# $\{k, r-k\}$-factors of $r$-regular graphs

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**Abstract**

For a set $\mathcal{S}$ of positive integers, a spanning subgraph $F$ of a graph $G$ is called an $\mathcal{S}$-factor of $G$ if $\deg_F(x) \in \mathcal{S}$ for all vertices $x$ of $G$, where $\deg_F(x)$ denotes the degree of $x$ in $F$. We prove the following theorem on $\{a, b\}$-factors of regular graphs. Let $r \geq 5$ be an odd integer and $k$ be either an even integer such that $2 \leq k < r/2$ or an odd integer such that $r/3 \leq k < r/2$. Then every $r$-regular graph $G$ has a $\{k, r-k\}$-factor. Moreover, for every edge $e$ of $G$, $G$ has a $\{k, r-k\}$-factor containing $e$ and another $\{k, r-k\}$-factor avoiding $e$.

## 1 Introduction

In this paper we consider a *general graph*, which may have multiple edges and loops, because our theorem holds for a general graph and also this relaxation makes the proof simpler. A general graph without loops is called a *multigraph*, and a general graph without multiple edges or loops is called a *simple graph*.

For a general graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. Let $H$ be a subgraph of a general graph $G$. Then $H$ may have multiple edges and loops, and for a vertex $v$ of $H$, the degree of $v$ in $H$ is denoted by $\deg_H(v)$. For a set $\mathcal{S}$ of positive integers, a general graph $G$ is called a *general $\mathcal{S}$-graph* if $\deg_G(x) \in \mathcal{S}$ for all $x \in V(G)$. An $\mathcal{S}$-subgraph is similarly defined. A spanning $\mathcal{S}$-subgraph is called an $\mathcal{S}$-*factor*. For convenience, if $\mathcal{S} = \{k\}$, (i.e., if $\mathcal{S}$ consists of one integer $k$), then an $\mathcal{S}$-factor is briefly called a *k-factor*. In this paper we prove the following Theorem 1. It is known that for a simple graph $G$ and positive integers $1 \le a < b$ such that $b - a \ge 3$, the problem of determining whether $G$ has an $\{a, b\}$-factor or not is NP-complete [7]. So if $b - a \ge 3$, then there might exist no good criterion for a graph to have an $\{a, b\}$-factor. Thus the next theorem shows a special property of regular graphs.

**Theorem 1** *Let $r \ge 5$ be an odd integer and $k$ be an integer satisfying one of the following two conditions:*
*(i) $k$ is even and $2 \le k < r/2$.*
*(ii) $k$ is odd and $r/3 \le k < r/2$.*
*Then every $r$-regular general graph $G$ has a $\{k, r-k\}$-factor. Moreover, for every edge $e$ of $G$, $G$ has a $\{k, r-k\}$-factor containing $e$ and another $\{k, r-k\}$-factor avoiding $e$.*

Many results on factors of regular graphs were obtained. In particular, regular factors of $n$-edge connected regular graphs with $n \ge 2$ are extensively studied. We now show some results on $\mathcal{S}$-factors of regular graphs, which are related to our theorem. Notice that the following theorems including their proofs can be found in a recent book [1].

**Theorem 2 (Petersen [9])** *Let $r$ and $k$ be both even integers such that $2 \le k < r$. Then every $r$-regular general graph has a $k$-factor.*

The following theorem was obtained by Tutte, and its elementary and simple proof was given by Thomassen [10].

**Theorem 3 (Tutte [11])** *Let $r$ be an odd integer and $k$ be an integer such that $1 \le k < r$. Then every $r$-regular general graph has a $\{k-1, k\}$-factor.*

The following two theorems are detailed results of the above Theorem 3.

