Spanning Caterpillars Having at most k Leaves

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Abstract. A tree is called a caterpillar if all its leaves are adjacent to the same its path, and the path is called a spine of the caterpillar. Broersma and Tuinstra proved that if a connected graph G satisfies $\sigma_2(G) \geq |G| - k + 1$ for an integer $k \geq 2$, then G has a spanning tree having at most k leaves. In this paper we improve this result as follows. If a connected graph G satisfies $\sigma_2(G) \geq |G| - k + 1$ and $|G| \geq 3k - 10$ for an integer $k \geq 2$, then G has a spanning caterpillar having at most k leaves. Moreover, if $|G| \geq 3k - 7$, then for any longest path, G has a spanning caterpillar having at most k leaves such that its spine is the longest path. These three lower bounds on $\sigma_2(G)$ and $|G|$ are sharp.

1 Introduction

We consider simple graphs, which have neither loops nor multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. We write $|G|$ for the order of G (i.e., $|G| = |V(G)|$). For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. We define $\sigma_2(G)$ to be the minimum degree sum of two nonadjacent vertices of G. An end-vertex of a tree is often called a *leaf*. A tree T is called a *caterpillar* if T contains a path such that all the vertices not contained in the path are adjacent to the path. In other words, a tree is a caterpillar if the removal of its leaves results in a path. Let T be a caterpillar. Then T has a path P connecting two leaves such that all the leaves of T not contained in P are adjacent to P. This path P is called a *spine* of T. Notice that the path Q obtained from T by removing all the leaves of T is often called the spine, however, for convenience, in this paper a spine of a caterpillar connects two leaves of a caterpillar and includes the path Q.

Recall the classic theorem of Ore [6] on a hamiltonian cycle.

Theorem 1 (Ore [6]). *Let G be a connected graph. If $\sigma_2(G) \geq |G|$, then G has a hamiltonian cycle.*

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This theorem implies the following corollary on a hamiltonian path.

Corollary 1. *Let G be a connected graph. If $\sigma_2(G) \geq |G| - 1$, then G has a hamiltonian path.*

This corollary was generalized as follows by introducing a tree having at most k leaves. Notice that a hamiltonian path is a spanning tree with two leaves.

Theorem 2 (Broersma and Tuinstra [3]). *Let $k \geq 2$ be an integer and G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning tree with at most k leaves.*

Our main result is the following theorem, which says that under the same condition of Theorem 2, if the order of G is sufficiently large, then G has a spanning caterpillar having at most k leaves.

Theorem 3. *Let $k \geq 2$ be an integer and G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$ and $|G| \geq 3k - 10$, then G has a spanning caterpillar having at most k leaves.*

Furthermore, we obtain the following result, which requires the spine of a spanning caterpillar to be a given longest path.

Theorem 4. *Let $k \geq 2$ be an integer and let G be a connected graph. Let P be a longest path of G. If $\sigma_2(G) \geq |G| - k + 1$ and $|G| \geq 3k - 7$, G has a spanning caterpillar having at most k leaves such that its spine is P.*

We first show that the degree conditions of Theorems 3 and 4 are sharp. It is shown in [3] that the condition $\sigma_2(G) \geq |G| - k + 1$ is sharp for a graph to have a spanning tree with k leaves.

We now show that the order condition of Theorem 3 is sharp. Let K_m denote the complete graph of order m. Assume that $k \geq 6$. For each $1 \leq i \leq 3$, let H_i be a copy of K_{k-5}. We construct a graph G as follows: $V(G) = \{w, v_1, v_2, v_3\} \cup V(H_1) \cup V(H_2) \cup V(H_3)$ (disjoint union), w is adjacent all the vertices of $H_1 \cup H_2 \cup H_3$ and v_i is adjacent to all the vertices of H_i for each $1 \leq i \leq 3$. Then $|G| = 3(k-5) + 4 = 3k - 11$ and

$$\sigma_2(G) = \deg_G(v_1) + \deg_G(v_2) = 2(k-5) = |G| - k + 1.$$

However G has no spanning caterpillar. Thus the condition $|G| \geq 3k - 10$ is sharp.

We next show that the order condition of Theorem 4 is sharp. Assume that $k \geq 5$. Let H_i be a copy of K_{k-3} for each $i \in \{1, 2\}$ and let H_3 be a copy of K_{k-4}. We construct a graph G as follows: $V(G) = \{w, v_3\} \cup V(H_1) \cup V(H_2) \cup V(H_3)$ (disjoint union), w is adjacent all the vertices of $H_1 \cup H_2 \cup H_3$ and v_3 is adjacent to all the vertices of H_3. Then $|G| = 2(k-3) + (k-4) + 2 = 3k - 8$ and

$$\sigma_2(G) = \deg_G(v_3) + \deg_G(v) = k - 4 + k - 3 = |G| - k + 1,$$

where $v \in V(H_1) \cup V(H_2)$. However for a longest path P containing all vertices of $V(H_1) \cup V(H_2) \cup \{w\}$, G has no spanning caterpillar whose spine is P. Thus the condition $|G| \geq 3k - 7$ is sharp.

