m-dominating k-ended trees of graphs

Mikio Kano[1]*, Masao Tsugaki[2]† and Guiying Yan[3]‡

[1] Department of computer and information sciences
Ibaraki University, Hitachi, Ibaraki, Japan
kano@mx.ibaraki.ac.jp
http://gorogoro.cis.ibaraki.ac.jp

[2,3] Academy of Mathematics and System Science
Chinese Academy of Science, Beijing, P. R. China
tsugaki@amss.ac.cn
yangy@amss.ac.cn

Abstract

Let $k \geq 2$, $l \geq 2$ and $m \geq 0$ be integers, and let G be a connected graph. If there exists a subgraph X of G such that for every vertex v of G, the distance between v and X is at most m, then we say that X m-dominates G. Define $\alpha^l(G) = \max\{|S| : S \subseteq V(G),\ d_G(x,y) \geq l$ for all distinct $x, y \in S\}$, where $d_G(x,y)$ denotes the distance between x and y in G. We prove the following theorem and show that the condition is sharp. If $\alpha^{2(m+1)}(G) \leq k$, then G has a tree that has at most k leaves and m-dominates G. This is a generalization of some related results.

Keywords: tree with few leaves, dominating set, k-ended tree

*Partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C) No.25400187

†Supported by Chinese Academy of Sciences Fellowship for Young International Scientists; Grant No. 2012Y1JA0004.

‡Supported by the National Natural Science of Foundation of China under Grant No. 10721101.

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For two vertices u and v of G, let $d_G(u, v)$ denote the *distance* between u and v in G, which is the length of a shortest path of G connecting u and v. For a subgraph X or a vertex set X of G and a vertex v of G, the distance between v and X is defined to be the minimum value of $d_G(v, x)$ for all $x \in V(X)$ or $x \in X$, and denoted by $d_G(v, X)$. Thus $d_G(v, X) = 0$ implies that v is contained in X.

Let $m \geq 0$ be an integer and X be a subgraph or a vertex set of G. Then the *m-th dominating set* of X, denoted by $Domi^m(X)$, is defined to be the following vertex set of G.

$$Domi^m(X) = \{v \in V(G) \ : \ d_G(v, X) \leq m\}.$$

If all the vertices of a subgraph Y or a vertex set Y of G are included in $Domi^m(X)$, then we say that X *m-dominates* Y.

For an integer $l \geq 2$, the invariant $\alpha^l(G)$ of a graph G is defined as follows:

$$\alpha^l(G) = \max\{|S| : S \subseteq V(G), \ \ d_G(x, y) \geq l \text{ for all distinct } x, y \in S\}.$$

Thus the independence number $\alpha(G)$ is equal to $\alpha^2(G)$. Let T be a tree. An end-vertex of T, which has degree one, is often called a *leaf* of T. A tree is called a *k-ended tree* if it has at most k leaves. On the other hand, a tree whose maximum degree is at most k is called a *k-tree.* The following theorem is our main result of this paper.

Theorem 1 *Let $k \geq 2$ and $m \geq 0$ be integers, and let G be a connected graph. If $\alpha^{2(m+1)}(G) \leq k$, then G has a k-ended tree that m-dominates G.*

We now explain a conjecture and some results related to our theorem. Concerning m-dominating k-ended trees, the following two theorems are known.

Theorem 2 (Broersma [2]) *Let $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If $\alpha^{2(m+1)}(G) \leq l + 1$, then G has a path that m-dominates G, that is, G has a 2-ended tree which m-dominates G.*

Theorem 3 (Win [6]) *Let $k \geq 2$ and $l \geq 1$ be integers, and let G be an l-connected graph. If $\alpha^2(G) \leq k + l - 1$, then G has a k-ended tree that 0-dominates G, which is a spanning k-ended tree of G.*

By Theorems 1, 2 and 3, we propose the following conjecture.

Conjecture 4 *Let $k \geq 2$, $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If $\alpha^{2(m+1)}(G) \leq k + l - 1$, then G has a k-ended tree that m-dominates G.*

Notice that this conjecture is true in the case of $l = 1$ (by Theorem 1), in the case of $k = 2$ (by Theorem 2), and in the case of $m = 0$ (by Theorem 3).

