Star-cycle factors of graphs

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Abstract

A *star-cycle factor* of a graph is its spanning subgraph each of whose components is a star or cycle. Let G be a graph and $f : V(G) \to \{1, 2, 3, \ldots\}$ be a function. Let $W = \{v \in V(G) : f(v) = 1\}$. Under this notation, it was proved by Berge and Las Vergnas that G has a star-cycle factor F with the property that (i) if a component D of F is a star with center v, then $\deg_F(v) \le f(v)$, and (ii) if a component D of F is a cycle, then $V(D) \subseteq W$ if and only if $iso(G-S) \le \sum_{x \in S} f(x)$ for all $S \subset V(G)$, where $iso(G-S)$ denotes the number of isolated vertices of $G-S$. They proved this result by using circulation theory of flows and fractional factors of graphs. In this paper, we give an elementary and short proof of this theorem.

Keywords: star factor, cycle factor, star-cycle factor, factor of graph

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1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For a vertex set S of G, let $G[S]$ and $G-S$ denote the subgraph of G induced by S and $V(G)-S$, respectively. Let $iso(G)$ and $Iso(G)$ denote the number of isolated vertices and the set of isolated vertices of G, respectively. Thus $iso(G) = |Iso(G)|$.

For a set $\mathcal{S}$ of connected graphs, a spanning subgraph F of a graph G is called an $\mathcal{S}$-factor of G if each component of F is isomorphic to an element of $\mathcal{S}$. The cycle of order n is denoted by C_n. The complete bipartite graph with bipartition (A, B) of $|A| = m$ and $|B| = n$ is denote by $K_{m,n}$. A complete bipartite graph $K_{1,n}$ is called a *star*, and its vertex of degree n is called the *center*. For $K_{1,1}$, either of the two vertices can be regarded as its center.

Tutte obtained the following criterion for a graph to have a $\{K_2, C_n : n \geq 3\}$-factor, and its elementary and short proof can be found in a book [1].

Theorem 1 (Tutte [5], [1] Theorem 7.2) *A graph G has a $\{K_2, C_n : n \geq 3\}$-factor if and only if $iso(G - S) \leq |S|$ for all $S \subset V(G)$.*

On the other hand, a graph G that satisfies $iso(G - S) \leq n|S|$ for all $S \subset V(G)$ has the following property.

Theorem 2 (Las Vergnas[4], Amahashi and Kano [2]) *Let $n \geq 2$ be an integer. Then a graph G has a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,n}\}$-factor if and only if $iso(G - S) \leq n|S|$ for all $S \subset V(G)$.*

Berge and Las Vergnas generalized the above two theorems to the following theorem, and they proved it by using circulation theory of flows and fractional factors of graphs. In this paper, we give another elementary short proof of this theorem, where by a *star-cycle factor*, we mean a $\{K_{1,n},\ C_m : n \geq 1,\ m \geq 3\}$-factor.

Theorem 3 (Berge and Las Vergnas [3]) *Let G be a graph and $f : V(G) \to \{1, 2, 3, \ldots\}$ be a function, and let $W = \{v \in V(G) : f(v) = 1\}$. Then G has a star-cycle factor F having the property that*

(i) if a component of F is a star with center v, then $\deg_F(v) \leq f(v)$; and

(ii) if a component of F is a cycle, then its vertex set is included in W,

if and only if

$$iso(G - S) \leq \sum_{x \in S} f(x) \qquad \text{for all} \quad S \subset V(G). \tag{1}$$

It is easy to see that the condition (1) is equivalent to

$$|X| \leq \sum_{x \in N_G(X)} f(x) \qquad \text{for all independent subsets} \quad X \subset V(G),$$

which is given in [3]. We do not make use of Theorem 2 in our proof of Theorem 3, but the proof relies on Theorem 1. For completeness, we also include a proof of Theorem 1 which follows the same line of argument as our proof of Theorem 3, and is almost as short as the proof given in [1].

2 Proof of Theorem 3

We need some other notations. For two sets X and Y, $X \subset Y$ means that X is a proper subset of Y. Let G be a graph. For two vertices x and y of G, we write xy or yx for the edge joining x and y. For a vertex v of G, we denote by $N_G(v)$ the neighborhood of v. For a subset S of $V(G)$, we define $N_G(S) := \cup_{x \in S} N_G(x)$. A component of $G - S$ that is not an isolated vertex is called a *nontrivial component*, which has order at least two.

In order to prove Theorem 3, we need the following lemma.