Czygrinow, Fan, Hurlbert, Kierstead and Trotter [4] investigated a spanning caterpillar with bounded degree in the same direction.

Another results on spanning trees with at most k leaves can be found in [5], [8] and others. The interested reader is referred to the survey paper [7] and the book [1] for more information on spanning trees.

2 Proof of Theorem 3

In this section, we give a proof of Theorem 3. Our proof uses the following result on dominating paths of graphs. For a graph G, let $\sigma_3(G)$ is defined to be the minimum degree sum of three independent vertices of G, where a vertex set X is called independent if no two vertices of X are adjacent in G.

Lemma 1 (Broersma [2], Corollary 14 ($k = 1$ and $\lambda = 2$)). *Let G be a connected graph. If $\sigma_3(G) \geq |G| - 3$, then G has a spanning caterpillar.*

Proof of Theorem 3. Let $\{x, y, z\}$ be any set of three independent vertices of G. Then

$$\begin{aligned}
\sigma_3(G) &\geq \deg_G(x) + \deg_G(y) + \deg_G(z) \\
&= \frac{\deg_G(x) + \deg_G(y)}{2} + \frac{\deg_G(y) + \deg_G(z)}{2} + \frac{\deg_G(z) + \deg_G(x)}{2} \\
&\geq \frac{3\sigma_2(G)}{2} \geq \frac{3(|G| - k + 1)}{2} \\
&\geq |G| - \frac{7}{2}. \qquad \text{(by } |G| \geq 3k - 10)
\end{aligned}$$

Hence by Lemma 1, G has a spanning caterpillar.

Choose a spanning caterpillar T of G so that its spine is as long as possible. Let P be a spine of T, and let u and v be the two end-vertices of P, which are leaves of T. We assign an orientation in P from u to v, and for a vertex x of P, its *successor* x^+ and the *predecessor* x^- are defined, if they exist. By the choice of the spanning caterpillar, G has no cycle C with $V(C) = V(P)$, and it follows that $N_G(u) \cap (V(G) - V(P)) = \emptyset$, $N_G(v) \cap (V(G) - V(P)) = \emptyset$ and $N_G(u)^- \cap N_G(v) = \emptyset$. Since $N_G(u)^- \cup N_G(v) \subseteq V(P) - \{v\}$, we obtain

$$\deg_G(u) + \deg_G(v) \leq |P| - 1.$$

Since $\sigma_2(G) \geq |G| - k + 1$, we have $|G| - k + 1 \leq |P| - 1$, which implies $|G| - |P| \leq k - 2$. Therefore the spanning caterpillar T has at most k leaves. $\square$

3 Proof of Theorem 4

In this section, we give a proof of Theorem 4. We denote by $P[u, v]$ a path connecting two vertices u and v, which are the end-vertices of P. For a vertex set X of a graph G, let $\langle X\rangle_G$ denote the subgraph of G induced by X.

Proof of Theorem 4. If G has a hamiltonian path, we are done, and so we may assume that G does not have a hamiltonian path. Let P be a longest path in G, and let u and v be the end-vertices of P. We assign an orientation in P from u to v, and for a vertex x of P, we denote its successor and predecessor, if any, by x^+ and x^-, respectively. The following claim holds immediately by the fact that P is a longest path of G.

Claim 1. (i) $N_G(u) \cup N_G(v) \subseteq V(P)$.
(ii) G has no cycle C with $V(C) = V(P)$.
(iii) $N_G(u)^- \cap N_G(v) = \emptyset$ and $\{v\} \cup N_G(u)^- \cup N_G(v) \subseteq V(P)$.

By Claim 1, we have

$$\begin{aligned} |V(P)| &\geq |N_G(u)^-| + |N_G(v)| + |\{v\}| = \deg_G(u) + \deg_G(v) + 1 \\ &\geq \sigma_2(G) + 1 \geq |G| - k + 2. \end{aligned} \tag{1}$$

Hence $|G| - |V(P)| \leq k - 2$. Since G is a connected graph, by connecting all the vertices in $V(G) - V(P)$ to P by edges or paths, we can obtain a spanning tree T of G with at most k leaves.

Next, we prove that T is a caterpillar. Otherwise, there exists a vertex $w \in V(G) - V(P)$ such that $N_G(w) \cap V(P) = \emptyset$. By the choice of w and by Claim 1, the following claim easily holds.

Claim 2. (i) $\{w, u, v\}$ is an independent set of G.
(ii) $N_G(w) \subseteq V(G) - V(P) - \{w\}$.

