Spanning trees whose stems have at most k leaves

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Abstract

For a tree T, the set of leaves of T is denoted by $Leaf(T)$, and the subtree $T - Leaf(T)$ is called the stem of T. We prove that if a connected graph G satisfies $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has at most k leaves, where $\sigma_{k+1}(G)$ denotes the minimum degree sum of $k+1$ independent vertices of G. Moreover, we show that the condition on σ_{k+1} is sharp. Also we give another similar sufficient condition for a claw-free graph to have such a spanning tree.

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G.

Let T be a tree. A vertex of T with degree one is often called a *leaf*, and the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the *stem* of T and denoted by $Stem(T)$. A spanning tree with specified stem was first considered in [3].

Let $k \geq 2$ be an integer. A tree whose maximum degree at most k is called a *k-tree*. Similarly, a stem whose maximum degree at most k is called

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a k-*stem*, and a tree whose stem is k-tree is called a *tree with k-stem* (see Figure 1). For a graph G, $\sigma_k(G)$ denotes the minimum degree sum of k independent vertices of G. The following theorem gives a sufficient condition using $\sigma_k(G)$ for a graph G to have a spanning tree with k-stem.

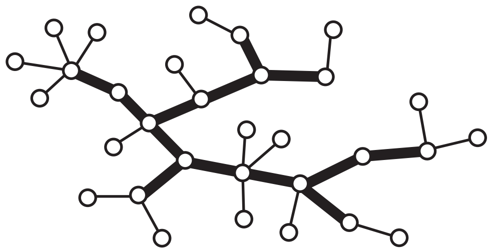

Figure 1: A tree with 3-stem, which is also a tree with 6-ended stem.

Theorem 1 (Kano, Tsugaki and Yan [3]) *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree with k-stem.*

A tree having at most k leaves is called a k-*ended tree*, and a stem having at most k leaves is called a k-*ended stem*. A tree whose stem has at most k leaves is called a *tree with k-ended stem* (see Figure 1). In [4], Tsugaki and Zhang give a sufficient condition using $\sigma_3(G)$ for a graph to have a spanning tree with k-ended stem as follows.

Theorem 2 (Tsugaki and Zhang [4]) *Let G be a connected graph and $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree with k-ended stem.*

Note that if G has no set of $k + 1$ independent vertices, then we define $\sigma_{k+1}(G) = \infty$. In this paper, we prove the following two theorems.

Theorem 3 *Let G be a connected graph and $k \geq 2$ be an integer. If*

$$\sigma_{k+1}(G) \geq |G| - k - 1, \tag{1}$$

then G has a spanning tree with k-ended stem.

Theorem 4 *Let G be a connected claw-free graph and $k \geq 2$ be an integer. If*

$$\sigma_{k+1}(G) \geq |G| - 2k - 1, \tag{2}$$

then G has a spanning tree with k-ended stem.

Let us define $\alpha^4(G)$ as

$$\alpha^4(G) = \max\{|S| : S \subset V(G), d_G(x,y) \geq 4 \text{ for all distinct } x,y \in S\},$$

where $d_G(x,y)$ denotes the distance between x and y in G. By the proof of Theorem 3 in the next section (see Claim 3 in the proof of Theorem 3), we can obtain the next theorem.

Theorem 5 *Let G be a connected graph and $k \geq 2$ be an integer. If $\alpha^4(G) \leq k$, then G has a spanning tree with k-ended stem.*

It is clear that our Theorem 3 implies Theorem 1 since a k-ended stem is a k-stem. Notice that the condition of Theorem 1 is also best possible. Moreover, if the order of G is sufficiently large, then our Theorem 3 implies Theorem 2. Namely, if $k = 2$, then (1) is equivalent to the condition of Theorem 2, and if $k \geq 3$, $\sigma_3(G) \geq |G| - 2k + 1$ and $|G| \geq 2k + 8$, then

$$\begin{aligned}\sigma_{k+1}(G) &\geq \frac{(k+1)\sigma_3(G)}{3} \geq \frac{(k+1)(|G| - 2k + 1)}{3} \\ &\geq |G| - k + 1. \qquad \text{(by } |G| \geq 2k + 8 \text{ and } k \geq 3)\end{aligned}$$

Hence the condition of Theorem 2 and $|G| \geq 2k + 8$ imply (1).

A sufficient condition for a graph to have a spanning k-ended tree were obtained as follows.

Theorem 6 (Broersma and Tuinstra [2]) *Let $k \geq 2$ be an integer and G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

Some other results on spanning trees can be found in a survey paper [5] and book [1]. We conclude this section by showing that the two conditions in Theorems 3 and 4 are sharp.