**Theorem 4 (Kano [5])** *Let $r \ge 3$ be an odd integer and $k$ be an integer such that $1 \le k \le 2r/3$. Then every $r$-regular multigraph has a $\{k-1, k\}$-factor each of whose components is regular.*

**Theorem 5 (Egawa and Kano [3])** *Let $r$ and $k$ be integers such that $1 \le k < r$, and $G$ be an $r$-regular simple graph. Let $W$ be a maximal independent set of $G$, where $W \subset V(G)$. Then $G$ has a $\{k-1, k\}$-factor $F$ such that $\deg_F(x) = k-1$ for all $x \in V(G) - W$, as well as a $\{k-1, k\}$-factor $H$ such that $\deg_H(y) = k$ for all $y \in V(G) - W$.*

We conclude this section with a conjecture and a problem. Notice that our theorem shows that the following conjecture holds except in the case where $a$ is odd and $1 \le a < r/3$. So, for example, it is unsolved whether every 5-regular graph has a $\{1,4\}$-factor or not.

**Conjecture 6** *Let $r$ be an odd integers and $a$ and $b$ are positive integers such that $a+b=r$ and $a<b$. Then every $r$-regular simple graph (or general graph) has an $\{a,b\}$-factor.*

**Problem 7** *Let $r$ be an even integer and $k$ be an odd integer such that $1 \le k < r/2$. Is it true that every connected $r$-regular simple graph (or general graph) of even order has a $\{k, r-k\}$-factor?*

Notice that the above Problem 7 was recently solved negatively by Lu and Wang [8].

## 2 Proof of Theorem 1

Let $G$ be a general graph. The order of $G$ is denoted by $|G|$. For two vertices $x$ and $y$ of $G$, which may be the same, an edge joining $x$ to $y$ is denoted by $xy$, and if $x=y$, then it denotes a loop incident with $x$.

For a vertex set $X$ of $G$, the subgraph of $G$ induced by $X$ is denoted by $\langle X \rangle_G$. For disjoint subsets $X, Y \subset V(G)$, $e_G(X,Y)$ denotes the number of edges of $G$ joining $X$ to $Y$. For a subset $X \subset V(G)$, $e_G(X, V(G)-X)$ is briefly denoted by $\partial(X)$.

If an $r$-regular general graph $G$ has a $\{k, r-k\}$-factor $F$, then the set of edges of $G$ not contained in $F$ induces also a $\{k, r-k\}$-factor of $G$, and thus $G$ is decomposed into two edge disjoint $\{k, r-k\}$-factors. In particular, the latter part of Theorem 1 follows from the existence of a $\{k, r-k\}$-factor, that is, the former statement of the theorem guarantees the latter statement.

We now give some results on factors, which will be used here, and their proofs are found in [1].

**Lemma 8 ((i) [4], (ii) [2], [1] Theorem 3.4)** *Let $\lambda \ge 1$ be an integer, $r \ge 3$ be an odd integer, and $k \ge 2$ be an integer satisfying one of the following two conditions:*

*(i) $k$ is even and $2 \le k \le r(1-(1/\lambda))$.*

*(ii) $k$ is odd and $r/\lambda^* \le k$, where $\lambda^* \in \{\lambda, \lambda+1\}$ and $\lambda^* \equiv 1 \pmod 2$.*

*Then every $\lambda$-edge connected $r$-regular general graph has a $k$-factor. In particular, every 2-edge connected $r$-regular general graph has a $k$-factor for every even integer $k$, $2 \le k < r/2$, and for every odd integer $k$, $r/3 \le k$.*

For a function $f : V(G) \to \{0, 1, 2, 3. \dots\}$, a spanning subgraph $F$ of $G$ is called an *$f$-factor* if $\deg_F(x) = f(x)$ for all vertices $x$ of $G$.

