A Generalization of Zero-Sum Flows in Graphs *†

S. Akbari[a,d], M. Kano[c], G.B. Khosrovshahi[d], S. Zare[b,d]

[a] *Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*

[b] *Department of Mathematical Sciences, Amirkabir University of Technology, Tehran, Iran*

[c] *Department of Computer and Information Science, Ibaraki University, Hitachi, Ibaraki, Japan*

[d] *School of Mathematics, Institute for Research in Fundamental Sciences* (IPM),

P.O. Box 19395-5746, Tehran, Iran ‡

Abstract

Let G be a graph and H be an abelian group. For every subset $S \subseteq H$ a map $\phi : E(G) \to S$ is called an S-flow. For a given S-flow of G, and every $v \in V(G)$, define $s(v) = \sum_{uv \in E(G)} \phi(uv)$. Let $k \in H$. We say that a graph G admits a k-sum S-flow if there is an S-flow such that for each vertex v, $s(v) = k$. We prove that if G is a connected bipartite graph with two parts $X = \{x_1, \ldots, x_r\}$, $Y = \{y_1, \ldots, y_s\}$ and $c_1, \ldots, c_r$, $d_1, \ldots, d_s$ are real numbers, then there is an $\mathbb{R}$-flow such that $s(x_i) = c_i$ and $s(y_j) = d_j$, for $1 \leq i \leq r$, $1 \leq j \leq s$ if and only if $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$. Also, it is shown that if G is a connected non-bipartite graph and $c_1, \ldots, c_n$ are arbitrary integers, then there is a $\mathbb{Z}$-flow such that $s(v_i) = c_i$, for $i = 1, \ldots, n$ if and only if the number of odd c_i is even.

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‡ *E-mail addresses*: s_akbari@sharif.edu, kano@mx.ibaraki.ac.ip, rezagbk@ipm.ir, sa_zare_f@yahoo.com.

1 Introduction

A *simple graph* is a graph without loops or multiple edges. Throughout this paper all graphs are simple. Let G be a graph. The number of vertices and the number of edges of G is called the *order* and the *size* of G, respectively. A graph is *k-edge connected* if the minimum number of edges whose removal would disconnect the graph is at least k.

Let G be a graph, $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_m\}$ be the vertex set and the edge set of G, respectively. The *adjacency matrix* of G, $A = (a_{ij})$, is an $n \times n$ matrix, where $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$, otherwise. Also, the *incidence matrix* of G, $N = (n_{ij})$, is an $n \times m$ matrix, where $n_{ij} = 1$ if the vertex v_i is incident with the edge e_j, and $n_{ij} = 0$, otherwise. Let $\boldsymbol{j}$ be an $n \times 1$ matrix whose all entries are one. An $n \times n$ non-negative matrix $A = (a_{ij})$ is said to be *primitive* if $A^k > 0$, for some positive integer k. A graph G is said to be *primitive* if there exists an integer $k > 0$ such that for all ordered pairs of vertices $i, j \in V(G)$ (not necessarily distinct), there is a walk from i to j of length k. If A is the adjacency matrix of a graph G, then the $(i,j)th$ entry of A^k is the number of walks of length k from v_i to v_j in G; see Theorem 1.7 of [2]. So, a graph G with the adjacency matrix A is *primitive* if $A^k > 0$, for some positive integer k.

Let G be a graph and H be an abelian group. Let $H^* = H \setminus \{0\}$. For every subset $S \subseteq H$ a map $\phi : E(G) \to S$ is called an *S-flow*. For a given S-flow of G and every $v \in V(G)$, define $s(v) = \sum_{uv \in E(G)} \phi(uv)$. Let k be an element of H. We say that a graph G admits a *k-sum S-flow* if there exists an S-flow ϕ of G such that for each vertex v, $s(v) = k$. A 0-sum S-flow was first defined in [1].

In this paper we obtain some necessary and sufficient conditions under which a graph admits a 1-sum $\mathbb{R}$-flow or a 1-sum $\mathbb{Z}^*$-flow. Also, we shall generalize the concept of k-sum S-flow.

2 A generalization of k-sum S-flows

Let G be a graph and $V(G) = \{v_1, \ldots, v_n\}$. In this section we generalize the concept of k-sum S-flows. We would like to study those graphs with the property that for any given real numbers $c_1, \ldots, c_n$, there exists an $\mathbb{R}$-flow such that $s(v_i) = c_i$, for $i = 1, \ldots, n$.