Let $k \geq 2$ and $m \geq 3$ be integers, and let $D_1, D_2, \ldots, D_{k+1}$ be $k+1$ disjoint copies of the complete graph K_m of order m. Let $w, v_1, \ldots, v_{k+1}$ be $k+2$ vertices not contained in $D_1 \cup D_2 \cup \cdots \cup D_{k+1}$. Join w to all the vertices of $D_1 \cup D_2 \cup \cdots \cup D_{k+1}$ by edges, and join v_i to all the vertices of D_i by edges for every $1 \leq i \leq k+1$. Let G_1 denote the resulting graph (see Figure 2). Then $|G_1| = (k+1)m + k + 2$ and $\sigma_{k+1}(G_1) = (k+1)m = |G_1| - k - 2$, but G_1 has no spanning tree with k-ended stem. Hence the condition on $\sigma_{k+1}(G)$ in Theorem 3 is sharp.

Let $k \geq 2$ and $m \geq 3$ be integers. Let H be a copy of the complete graph K_{k+1} with vertex set $V(H) = \{u_1, u_2, \ldots, u_{k+1}\}$, and let $D_1, D_2, \ldots, D_{k+1}$

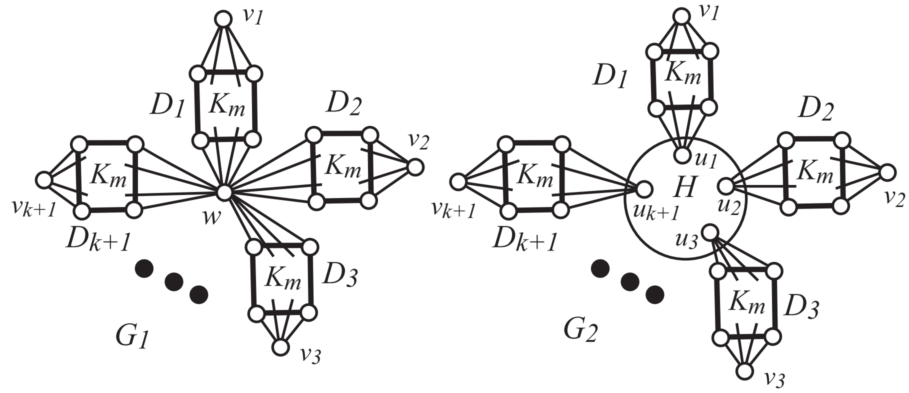

Figure 2: The graph G_1 that has no spanning tree with k-ended stem and satisfies $\sigma_{k+1}(G) = |G| - k - 2$.

be $k+1$ disjoint copies of the complete graph K_m. We construct a graph G_2 as follows: $V(G_2) = V(H) \cup V(D_1) \cup \cdots \cup V(D_{k+1}) \cup \{v_1, \ldots, v_{k+1}\}$ (disjoint union). For every $1 \leq i \leq k+1$, join u_i and v_i to all the vertices of D_i. Then the resulting graph is G_2 (see Figure 2). It is immediate that $|G_2| = k+1+(k+1)(m+1)$ and G_2 is claw-free. Moreover,

$$\sigma_{k+1}(G_2) = \sum_{i=1}^{k+1} \deg_{G_2}(v_i) = (k+1)m = |G_2| - 2k - 2.$$

It is easy to see that G_2 has no spanning tree with k-ended stem. Therefore the condition on $\sigma_{k+1}(G)$ in Theorem 4 is sharp.

2 Proofs of Theorems 3, 4 and 5

In this section we first prove Theorem 3. As we mentioned before, the Theorem 5 follows from Claim 3 in the following proof of Theorem 3. For a vertex v of a graph G, let $N_G(v)$ denote the neighborhood of v in G. Thus $\deg_G(v) = |N_G(v)|$.

Proof of Theorem 3. If $k = 2$, the theorem follows from Theorem 1 since a spanning tree with 2-stem is a spanning tree whose stem is a path. Thus we may assume $k \geq 3$.

Suppose that G satisfies the condition (1) in Theorem 3 but does not have a spanning tree with k-ended stem. Choose a tree T with k-ended stem of G so that

(T1) $|T|$ is as large as possible, and

(T2) $|Stem(T)|$ is as small as possible subject to (T1),

By the choice (T1), we can obtain the following claim.

Claim 1. For every $v \in V(G) - V(T)$, $N_G(v) \subseteq Leaf(T) \cup (V(G) - V(T))$.

Since G is connected and T is not a spanning tree of G, there exist two vertices $v_1 \in V(G) - V(T)$ and $v_2 \in Leaf(T)$ which are adjacent in G. It is immediate that $Stem(T)$ has precisely k leaves since otherwise $T + v_1v_2$ is a larger tree with k-ended stem than T. Let $x_1, x_2, \ldots, x_k$ be the leaves of $Stem(T)$.