**Lemma 9 ([6], [1] Theorem 3.12 )** *Let $G$ be a connected general graph, $\theta$ be a real number such that $0 < \theta < 1$, and $f : V(G) \to \{0, 1, 2, \dots\}$ be a function. If the following four conditions hold, then $G$ has an $f$-factor.*

*(1) $\sum_{x \in V(G)} f(x)$ is even.*

*(2) $\sum_{x \in V(G)} |f(x) - \theta \deg_G(x)| < 2$.*

*(3) $\theta \partial(X) \geq 1$ for all $X \subset V(G)$ such that $\langle X \rangle_G$ is connected and $\sum_{x \in X} f(x)$ is odd.*

*(4) $(1-\theta)\partial(X) \geq 1$ for all $X \subset V(G)$ such that $\langle X \rangle_G$ is connected and $\sum_{x \in X} f(x) + \partial(X)$ is odd.*

**Lemma 10** *Let $r \geq 5$ be an odd integer, and $k$ be an even integer such that $2 \leq k < r/2$. Then every 2-edge connected $\{r-1, r\}$-general graph having at most four vertices of degree $r-1$ has a $k$-factor.*

*Proof.* We use Lemma 9. Let $\theta = k/r$ and define $f(x) = k$ for all vertices $x$ of $G$. Then the conditions (1) and (3) of Lemma 9 hold. It follows that

$$\sum_{x \in V(G)} |k - \theta \deg_G(x)| \leq \Big(k - \frac{k}{r}(r-1)\Big) \times 4 = \frac{4k}{r} < 2.$$

Let $X \subset V(G)$ be a vertex set such that $\langle X \rangle_G$ is connected and $\sum_{x \in X} f(x) + \partial(X) \equiv 1 \pmod 2$. Then $\partial(X) \geq 3$ since $\partial(X) \equiv 1 \pmod 2$ and $G$ is 2-edge connected. Thus

$$(1-\theta)\partial(X) \geq \Big(1 - \frac{k}{r}\Big) \times 3 \geq 1.$$

Hence (2) and (4) hold, and thus $G$ has an $f$-factor, which is a $k$-factor. □

**Lemma 11** *Let $r \geq 7$ be an odd integer, and $k$ be an odd integer such that $r/3 \leq k < r/2$. Let $G$ be a 2-edge connected general $\{r-1, r\}$-graph that has exactly one vertex $w$ of degree $r-1$. Then $G$ has a $\{k-1, k\}$-factor $F$ such that $\deg_F(w) = k-1$ and $\deg_F(x) = k$ for all $x \in V(G) - \{w\}$.*

*Proof.* We use Lemma 9. Let $\theta = k/r$ and define a function $f$ on $V(G)$ so that $f(w) = k-1$ and $f(x) = k$ for all $x \in V(G) - \{w\}$. Since $G$ is of odd order, the conditions (1) of Lemma 9 hold. (2) follows from the inequality that

$$\sum_{x \in V(G)} |f(x) - \theta \deg_G(x)| = \Big|k - 1 - \frac{k}{r}(r-1)\Big| = 1 - \frac{k}{r} < 2.$$

Let $X \subset V(G)$ be a vertex set such that $\langle X \rangle_G$ is connected and $\sum_{x \in X} f(x)$ is odd. Then $|X \setminus \{w\}|$ is odd, and so

$$2|E(\langle X \rangle_G)| + \partial(X) = \sum_{x \in X} \deg_G(x) \equiv 1 \pmod 2.$$

Thus $\partial(X) \equiv 1 \pmod 2$, which implies $\partial(X) \geq 3$ since $G$ is 2-edge connected. Thus

$$\theta\partial(X) \geq \frac{k}{r} \times 3 \geq 1.$$

Hence (3) hold.

Let $X \subset V(G)$ be a vertex set such that $\langle X \rangle_G$ is connected. Since $G$ is 2-edge connected and $k < r/2$, we have

$$(1-\theta)\partial(X) \geq \left(1 - \frac{k}{r}\right) \times 2 \geq 1.$$

Hence (4) hold, and thus $G$ has an $f$-factor, which is the desired $\{k-1, k\}$-factor. □

*Proof of Theorem 1.* We shall prove Theorem 1 by induction on the order $|G|$. Hereafter we briefly call a general graph *a graph*. There exist $(r+1)/2$ $r$-regular connected graphs of order two as shown in Figure 1, and each of them has a $k$-factor. So the theorem holds if $|G| = 2$. We may assume $|G| \geq 4$ since $G$ is of even order. If $G$ is 2-edge connected, then by Lemma 8 $G$ has a $k$-factor, which is of course a $\{k, r-k\}$-factor. Hence we may assume that $G$ has a bridge.