The following interesting result was proved in [3, p.63].

Theorem 1. *The incidence matrix of a connected graph of order n has rank n if it has an odd cycle and has rank $n-1$, otherwise.*

Now, we have the following result.

Theorem 2. *Let G be a connected non-bipartite graph with the vertex set $\{v_1, \ldots, v_n\}$ and $c_1, \ldots, c_n$ be real numbers. Then there is an $\mathbb{R}$-flow of G such that $s(v_i) = c_i$, for $i = 1, \ldots, n$.*

Proof. Assume that N is the incidence matrix of G and $E(G) = \{e_1, \ldots, e_m\}$. By Theorem 1, $rank(N) = n$. Since, N is full rank, the columns of N generate $\mathbb{R}^n$. Thus there is a real vector $Z = [z_1, \ldots, z_m]^T$ such that $NZ = [c_1, \ldots, c_n]^T$. Now, define $\phi(e_i) = z_i$, for $i = 1, \ldots, m$. Obviously, $s(v_i) = c_i$, for $i = 1, \ldots, n$. $\square$

Remark 1. By the proof of Theorem 2, one can find an $\mathbb{R}$-flow of G with the desired property in which at most n edges have zero values.

Remark 2. In Theorem 2 one can replace the real numbers with any field of characteristic zero.

Theorem 3. *Let G be a connected bipartite graph with two parts $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_s\}$. Let $c_1, \ldots, c_r$ and $d_1, \ldots, d_s$ be real numbers. Then there is an $\mathbb{R}$-flow of G such that $s(x_i) = c_i$ and $s(y_j) = d_j$, for $1 \leq i \leq r$, $1 \leq j \leq s$ if and only if $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$.*

Proof. The necessity is obvious, and so we shall prove the sufficiency. Let G be a bipartite graph of order n and size m and N be the incidence matrix of G. By Theorem 1, $rank(N) = n - 1$. Suppose that N' is the column reduced echelon form of the matrix N. We note that the column spaces of N and N' are the same. Since $rank(N) = n - 1$, N' has the following form:

$$N' = \left[\begin{array}{c|c} I_{n-1} & \\ \hline * \quad * \quad \cdots \quad * & 0 \end{array}\right],$$

Figure 1

where the size of zero matrix is n by $m-n+1$. Clearly, $[c_1,\, c_2,\, \ldots,\, c_r,\, d_1,\, d_2,\, \ldots,\, d_{s-1},\, x]^T$ is contained in the column space of N, for some $x \in \mathbb{R}$. Thus there exists an $\mathbb{R}$-flow of G such that $s(x_i) = c_i$, $s(y_j) = d_j$, for $1 \leq i \leq r$, $1 \leq j \leq s - 1$. Since $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$, we conclude that $s(y_s) = x = d_s$ and the proof is complete. $\square$

Now, we state Theorem 3 for the integers.

Theorem 4. *Let G be a connected bipartite graph with two parts $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_s\}$. Let $c_1, \ldots, c_r$ and $d_1, \ldots, d_s$ be integers. Then there is a $\mathbb{Z}$-flow of G such that $s(x_i) = c_i$ and $s(y_j) = d_j$, for $1 \leq i \leq r$, $1 \leq j \leq s$ if and only if $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$.*

Proof. One side is clear. We prove the other side by induction on $t = \sum_{i=1}^{r} |c_i| + \sum_{j=1}^{s} |d_j|$. If $t = 0$, then the assertion is trivial. Let $t > 0$. First assume that there are two elements in the set $\{c_1, \ldots, c_r\}$ with the different signs. Assume that c_{r-1} is positive and c_r is negative. By the induction hypothesis, there exists a $\mathbb{Z}$-flow of G such that for each i, $s(x_i) = c_i$, $s(y_j) = d_j$, for $1 \leq i \leq r - 2$, $1 \leq j \leq s$ and $s(x_{r-1}) = c_{r-1} - 1$, $s(x_r) = c_r + 1$. Now, since G is connected, there exists a path of even length between x_{r-1}