We will show that the condition of Theorem 1 is sharp, and if the Conjecture 4 holds, then its condition is also sharp in some sense. Let $k \geq 2$, $l \geq 1$ and $m \geq 1$ be integers. Let $D_{i,1}, D_{i,2}, \ldots, D_{i,m}$ be disjoint copies of the complete graph K_l of order l, where $1 \leq i \leq k + l$. For every $1 \leq j \leq m - 1$, join all the vertices of $D_{i,j}$ and all the vertices of $D_{i,j+1}$ by edges. Let v_i be a vertex not contained in $D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,m}$, and join v_i and all the vertices of $D_{i,m}$ by edges. We denote by H_i the resulting graph (see Fig. 1). Let H be a complete graph K_l. For every $1 \leq i \leq k + l$, join all the vertices of H and all the vertices of $D_{i,1}$ by edges. Let G_1 denote the resulting graph (see Fig. 1). Then it is easy to see that G_1 is an l-connected graph, and has no k-ended tree that m-dominates G_1. On the other hand, we can see that $\alpha^{2(m+1)}(G_1) = |\{v_i : 1 \leq i \leq k + l\}| = k + l$. Therefore, the conditions in Theorem 1 ($l = 1$) and Conjecture 4 ($l \geq 1$) are sharp in the above sense.

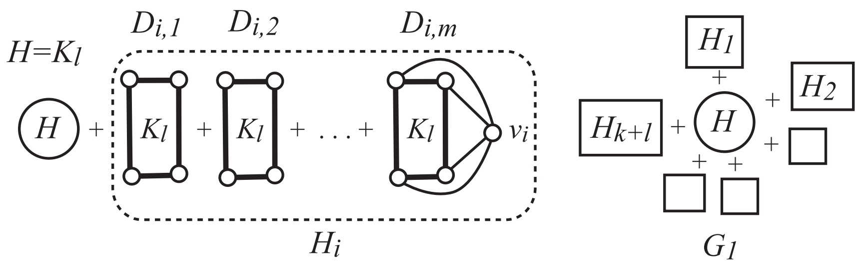

Figure 1: The graphs H, H_i and G_1, where $H = K_l$.

Next, we explain some results related to our Theorem 1. Let x and y be two vertices of a graph G. If $d_G(x,y) \geq 2(m+1)$, then $Domi^m(x)$ and $Domi^m(y)$ are disjoint and there exists no edge joining $Domi^m(x)$ and $Domi^m(y)$. Therefore the following lemma holds.

Lemma 5 *Let $m \geq 0$ be an integer, and let G be a connected graph. If a vertex set S of G satisfies that $|S| \geq 2$ and $d_G(x,y) \geq 2(m+1)$ for all two distinct $x, y \in S$, then $\sum_{x \in S} |Domi^m(x)| < |G|$.*

From Theorem 1 and Lemma 5 we can obtain the following theorem.

Theorem 6 *Let $k \geq 2$ and $m \geq 0$ be integers, and let G be a connected graph. If $\sum_{x \in S} |Domi^m(x)| \geq |G|$ for every independent set $S \subseteq V(G)$ of size $k+1$, then G has a k-ended tree that m-dominates G.*

Proof. Suppose that G satisfies the assumption of Theorem 6, but does not have a k-ended tree that m-dominates G. Then, by Theorem 1, there exists $S \subseteq V(G)$ such that $|S| = k+1$ and $d_G(x,y) \geq 2(m+1)$ for all two distinct vertices x, y of S. By Lemma 5, $\sum_{x \in S} |Domi^m(x)| < |G|$, which contradicts the assumption of Theorem 6. Hence Theorem 6 holds. □

Theorem 6 is a generalization of some known results. We need some notations and definitions to explain it. For a tree T, the set of leaves of T is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of T is called the *stem* of T and denoted by $Stem(T)$. A stem with maximum degree at most k is called a *k-stem*, and so a tree whose stem has maximum degree at most k is called a *tree with k-stem* (see Fig. 2). Notice that a caterpillar is nothing but a tree whose stem is a path. It is obvious that G has a spanning tree whose stem is a k-ended tree (or a k-tree) if and only if G has a k-ended tree (or a k-tree) that 1-dominates G.