Claim 2. For every x_i, $1 \leq i \leq k$, there exists a leaf y_i of T such that y_i is adjacent to x_i in T and satisfies $N_G(y_i) \subseteq leaf(T) \cup \{x_i\}$.

Let x_a be a leaf of $Stem(T)$, where $1 \leq a \leq k$. It is obvious that there exists a leaf of T which is adjacent to x_a in T. Assume that every leaf y of T adjacent to x_a satisfies $N_G(y) \cap (Stem(T) - \{x_a\}) \neq \emptyset$. Then for every leaf y of T adjacent to x_a in T, remove the edge yx_a from T and add an edge yz of G, where z is a vertex of $N_G(y) \cap (Stem(T) - \{x_a\})$. Denote the resulting tree of G by T_1. Then T_1 is a tree such that $|T_1| = |T|$ and $Stem(T_1) = Stem(T) - \{x_a\}$, which contradicts the condition (T2). Therefore, for every x_i, there exists a leaf y_i adjacent to x_i such that $N_G(y_i) \cap (Stem(T) - \{x_i\}) = \emptyset$. Hence the claim holds.

Let $\{v_1, y_1, y_2, \ldots, y_k\}$ be a set of $k+1$ vertices, where y_i is a vertex given in Claim 2.

Claim 3. For any two distinct vertices $y, z \in \{v_1, y_1, y_2, \ldots, y_k\}$, $d_G(y, z) \geq 4$.

First, we show that $d_G(v_1, y_i) \geq 4$ for every $1 \leq i \leq k$. Let $P(v_1, y_a)$ be a shortest path in G connecting v_1 and y_a. If all the vertices of $P(v_1, y_a)$ are contained in $Leaf(T) \cup (V(G) - V(T))$, then add $P(v_1, y_a)$ to T and remove the edges of T joining $P(v_1, y_a) \cap Leaf(T)$ to $Stem(T)$ except the edge y_ax_a. Then the resulting tree of G is a tree with k-ended stem of order greater than $|T|$, which contradicts the condition (T1). Hence $P(v_1, y_a)$ passes through $Stem(T)$. Choose the vertex s of $Stem(T) \cap P(v_1, y_a)$ that is nearest to v_1.

If $s = x_r$ for some $1 \leq r \leq k$, then add $P(v_1, x_r)$ to T and remove the edges of T joining $P(v_1, x_r) \cap Leaf(T)$ to $Stem(T)$ except x_ry_r. Then the resulting tree is a tree with k-ended stem of order greater than $|T|$, which is a contradiction. Thus $s \in Stem(T) - \{x_1, \ldots, x_k\}$. By Claims 1 and 2, $d_G(v_1, s) \geq 2$ and $d_G(s, y_a) \geq 2$, which implies $d_G(v_1, y_a) = d_G(v_1, s) + d_G(s, y_a) \geq 4$.

Next, we show that $d_G(y_i, y_j) \geq 4$ for all $1 \leq i < j \leq k$. Let $P(y_b, y_c)$ be a shortest path connecting y_b and y_c in G. Assume first that all the vertices of $P(y_b, y_c)$ are contained in $Leaf(T) \cup (V(G) - V(T))$. Then add $P(y_b, y_c)$ to T and remove the edges of T joining $P(y_b, y_c) \cap Leaf(T)$ to $Stem(T)$ except the edges $y_b x_b$ and $y_c x_c$. Then the resulting subgraph of G includes a unique cycle, which contains an edge e_1 of $Stem(T)$ incident with a vertex of degree at least three in $Stem(T)$. By removing the edge e_1 and by adding an edge $v_1 v_2$, we obtain a tree with k-ended stem of order greater than $|T|$, which contradicts (T1). Hence $P(y_b, y_c)$ passes through a vertex s of $Stem(T)$.

If $s \notin Stem(T) - \{x_b, x_c\}$, then $d_G(y_b, s) \geq 2$ and $d_G(s, y_c) \geq 2$ by Claim 2, and thus $d_G(y_b, y_c) = d_G(y_i, s) + d_G(s, y_j) \geq 4$. So we may assume that $s = x_b$ by symmetry. Namely, $P(y_b, y_c) = y_b x_b + P(x_b, y_c)$. If $P(x_b, y_c)$ passes through a vertex, say t, of $Stem(T) - \{x_c\}$, then $d_G(y_b, y_c) = d_G(y_b, x_b) + d_G(x_b, t) + d_G(t, y_c) \geq 4$ by Claim 2. Thus $P(x_b, y_c)$ does not pass through $Stem(T) - \{x_c\}$.

Add $P(x_b, y_c)$ to T and remove the edges of T joining $P(x_b, y_c) \cap Leaf(T)$ to $Stem(T)$ except $y_c x_c$. Then the resulting subgraph of G includes a unique cycle, which contains an edge e_2 of $Stem(T)$ incident with a vertex degree at least three in $Stem(T)$. By removing the edge e_2 and by adding an edge $v_1 v_2$, we obtain a tree with k-ended stem of order greater than $|T|$, which contradicts (T1). Therefore Claim 3 holds.