and x_r. Now, add $+1$ and -1 to the values of all edges of this path alternatively starting from x_{r-1} to obtain the desired $\mathbb{Z}$-flow. Now, assume that one element of $\{c_1, \ldots, c_r\}$ and one element of $\{d_1, \ldots, d_s\}$ have the same sign, say c_r and d_s, and they are positive. By induction hypothesis there exists a $\mathbb{Z}$-flow of G such that for each i, $s(x_i) = c_i$, $s(y_j) = d_j$, $1 \leq i \leq r-1$, $1 \leq j \leq s-1$, $s(x_r) = c_r - 1$, $s(y_s) = d_s - 1$. Now, since G is connected, there exists a path of odd length between x_r and y_s. Add $+1$ and -1 to the values of all edges of this path alternatively starting from x_r to obtain the desired $\mathbb{Z}$-flow. Now, assume that both c_r and d_j are negative. By induction hypothesis there exists a $\mathbb{Z}$-flow of G such that for each i, $s(x_i) = c_i$, $s(y_j) = d_j$, $1 \leq i \leq r-1$, $1 \leq j \leq s-1$, $s(x_r) = c_r + 1$, $s(y_s) = d_s + 1$. Consider a path between x_r and y_s and add -1 and $+1$ to the values of all edges of this path alternatively starting from x_r to obtain the desired $\mathbb{Z}$-flow. Note that since $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$, one of the above cases occurs and the proof is complete. □

In [1], the following theorem was proved.

Theorem 5. *(i) If G is a connected bipartite graph, then G has a 0-sum $\mathbb{Z}^*$-flow if and only if it is 2-edge connected.*
(ii) Suppose G is not a bipartite graph. Then G has a 0-sum $\mathbb{Z}^*$-flow if and only if for any edge e of G, $G\backslash\{e\}$ has no bipartite component.

Now, we are ready to state the next result.

Theorem 6. *Let G be a 2-edge connected bipartite graph with two parts $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_s\}$. Suppose that $c_1, \ldots, c_r, d_1, \ldots, d_s$ are integers. Then there is a $\mathbb{Z}^*$-flow of G such that $s(x_i) = c_i$ and $s(y_j) = d_j$, for $1 \leq i \leq r$, $1 \leq j \leq s$ if and only if $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$.*

Proof. One side is clear. Now, assume that $\sum_{i=1}^{r} c_i = \sum_{j=1}^{s} d_j$. Let $|E(G)| = m$. By Theorem 4, there is $U \in \mathbb{Z}^m$ such that $NU = [c_1, \ldots, c_r, d_1, \ldots, d_s]^T$. By Theorem 5, G admits a 0-sum $\mathbb{Z}^*$-flow. So, there exists a nowhere-zero vector $V \in \mathbb{Z}^m$ such that $NV = 0$. Clearly, there is $a \in \mathbb{Z}$ such that no entry of $U + aV$ is zero. Thus $N(U + aV) = [c_1, \ldots, c_r, d_1, \ldots, d_s]^T$ and the proof is complete. □

Before proving the next result we need the following theorem.

Theorem 7.[5] *A graph G is primitive if and only if G is connected and contains an odd cycle.*

Now, we prove the following result.

Theorem 8. *Let G be a connected non-bipartite graph with the vertex set $\{v_1, \ldots, v_n\}$ and $c_1, \ldots, c_n$ be integers. Then there is a $\mathbb{Z}$-flow such that $s(v_i) = c_i$, for $i = 1, \ldots, n$ if and only if the number of odd c_i is even.*

Proof. First suppose that G admits a $\mathbb{Z}$-flow, $\phi : E(G) \to \mathbb{Z}$, such that $s(v_i) = c_i$, for $i = 1, \ldots, n$. We have

$$2 \sum_{e \in E(G)} \phi(e) = \sum_{i=1}^{n} s(v_i) = \sum_{i=1}^{n} c_i.$$

Clearly, this implies that the number of odd c_i is even.