For an integer $k \geq 2$, the invariant $\sigma_k(G)$ of a graph G is defined to be the minimum degree sum of k independent vertices of G, that is,

$$\sigma_k(G) = \min\{\sum_{x \in S} \deg_G(x) \ : \ S \subseteq V(G), \ |S| = k, \ S \text{ is independent}\}.$$

The following theorem is the first result on a spanning tree with specified stem.

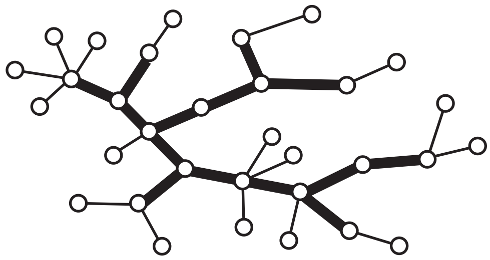

Figure 2: A tree with 3-stem, where bold edges form its stem.

Theorem 7 (Kano, Tsugaki, Yan [3]) *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree with k-stem.*

It is clear that $|Domi^1(x)| = \deg_G(x)+1$ for every vertex x, and a k-ended tree is a k-tree. Therefore Theorem 6 with $m = 1$ implies Theorem 7.

On the other hand, a sufficient condition for a graph to have a spanning tree whose stem is a k-ended tree was obtained as follows.

Theorem 8 (Tsugaki and Zhang [5]) *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree whose stem is a k-ended tree.*

If G satisfies $\sigma_3(G) \geq |G| - 2k + 1$, then $\sigma_{k+1}(G) \geq \sigma_3(G) + (k-2) \geq |G| - 2k + 1 + (k-2) = |G| - k - 1$, which implies that for every independent set S of size $k+1$,

$$\sum_{x \in S} |Domi^1(x)| = \sum_{x \in S} (\deg_G(x) + 1) \geq \sigma_{k+1}(G) + k + 1 \geq |G|.$$

Therefore Theorem 6 with $m = 1$ also implies Theorem 8.

Finally, we explain another representation of our theorem. Let T be a tree. Then let $T_0 = T$ and, for an integer $m \geq 1$, define

$$T_{i+1} = Stem(T_i) = T_i - Leaf(T_i) \qquad \text{for all} \;\; 0 \leq i \leq m-1,$$

where it may happen that $T_j = \emptyset$ for some $1 \leq j \leq m$. The tree T_m is called the *m-th stem* of T and denoted by m-th $Stem(T)$. We need the following two lemmas.

Lemma 9 *Let $m \geq 0$ be an integer, and let T be a tree and R be a subtree of T. Then m-th stem of R is a subtree of m-th stem of T.*

Proof. It follows that $Stem(R)$ is a subtree of $Stem(T)$ since every leaf of T contained in R is a leaf of R. Similarly, $Stem(Stem(R))$ is a subtree of $Stem(Stem(T))$. By repeating this operation, we obtain the lemma. □

Lemma 10 *Let T be a tree and R be a subtree of T. Then the number of leaves of R is less than or equal to that of T.*

Proof. Let $W = \{v \in V(T) : \deg_T(v) \geq 3\}$. Then the number of leaves of T is $\sum_{x \in W}(\deg_T(x) - 2) + 2$. The number of leaves of R is obtained by a similar formula. Hence the lemma holds. □

By Lemmas 9 and 10, we can see the following equivalence relation.

Lemma 11 *Let $k \geq 2$ and $m \geq 0$ be integers, and let G be a graph. Then G has a k-ended tree that m-dominates G if and only if G has a spanning tree whose m-th stem is a k-ended tree.*

Proof. If G has a spanning tree whose m-th stem R is a k-ended tree, then R is a k-ended tree of G that m-dominates G. Therefore, we have only to prove that if G has a k-ended tree that m-dominates G, then G has a spanning tree whose m-th stem is a k-ended tree.

