the same color. Let a and b be the left and the right vertices on $rect(X)$, respectively. We sort all the points of X by their horizontal coordinate, and denote the sorted sequence by $(x_1, x_2, \ldots, x_{2n})$ so that $x_1 = a$ and $x_{2n} = b$. Since the two end-points x_1 and x_{2n} have the same color, by Lemma 1.8, there exists an even number p ($2 \leq p \leq 2n - 1$) such that for every $C \in \{R, B, G\}$,

$$|C \cap \{x_1, \ldots, x_p\}| \leq \frac{p}{2}, \quad |C \cap \{x_{p+1}, \ldots, x_{2n}\}| \leq \left\lceil \frac{2n-p}{2} \right\rceil.$$

This implies that the vertical line passing through x_p partitions $X \setminus \{x_p\}$ into $Left = \{x_1(= a), x_2, \ldots, x_{p-1}\}$ and $Right = \{x_{p+1}, \ldots, x_{2n-1}, x_{2n}(= b)\}$ so that $a \in Left$, $b \in Right$, $|Left \cup \{x_p\}| = p$, $|Right| = 2n - p$, and for every $C \in \{R, B, G\}$,

$$|C \cap (Left \cup \{x_p\})| \leq \frac{p}{2}, \quad |C \cap Right| \leq \left\lceil \frac{2n-p}{2} \right\rceil.$$

Since p is even, by applying the inductive hypothesis to each of $Left \cup \{x_p\}$ and $Right$, we can obtain the desired Perfect Matching.

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