Lemma 4 (Generalized Marriage Theorem, [1] Theorem 2.10) *Let G be a bipartite graph with bipartition (A, B), and let $f : A \to \{1, 2, 3 \ldots\}$. If $|B| = \sum_{x \in A} f(x)$, and*

$$|N_G(X)| \geq \sum_{x \in X} f(x) \qquad \textit{for all} \quad X \subseteq A,$$

then G has a star factor F such that $\deg_F(a) = f(a)$ for all $a \in A$, and $f(b) = 1$ for all $b \in B$.

Proof of Theorem 3. For briefly, we refer to a star-cycle factor satisfying conditions (i) and (ii) in Theorem 3 as an *SC-factor* with respect to f. We first prove the necessity. Assume that G has an SC-factor F with components $D_1, D_2, \ldots, D_m$. Let $\emptyset \neq S \subset V(G)$. By inspection, we see that $iso(D_i - S \cap V(D_i)) \leq \sum_{x \in S \cap V(D_i)} f(x)$ for each i, and hence

$$iso(G - S) \leq iso(F - S) = \sum_{i=1}^{m} iso(D_i - S \cap V(D_i))$$
$$\leq \sum_{i=1}^{m} \sum_{x \in S \cap V(D_i)} f(x) = \sum_{x \in S} f(x).$$

We shall prove the sufficiency of Theorem 3 by induction on $\sum_{x\in V(G)} f(x)$. We may assume that G is connected since otherwise by applying the induction hypothesis to each component, we can obtain the desired SC-factor of G.

Obviously, $\sum_{x\in V(G)} f(x) \geq |G|$ since $f(x) \geq 1$ for all $x \in V(G)$. If $\sum_{x\in V(G)} f(x) = |G|$, then $f(x) = 1$ for all $x \in V(G)$. Thus the condition (1) becomes

$$iso(G-S) \leq |S| \qquad \text{for all} \quad S \subset V(G).$$

By Theorem 1, G has a $\{K_{1,1}, C_n : n \geq 3\}$-factor, which is the desired SC-factor of G. So we may assume that $\sum_{x\in V(G)} f(x) \geq |G| + 1$. Then there exists a vertex $w \in V(G)$ such that $f(w) \geq 2$.

Let us define the number β by

$$\beta = \min\{\sum_{x\in X} f(x) - iso(G-X) \ : \ \emptyset \neq X \subset V(G)\}.$$

Then $\beta \geq 0$ by (1), and it follows from the definition of β that

$$iso(G-Y) \leq \sum_{x\in Y} f(x) - \beta \qquad \text{for all} \quad \emptyset \neq Y \subset V(G). \tag{2}$$

Take a maximal subset S of $V(G)$ such that

$$\sum_{x\in S} f(x) - iso(G-S) = \beta. \tag{3}$$

Case 1. $\beta \geq 1$.

Define $f^* : V(G) \to \{1, 2, 3, \ldots\}$ by

$$f^*(x) = \begin{cases} f(x) - 1 & \text{if } x = w; \\ f(x) & \text{otherwise.} \end{cases}$$

Let $\emptyset \neq X \subset V(G)$. Then we have

$$iso(G-X) \leq \sum_{x\in X} f(x) - \beta \leq \sum_{x\in X} f(x) - 1 \leq \sum_{x\in X} f^*(x).$$

Hence, G has an SC-factor F^* with respect to f^* by induction. If w is contained in a star of F^*, then F^* is also an SC-factor of G with respect to f. Assume that w is contained in an cycle of F^*. Note that $f(w) \geq 2$. If $|V(D)|$ is even, D has a $\{K_{1,1}\}$-factor; if $|V(D)|$ is odd, then D has a $\{K_{1,1}, K_{1,2}\}$-factor consisting of one copy of $K_{1,2}$ and $(|V(D)| - 3)/2$ copies of $K_{1,1}$ such that w is the center of $K_{1,2}$. Therefore G has the desired SC-factor.

Case 2. $\beta = 0$.

We start with a claim.

Claim 2.1 *Every nontrivial component of $G - S$ has an SC-factor with respect to f.*

Let D be a nontrivial component of $G - S$, and let $\emptyset \neq X \subset V(D)$. Then by (2), we have

$$iso(G - S) + iso(D - X) = iso(G - S \cup X)$$
$$\leq \sum_{x \in S \cup X} f(x) = \sum_{x \in S} f(x) + \sum_{x \in X} f(x).$$

Thus $iso(D - X) \leq \sum_{x \in X} f(x)$ by (3), which implies that D has an SC-factor by induction.

We construct a bipartite graph B with bipartition $(S, Iso(G - S))$ in which two vertices $x \in S$ and $y \in Iso(G - S)$ are adjacent if and only if x and y are joined by an edge of G.