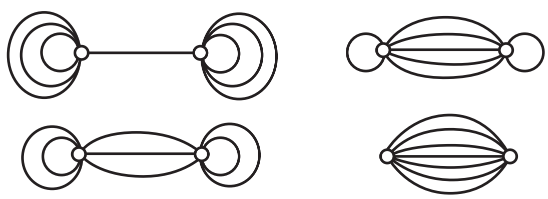

Figure 1: The four 7-regular general connected graphs of order two, which have both 2-factors and 3-factors.

**Case 1.** *$G$ has a bridge $e$ such that every component of $G - e$ has order at least two.*

Let $e$ be the bridge mentioned in Case 1. Let $D_1$ and $D_2$ be the two components of $G - e$, and let $v_i$ be the end-point of $e$ contained in $D_i$ for $i \in \{1, 2\}$. Let $H_1$ be an $r$-regular graph obtained from $D_1$ by adding the bridge $e$, its end-point $v_2$ and $(r-1)/2$ new loops incident with $v_2$. Then by the inductive hypothesis, $H_1$ has a $\{k, r-k\}$-factor avoiding $e$. Note that the existence of a $\{k, r-k\}$-factor implies a $\{k, r-k\}$-factor with this property as mentioned before. Thus $D_1$ has a $\{k, r-k\}$-factor $F_1$. Similarly, $D_2$ has a $\{k, r-k\}$-factor $F_2$. Then $F_1 \cup F_2$ is the desired $\{k, r-k\}$-factor of $G$.

**Case 2.** *For every bridge $e$ of $G$, one component of $G-e$ has order one.*

Assume first that $G$ has precisely one bridge. Let $e = vw$ the unique bridge, where $v$ and $w$ are the two end-points of $e$ and $w$ is contained in a big component $D_1$ of $G-e$ and $\{v\}$ is the vertex set of the other component of $G-e$. Since $G$ has exactly one bridge, $D_1$ is a 2-edge connected $\{r-1, r\}$-graph having exactly one vertex of degree $r-1$. Hence if $k$ is even, then by Lemma 10, $D_1$ has a $k$-factor $F_1$, and thus $G$ has a $k$-factor, which is obtained from $F_1$ by adding $k/2$ loops incident with $v$. If $k$ is odd, then Lemma 11, $G$ has a $\{k-1, k\}$-factor $F$ such that $\deg_F(w) = k-1$ and $\deg_F(x) = k$ for all $x \in V(D_1) - \{w\}$. Then by adding the bridge $e$ and $(k-1)/2$ loops incident with $v$ to $F$, we can obtain the desired $k$-factor of $G$.

Next assume that $G$ has at least two bridges. Let $e_1 = v_1w_1$ and $e_2 = v_2w_2$ be two distinct bridges of $G$, where $v_i$ and $w_i$ are the end-points of $e_i$ such that $w_i$ is contained in the big component of $G-e_i$ and $\{v_i\}$ is the vertex set of the other component of $G-e_i$. Note that it may occur that $w_1 = w_2$, but it always holds that $v_1 \neq v_2$. Let $H$ be an $r$-regular graph obtained from the big component of $G - \{e_1, e_2\}$ by adding a new edge or a new loop joining $w_1$ to $w_2$. Then by induction, $H$ has a $\{k, r-k\}$-factor avoiding the new edge $w_1w_2$. If $k$ is even, then by adding $k/2$ loops incident with $v_1$ together with $k/2$ loops incident with $v_2$ to $H$, we obtain the desired $\{k, r-k\}$-factor of $G$. If $k$ is odd, the by adding $(r-k)/2$ loops instead of $k$ loops, we can similarly obtain the desired $\{k, r-k\}$-factor of $G$. Consequently the theorem is proved. □

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