We prove the above statement by the induction on m. The case $m = 0$ is obviously true. Let $m \geq 1$. Suppose that G has a k-ended tree T that m-dominates G. Let $M = \{v \in V(G) : d_G(v, T) = m\}$. Then T is a k-ended tree of $G - M$ that $(m-1)$-dominates $G - M$. Hence, by the induction hypothesis, $G - M$ has a spanning tree S whose $(m-1)$-th stem is a k-ended tree. By the definition of M, $N_G(v) \cap S \neq \emptyset$ for every $v \in M$. Let S' be a spanning tree of G obtained from S by adding an edge joining v to S for every vertex $v \in M$. Then $S' - Leaf(S')$ is a subtree of S. By Lemma 9, this implies that $(m-1)$-th stem of $S' - Leaf(S')$ is a subtree of $(m-1)$-th stem of S. Since $(m-1)$-th stem of S is a k-ended tree, it follows from Lemma 10 that $(m-1)$-th stem of $S' - Leaf(S')$ is a k-ended tree. Therefore, S' is a spanning tree of G whose m-th stem is a k-ended tree. Therefore Lemma 11 holds. $\square$

By Lemma 11, Theorem 1 also can be represented as follows.

Theorem 12 *Let $k \geq 2$ and $m \geq 0$ be integers, and let G be a connected graph. If $\alpha^{2(m+1)}(G) \leq k$, then G has a spanning tree whose m-th stem is a k-ended tree.*

Many results on spanning k-ended trees and some other spanning trees can be found in a recent book [1] and a survey [4].

2 Proof of Theorem 1

In this section, we shall prove Theorem 1.

Proof of Theorem 1. Let $k \geq 2$ and $m \geq 0$ be integers, and let G be a connected graph. Suppose that $\alpha^{2(m+1)}(G) \leq k$ and G has no k-ended tree that m-dominates G. We choose a k-ended tree T of G so that

(T1) $|Domi^m(T)|$ is as large as possible, and
(T2) $|T|$ is as small as possible subject to (T1).

We first show that T has exactly k leaves. Suppose, to the contrary, that T has at most $k-1$ leaves. For a vertex v of $V(G) - Domi^m(T)$, take a shortest path P connecting v to T. Then $T + P$ is a k-ended tree and $Domi^m(T + P) \supseteq Domi^m(T) \cup \{v\}$, which contradicts (T1). Hence T has exactly k leaves.

Let $x_1, x_2, \ldots, x_k$ be the leaves of T. Since T does not m-dominate G and G is a connected graph, there exists a vertex v_0 in G such that $d_G(v_0, T) = m+1$. Then there exists a path of length $m+1$ that connects v_0 and a vertex of T, say w. We denote this path by $P(v_0, w)$. Note that $V(P(v_0, w)) \cap V(T) = \{w\}$ and $w \notin \{x_1, \ldots, x_k\}$ by (T1).

Claim 1 *There exists no path Q in G that satisfies the following two conditions.*

(i) *Q connects x_i and a vertex of $P(v_0, w) - w$ for some $1 \leq i \leq k$.*

(ii) $V(Q) \cap V(T) = \{x_i\}$.

Suppose that there exists a path Q in G which satisfies (i) and (ii). Let $T' = T + Q$. Then T' is a k-ended tree and $Domi^m(T') \supseteq Domi^m(T) \cup \{v_0\}$ since the length of $P(v_0, w)$ is $m+1$. This contradict (T1). Hence Claim 1 holds.

Claim 2 *For any $1 \leq i, j \leq k$ with $i \neq j$, there exists no path R in G that connects x_i and x_j and satisfies $V(R) \cap V(T) = \{x_i, x_j\}$. In particular, x_i and x_j are not adjacent in G.*

Suppose, to the contrary, that there exists a path R in G that connects two distinct vertices x_i and x_j and satisfies $V(R) \cap V(T) = \{x_i, x_j\}$. By Claim 1, $V(R) \cap V(P(v_0, w) - w) = \emptyset$. Since $T + P(v_0, w) + R$ contains exactly $k-1$ vertices of degree one and a unique cycle, we can obtain a k-ended tree T' from $T + P(v_0, w) + R$ by deleting one edge, which is incident with a vertex of degree at least 3 and lies on the unique cycle of $T + R$. Then $Domi^m(T') \supseteq Domi^m(T) \cup \{v_0\}$, which contradicts (T1). Hence Claim 2 holds.