Claim 2.2 *For every $Y \subset S$, we have $|N_B(Y)| \geq \sum_{x \in Y} f(x)$, and $|N_B(S)| = \sum_{x \in S} f(x)$.*

Since G is connected, $N_B(S) = Iso(G-S)$, and hence $|N_B(S)| = \sum_{x \in S} f(x)$ by (3) and $\beta = 0$. Let $\emptyset \neq Y \subset S$. We may assume that $N_B(Y) \subset Iso(G-S)$. Then $Iso(G - S) - N_B(Y)$ is a set of isolated vertices of $G - (S - Y)$. Thus it follows from (2) and (3) that

$$\sum_{x \in S} f(x) - \sum_{x \in Y} f(x) = \sum_{x \in S - Y} f(x) \geq iso(G - (S - Y))$$
$$\geq \; iso(G - S) - |N_B(Y)| = \sum_{x \in S} f(x) - |N_B(Y)|.$$

Hence Claim 2.2 holds.

By Claim 2.2, B has a star-factor described in Lemma 4. By combining this star-factor and SC-factors of nontrivial components of $G - S$ guaranteed by Claim 2.1, we obtain the desired SC-factor of G. $\square$

Proof of Theorem 1. The proof of the necessity is the same as that in Theorem 3. We prove the sufficiency by induction on $|E(G)|$. We may assume that $|V(G)| \geq 3$, G is connected, and G is not a star. Let

$$\beta = \min\{|X| - iso(G - X) \;:\; \emptyset \neq X \subset V(G)\}.$$

If $\beta = 0$, then we can argue exactly as in Case 2 of the proof of Theorem 3. Thus we may assume $\beta \geq 1$. Since G is not a star, G has an edge $e = ab$ such that $Iso(G-e) = \emptyset$. If $iso(G-e-X) \leq |X|$ for all $X \subset V(G)$, then we obtain the desired factor by applying the induction hypothesis to $G-e$. Thus we may assume that there exists $S \subset V(G)$ such that $iso(G-e-S) > |S|$. Since $Iso(G-e) = \emptyset$, we have $S \neq \emptyset$. Hence $iso(G-S) \leq |S| - \beta \leq |S| - 1$, which implies $|S| < iso(G-S-e) \leq iso(G-S) + 2 \leq |S| + 1$, and thus

$$Iso(G-e-S) = Iso(G-S) \cup \{a, b\}, \ iso(G-S) = |S| - 1 \ \text{ and } \ \beta = 1.$$

Define a bipartite graph B with bipartition $(S, Iso(G-S))$ as in the proof of Theorem 3. Arguing as in Claim 2.2, we see that $|N_B(Y)| \geq |Y|$ for all $Y \subset S$. Since $Iso(G-e) = \emptyset$, it follows that $N_G(a) \cap S \neq \emptyset$, $N_G(b) \cap S \neq \emptyset$ and $G[\{a, b\}]$ is a component of $G - S$. Take $u_1 \in N_G(a) \cap S$ and $u_2 \in N_G(b) \cap S$. By Lemma 4, $B - u_i$ has a $\{K_{1,1}\}$-factor F_i for each i. Let H be the graph defined by $V(H) = S \cup Ios(G-S)$ and $E(H) = F_1 \cup F_2$. Then one of the components of H is a path joining u_1 and u_2, and each of the other components is either a cycle or isomorphic to $K_{1,1}$. Consequently, adding three edges u_1a, ab, bu_2 to H, we get a $\{K_{1,1}, C_n : n \geq 3\}$-factor of $G[S \cup Iso(G-S) \cup \{a, b\}]$ (note that this argument works even if $u_1 = u_2$, in particular, it works in the case where $|S| = 1$ as well). Combining this factor with $\{K_{1,1}, C_n : n \geq 3\}$-factors of the components of $G - S - Iso(G-S) - \{a, b\}$, where existence follows from the induction hypothesis as in Claim 2.1, we obtain a $\{K_{1,1}, C_n : n \geq 3\}$-factor of G. □

References

[1] J. Akiyama and M. Kano, *Factors and Factorizations of Graphs*, Lecture Note in Mathematics (LNM **2031**), Springer (2011).

[2] A. Amahashi and M. Kano, On factors with given components, *Discrete Math.* **42** (1983) 1–6.

[3] C. Berge and M. Las Vergnas, On the existence of subgraphs with degree constraints, *Nederl. Akad. Wetensch. Indag. Math.* **40** (1978), 165–176.

[4] M. Las Vergnas, A extension of Tutte‘s 1-factor theorem, *Discrete Math.* **23** (1978) 241–255.

[5] W.T. Tutte, The 1-factors of oriented graphs, *Proc. Amer. Math. Soc.* **4** (1953) 922–931.