By Claim 3, we can obtain the following Claim 4.

Claim 4. $N_G(v_1) \cap N_G(y_i) = \emptyset$ for $1 \leq i \leq k$ and $N_G(y_i) \cap N_G(y_j) = \emptyset$ for $1 \leq i \neq j \leq k$.

Let $Y = \{y_1, \ldots, y_k\}$. By Claims 1-3, we have

$$N_G(v_1) \subseteq (V(G) - V(T) - \{v_1\}) \cup (Leaf(T) - Y)$$
$$\bigcup_{i=1}^{k} N_G(y_i) \subseteq (Leaf(T) - Y) \cup \{x_1, \ldots, x_k\}.$$

Hence by letting $m = |N_G(v_1) \cap (Leaf(T) - Y)|$, we have by Claim 4 that

$$\begin{aligned} &\deg_G(v_1) + \deg_G(y_1) + \ldots + \deg_G(y_k) \\ \leq &|G| - |T| - 1 + m + |Leaf(T)| - k - m + k \\ = &|G| - |T| - 1 + |Leaf(T)| \\ = &|G| - |Stem(T)| - 1. \end{aligned} \tag{3}$$

Since $Stem(T)$ is a tree having k leaves, it follows that $|Stem(T)| \geq k+1$.

Therefore, we obtain

$$\begin{aligned} &\deg_G(v_1) + \deg_G(y_1) + \ldots + \deg_G(y_k) \\ \le &|G| - |Stem(T)| - 1 \le |G| - k - 2. \end{aligned}$$

This contradicts the assumption $\sigma_{k+1}(G) \ge |G| - k - 1$. Consequently Theorem 3 is proved. $\square$

We next prove Theorem 4.

Proof of Theorem 4. Suppose that G has no spanning tree with k-ended stem. Choose a tree T with k-ended stem of G as in the proof of Theorem 3, and let v_1, v_2, x_i, y_i, $1 \le i \le k$, be the same vertices as in the proof of Theorem 3. In particular, all x_i's are the leaves of $Stem(T)$, and all y_i's are leaves of T. Moreover, all the claims in the proof of Theorem 3 hold.

Since v_2 is a leaf of T, there exists a vertex v_3 in $Stem(T)$ adjacent to v_2 in T. For every $1 \le i \le k$, $v_2x_i \notin E(G)$ since $d_G(v_1, y_i) \ge 4$. Thus v_3 is not a leaf of $Stem(T)$ since $\{x_1, \ldots, x_k\}$ is the set of leaves of $Stem(T)$, and so v_3 has degree at least two in $Stem(T)$. Let s be a vertex of $Stem(T)$ adjacent to v_3.

Assume that $v_3x_b \in E(T)$ for some $1 \le b \le k$. Since G is claw-free, we have $sv_2 \in E(G)$ or $sx_b \in E(G)$. In each case, $T + sv_2 - sv_3 + v_1v_2$ or $T + sx_b - sv_3 + v_1v_2$ is a tree with k-ended stem and larger than T. Note that x_b is not a leaf of $Stem(T + sx_b - sv_3 + v_1v_2)$. Hence v_3 is not adjacent to any leaf of $Stem(T)$.

Assume that $|Stem(T)| \le 2k$. Since $Stem(T)$ has exactly k leaves and v_3 is not adjacent to any leaf of $Stem(T)$, there must exist two distinct leaves x_c and x_d in $Stem(T)$ that are adjacent to the same vertex t_1 of $Stem(T)$. Let t_2 be a vertex of $Stem(T)$ adjacent to t_1. Since G is claw-free and x_c and x_d are not adjacent by $d_G(y_c, y_d) \ge 4$, we obtain $t_2x_c \in E(G)$ or $t_2x_d \in E(G)$. Then $T + t_2x_c - t_1t_2 + v_1v_2$ or $T + t_2x_d - t_1t_2 + v_1v_2$ is a tree with k-ended stem and larger than T, a contradiction. Note that x_c is not a leaf of $T + t_2x_c - t_1t_2 + v_1v_2$ and that x_d is not a leaf of $T + t_2x_d - t_1t_2 + v_1v_2$. Hence $|Stem(T)| \ge 2k + 1$.

Therefore, by (3) we obtain

$$\begin{aligned} &\deg_G(v) + \deg_G(y_1) + \ldots + \deg_G(y_k) \\ = &|G| - |Stem(T)| - 1 \le |G| - 2k - 2. \end{aligned}$$

This contradicts the condition (2) in Theorem 4. Consequently, Theorem 4 is proved. $\square$

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