By Claim 2 (i), we have

$$\deg_G(w) + \deg_G(u) + \deg_G(v) \geq \frac{3\sigma_2(G)}{2} \geq \frac{3}{2}(|G| - k + 1). \tag{2}$$

On the other hand, it follows from Claim 2 (ii) and Claim 1 that

$$\begin{aligned} &\deg_G(w) + \deg_G(u) + \deg_G(v) \\ &= |N_G(w)| + |N_G(u)^-| + |N_G(v)| \\ &\leq |G| - |P| - 1 + |P| - 1 = |G| - 2. \end{aligned} \tag{3}$$

By (2) and (3), we have $|G| \leq 3k-7$. Hence, the theorem holds when $|G| \geq 3k-6$.

Next we consider the case where $|G| = 3k - 7$. In this case, $\sigma_2(G) \geq |G| - k + 1 = 2k - 6$. Furthermore, if $k \leq 4$, then $|G| \leq 5$ and so the theorem holds. Hence we may assume that $k \geq 5$.

Assume that $|P| = 2k-5+t$, where $t \geq 0$ by (1). Then $\deg_G(w) \leq |G| - |P| - 1 = k-3-t$. Hence,

$$\deg_G(w) + \min\{\deg_G(u), \deg_G(v)\} \leq k-3-t + \frac{|P|-1}{2} = 2k-6-\frac{t}{2}.$$

Since $\sigma_2(G) \geq 2k-6$, we obtain $t = 0$, that is, $|P| = 2k-5$. Since the above inequality holds with equality, it follows from (1) that $\deg_G(w) = \deg_G(u) = \deg_G(v) = k-3$. Since $\sigma_2(G) \geq 2k-6$, we have

$$\deg_G(x) \geq k-3 \qquad \text{for every vertex} \quad x \in V(G) - V(P). \tag{4}$$

Since the inequality (1) holds with equality,

$$V(P) - \{v\} = N_G(u)^- \cup N_G(v) \quad \text{(disjoint union).} \tag{5}$$

Since G is connected, G has a path Q connecting w and a vertex of $V(P)$. Note that Q has at least two vertices. Let $\{z\} = V(P) \cap V(Q)$. Since P is a longest path, $z \notin N_G(u)^-$. By (5), we obtain $z \in N_G(v)$. Since P is longest, we have $z^+ \notin N_G(u)^-$; otherwise $Q[w,z] + P[z,u] + uz^{++} + P[z^{++}, v]$ is a longer path than P. Hence $z^+ \in N_G(v)$. Inductively by using (5) and Calim 1, we obtain that $s \in N_G(v)$ for every vertex $s \in V(P[z, v^-])$. It is immediate that $z^- \notin N_G(v)$, which implies $z^- \in N_G(u)^-$ by (5), and thus $z \in N_G(u)$. Inductively, we can show that $t \in N_G(u)$ for every vertex $t \in V(P[u^+, z])$. By the fact that P is a longest path of G, for every vertex $x \in V(G) - V(P)$, it follows that $N_G(u)^- \cap N_G(x) = \emptyset$ and $N_G(v)^+ \cap N_G(x) = \emptyset$. Therefore we obtain

$$N_G(x) \cap V(P) \subseteq \{z\} \qquad \text{for every vertex} \quad x \in V(G) - V(P). \tag{6}$$

Claim 3. $H = \langle (V(G) - V(P)) \cup \{z\} \rangle_G$ has a hamiltonian path with an end-vertex z.

We prove Claim 3. If $\langle V(G) - V(P) \rangle_G$ is a complete graph, then we are done. We may assume that $\langle V(G) - V(P) \rangle_G$ is not complete. By (6), (4) and $k \geq 5$, we have $\deg_H(x) = \deg_G(x) \geq k-3 \geq 2$ for each vertex $x \in V(G) - V(P)$.

Since $\langle V(G) - V(P) \rangle_G$ is not complete, there exists two non-adjacent vertices s and t in it, which are adjacent to z since $\deg_H(s) + \deg_H(t) \geq \sigma_2(G) \geq 2(|H| - 2)$, and hence $\deg_H(z) \geq 2$. Therefore $\deg_H(x) + \deg_H(y) \geq k-3+2 = |H|$ for all non-adjacent two vertices $x, y \in V(H)$. By Theorem 1, H has a hamiltonian cycle, and so H has a hamiltonian path with end-vertex z. Therefore Claim 3 is proved.

Let R be a hamiltonian path with end-vertex z in H, and x be another end-vertex of R. Then $R[x,z] + P[z,u]$ or $R[x,z] + P[z,v]$ is a path of order at least $k-2+(|V(P)|+1)/2 > |V(P)|$, a contradiction.

Consequently, Theorem 4 is proved. $\square$

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