Now, assume that the number of odd c_i is even, for $i = 1, \ldots, n$. Let A be the adjacency matrix of G. Since, G is connected and contains an odd cycle, by Theorem 7, there exists a positive integer k such that $A^k > 0$. Since $A^k > 0$ implies that $A^{k+1} > 0$, we can assume that k is odd and $A^k > 0$. So for every v_i there is a closed walk, say C_i, of length k which contains v_i, for $i = 1, \ldots, n$, see Theorem 1.7 of [2]. With no loss of generality assume that $c_1, \ldots, c_t$ are even integers and $c_{t+1}, \ldots, c_n$ are odd integers. Now, assign $\frac{c_i}{2}$ and $-\frac{c_i}{2}$ to $E(C_i)$ alternatively, for $i = 1, \ldots, t$ and assign $\lfloor \frac{c_i}{2} \rfloor$ and $-\lfloor \frac{c_i}{2} \rfloor$ to $E(C_i)$ alternatively, for $i = t + 1, \ldots, n$. If C_i and C_j have some common edges, then add the two values of each edge which is contained in both C_i and C_j, for $1 \leq i, j \leq n$. So, $s(v_i) = c_i$, for $i = 1, \ldots, t$ and $s(v_i) = c_i - 1$, for $i = t + 1, \ldots, n$. On the other hand, there exists a walk of length k between v_i and v_{i+1}, for $i = t + 1, \ldots, n - 1$. Let W_i be a walk of length k between v_i and v_{i+1}, for $i = t + 1, t + 3, \ldots, n - 1$ (Note that $t + 1$ and $n - 1$ have the same parity). Assign 1 and -1 to all edges of W_i, alternatively, for $i = t + 1, t + 3, \ldots, n - 1$. If W_i and W_j have some common edges, then add two values of each edge which is contained in both

W_i and W_j. By continuing this procedure and assigning zero to each edge of G which is contained in no W_i or C_i, we obtain a labeling with the desired property. □

Corollary 1. *Let G be a connected non-bipartite graph with the vertex set $\{v_1, \ldots, v_n\}$ such that the removing of no edge does not make bipartite component and $c_1, \ldots, c_n$ be arbitrary integers. Then there is a $\mathbb{Z}^*$-flow such that $s(v_i) = c_i$, for $i = 1, \ldots, n$ if and only if the number of odd c_i is even.*

Proof. Let N be the incidence matrix of G. Suppose that the number of c_i is even. By the previous theorem there exists $U \in \mathbb{Z}^m$ such that $NU = [c_1, \ldots, c_n]^T$, where m is the size of G. By Theorem 5, Part (ii), there exists a nowhere-zero vector $U' \in \mathbb{Z}^m$ such that $NU' = 0$. By considering a vector $U + rU'$, for some suitable $r \in \mathbb{Z}$, we obtain a $\mathbb{Z}^*$-flow such that $s(v_i) = c_i$, for $i = 1, \ldots, n$. □

3 1-sum S-flows in graphs

The next lemma provides a necessary condition for the existence of a 1-sum $\mathbb{Z}$-flow in a graph.

Lemma 1. *Let G be a graph of order n and k be an odd integer. If G admits a k-sum $\mathbb{Z}$-flow, then n is even.*

Proof. Let ϕ be a k-sum $\mathbb{Z}$-flow. We have

$$kn = \sum_{v \in V(G)} s(v) = 2 \sum_{e \in E(G)} \phi(e).$$

Thus n is even. □

Before stating the next theorem we need one lemma.

Lemma 2. *Let G be a graph such that for every $e \in E(G)$, there exists an even cycle containing e. Then G admits a 0-sum $\mathbb{Z}^*$-flow.*

Proof. Assume that $E(G) = \{e_1, \ldots, e_m\}$. By assumption each e_i is contained in an even cycle, say C_i. Now, assign 2 and -2 to $E(C_1)$, alternatively and assign 0 to the remaining edges of G. In the new edge labeling of G add 2^2 and -2^2 to the values of $E(C_2)$, alternatively and keep the values of the remaining edges of G. Continue this procedure for every e_i and add 2^i and -2^i to the values of $E(C_i)$, alternatively and keep the values of the remaining edges of G in each step, for $i = 3, \ldots, m$. By this method we obtain a 0-sum $\mathbb{Z}^*$-flow for G. □

Theorem 9. *Let G be a connected non-bipartite graph such that for every $e \in E(G)$, there exists an even cycle containing e. Then G admits a 1-sum $\mathbb{Q}^*$-flow.*