Claim 3 *For every $1 \leq i \leq k$, there exists a vertex y_i in $Domi^m(T)$ such that $d_G(y_i, x_i) = m$ and $d_G(y_i, T - x_i) \geq m+1$. Notice that $y_i = x_i$ if $m = 0$.*

Let $Y = \{y \in Domi^m(T) : d_G(y, x_i) = m\}$. If either $Y = \emptyset$ or $d_G(y, T - x_i) \leq m$ for every $y \in Y$, then $T - x_i$ is a k-ended tree such that $Domi^m(T - x_i) = Domi^m(T)$, which contradicts (T2). Hence Claim 3 holds.

Let $S = \{v_0, y_1, y_2, \ldots, y_k\}$.

Claim 4 *For any two distinct vertices of S, the distance between them in G is at least $2(m+1)$.*

Without loss generality, we first consider the distance between y_1 and y_2. Suppose, to the contrary, that $d_G(y_1, y_2) \leq 2m+1$. Let P be a shortest path in G that connects y_1 and y_2.

By Claim 2, $V(P) \cap V(T) \neq \emptyset$. For $i = 1, 2$, choose a vertex z_i of $V(P) \cap V(T)$ so that $d_G(y_i, z_i)$ is minimum. First, suppose that $z_1 \neq z_2$. Then

$d_G(y_1, y_2) = d_G(y_1, z_1) + d_G(z_1, z_2) + d_G(z_2, y_2) \geq 2m + 1$. Since $d_G(y_1, y_2) \leq 2m + 1$, equality holds in the above inequality. Therefore $d_G(y_i, z_i) = m$, and so $z_i = x_i$ by Claim 3 for $i = 1, 2$. Since $d_G(z_1, z_2) = 1$, x_1 and x_2 are adjacent in G. This contradicts Claim 2. Therefore $z_1 = z_2$.

By Claim 2, $z_1 = z_2 \notin \{x_1, x_2\}$. Also by Claim 3, $d_G(y_1, y_2) = d_G(y_1, z_1) + d_G(z_1, y_2) \geq 2(m + 1)$, which is again a contradiction. Therefore, for any $1 \leq i, j \leq k$ with $i \neq j$, the distance between y_i and y_j is at least $2(m + 1)$.

Without loss generality, we next consider the distance between v_0 and y_1. Let Q be a shortest path in G that connects y_1 and v_0. Suppose, to the contrary, that $d_G(y_1, v_0) \leq 2m + 1$. By Claim 1, $V(Q) \cap V(T) \neq \emptyset$. Choose two vertices z_1 and z_2 of $V(Q) \cap V(T)$ such that $d_G(y_1, z_1)$ and $d_G(z_2, v_0)$ are minimum. By the choice of v_0, $d_G(z_2, v_0) \geq m+1$. If $z_1 \neq x_1$, then by Claim 3, $d_G(y_1, v_0) \geq d_G(y_1, z_1) + d_G(z_2, v_0) \geq 2(m + 1)$, a contradiction. Hence $z_1 = x_1$. By Claim 1, $z_2 \neq z_1 = x_1$. Therefore, $d_G(y_1, v_0) = d_G(y_1, z_1) + d_G(z_1, z_2) + d_G(z_2, v_0) \geq m + 1 + (m + 1) = 2(m + 1)$, a contradiction. Therefore, for every $1 \leq i \leq k$, the distance between y_i and v_0 is at least $2(m + 1)$. Hence Claim 4 holds.

By Claim 4, $\alpha^{2(m+1)}(G) \geq |S| = k + 1$. This contradicts the assumption of the theorem. Consequently Theorem 1 is proved. □

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