Proof. Let N be the incidence matrix of G and $E(G) = \{e_1, \ldots, e_m\}$. By Lemma 2, there exists a nowhere-zero integer vector Y such that $NY = 0$. Also, by Theorem 2 and Remark 2, there exists $X \in \mathbb{Q}^m$ such that $NX = \boldsymbol{j}$. Clearly, there is an integer a such that no entry of $X + aY$ is zero. So, $N(X + aY) = \boldsymbol{j}$. Hence G admits a 1-sum $\mathbb{Q}^*$-flow. □

Theorem 10. *Let G be a connected non-bipartite graph of even order such that every edge is contained in an even cycle. Then G admits a 1-sum $\mathbb{Z}^*$-flow.*

Proof. Let A be the adjacency matrix of G. Since, G is connected and contains an odd cycle, by Theorem 7, we conclude that there exists a positive integer k such that $A^k > 0$. Since $A^k > 0$ implies that $A^{k+1} > 0$, we can assume that k is odd and $A^k > 0$.

Let $V(G) = \{v_1, \ldots, v_{2n}\}$ and $|E(G)| = m$. For each i, $1 \leq i \leq n$, there exists a walk of length k, say W_{2i-1}, between v_{2i-1} and v_{2i}. Assign 1 and -1, alternatively to all edges of W_1. Then assign $1, -1$, alternatively to all edges of W_3. If W_1 and W_3 have some common edges, then add the two values of each edge which is contained in both W_1 and W_3. By continuing this procedure and assigning zero to each edge of G which is contained in no

W_{2j-1}, we obtain a 1-sum $\mathbb{Z}$-flow for G. Let N be the incidence matrix of G. Thus there exists $X \in \mathbb{Z}^m$ such that $NX = \boldsymbol{j}$. By Lemma 2, G admits a 0-sum $\mathbb{Z}^*$-flow, so there exists a nowhere-zero vector $Y \in \mathbb{Z}^m$ such that $NY = 0$. On the other hand, for every $a \in \mathbb{Z}$, $N(X + aY) = \boldsymbol{j}$. Therefore G admits a 1-sum $\mathbb{Z}^*$-flow and the proof is complete. □

In the sequel, we want to determine those bipartite graphs which admit a 1-sum $\mathbb{Z}^*$-flow. The next result is an immediate consequence of Theorem 6.

Theorem 11. *Let G be a 2-edge connected bipartite graph with two parts X and Y. Then G admits a 1-sum $\mathbb{Z}^*$-flow if and only if $|X| = |Y|$.*

Remark 3. The 2-edge connectivity in Theorem 11 is not superfluous. Let G be the graph shown in the Figure 2. It is not hard to check that G does not admit a 1-sum $\mathbb{Z}^*$-flow.

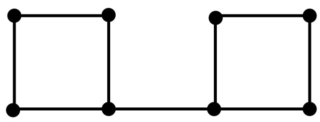

Figure 2

Question. Determine a necessary and sufficient condition under which a bipartite graph admits a 1-sum $\mathbb{R}^*$-flow or a 1-sum $\mathbb{Z}^*$-flow?

A matrix is said to be *totally unimodular* if every square submatrix of it has determinant $-1, 0$ or 1.

In 1931, Egervary [4] proved the following theorem.

Theorem 12. *Let G be a graph with the incidence matrix N. Then G is bipartite if and only if N is totally unimodular.*

Now, we have the following theorem.

Theorem 13. *Let G be a bipartite graph and k be an integer. If G admits a k-sum $\mathbb{R}$-flow, then G admits a k-sum $\mathbb{Z}$-flow.*

Proof. Let N be the incidence matrix of G and $rank(N) = r$. Then we can assume that $N = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A is an r by r matrix and $rank(A) = r$. Since N is totally unimodular, we have $A^{-1} \in M_r(\mathbb{Z})$ and this implies that

$$\begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix} N = \begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix} \in M_{n\times m}(\mathbb{Z}).$$

Since $rank(N) = r$, we find that $D - CA^{-1}B = 0$. By assumption the equation

$$\begin{bmatrix} I \mid A^{-1}B \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} A^{-1} & 0 \\ -CA^{-1} & I \end{bmatrix} k\mathbf{j}$$

has a real solution. Thus the equation $NX = k\boldsymbol{j}$ has an integer solution and the proof is complete. $\square$

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