Spanning k-ended trees of Bipartite Graphs

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Abstract

A tree is called a k-ended tree if it has at most k leaves, where a leaf is a vertex of degree one. We prove the following theorem. Let $k \geq 2$ be an integer, and let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A|+k-1$. If $\sigma_2(G) \geq (|G|-k+2)/2$, then G has a spanning k-ended tree, where $\sigma_2(G)$ denotes the minimum degree sum of two non-adjacent vertices of G. Moreover, the condition on $\sigma_2(G)$ is sharp. It was shown by Las Vergnas, and Broersma and Tuinstra, independently that if a graph H satisfies $\sigma_2(H) \geq |H|-k+1$ then H has a spanning k-ended tree. Thus our theorem shows that the condition becomes much weaker if a graph is bipartite.

Keywords: spanning tree, spanning k-ended tree, spanning tree with at most k leaves

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, let $N_G(v)$ denote the neighborhood of v in G, and denote the degree of v in

G by $\deg_G(v)$, in particular, $\deg_G(v) = |N_G(v)|$. For two vertices x and y of G, an edge joining them is denoted by xy or yx. A vertex of a tree is called a *leaf* if its degree is one. For an integer $k \geq 2$, a tree is called a *k-ended tree* if it has at most k leaves.

The invariant $\sigma_2(G)$ is defined to be the minimum degree sum of two non-adjacent vertices of G, i.e.,

$$\sigma_2(G) = \min_{xy \notin E(G)} \{\deg_G(x) + \deg_G(y)\}.$$

By using $\sigma_2(G)$, Ore obtained the following famous theorem on Hamilton path. Notice that a Hamilton path is a spanning 2-ended tree.

Theorem 1 (Ore [7]). *Let G be a connected graph. If $\sigma_2(G) \geq |G| - 1$, then G has a Hamilton path.*

The following theorem gives a similar sufficient condition for a graph to have a spanning k-ended tree.

Theorem 2 (Las Vergnas [5], Broersma and Tuinstra [2]). *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

Our main result of this paper is the following theorem, which shows that the lower bound on $\sigma_2(G)$ in Theorem 2 can be much weakened for bipartite graphs.

Theorem 3. *Let $k \geq 2$ be an integer, and let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + k - 1$. If*

$$\sigma_2(G) \geq \frac{|G| - k}{2} + 1, \tag{1}$$

then G has a spanning k-ended tree.

Note that the condition $|B| \leq |A| + k - 1$ is necessary for the bipartite graph G to have a spanning k-ended tree. Moreover, the degree sum condition is sharp in the sense that we cannot replace the lower bound on $\sigma_2(G)$ by $(|G| - k + 1)/2$. We show this sharpness in the last section.

On the other hand, one might conjecture that $\sigma_2(G)$ can be replaced by

$$\sigma_{1,1}(G) = \min_{xy \notin E(G)} \{\deg_G(x) + \deg_G(y) \mid x \in A,\ y \in B\}.$$

In fact, we obtain the following theorem.

Theorem 4. *Let $k \geq 2$ be an integer, and let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + k - 1$. If*

$$\sigma_{1,1}(G) \geq |B|, \tag{2}$$

then G has a spanning k-ended tree.

The above Theorem 4 is a generalization of the following Theorem 5 on Hamilton path.

Theorem 5 (Moon and Moser [4]). *Let G be a connected bipartite graph with bipartition (A, B) such that $|A| \leq |B| \leq |A| + 1$. If $\sigma_{1,1}(G) \geq |B|$, then G has a Hamilton path.*

Many results on spanning k-ended trees related to our theorems can be found in the book [1] and papers [3], [6] and so on. In particular, a survey article [8] contains many current results on spanning trees including spanning k-ended trees.

2 Proof of Theorem 3

We begin with some notation. A set X of vertices of G is called an independent set if no two vertices of X are adjacent in G. Let T be a tree. We denote the set of leaves of T by $Leaf(T)$. For two vertices u and v of T, there exists a unique path connecting u and v in T, and it is denoted by $P_T(u, v)$. We need the next lemma.

Lemma 2.1. *Let T be a tree whose vertices are colored with red and blue so that no two adjacent vertices have the same color. If all the leaves of T are red, then the number of red vertices of T is greater than or equal to the number of blue vertices of T.*

Proof. We prove the lemma by induction on $|T|$. It is easy to see that the lemma holds for small trees. Let T be a tree. We remove all the leaves from T, and denote the resulting tree by T_1. Again remove all the leaves of T_1 from T_1, and denote the resulting tree by T_2. Since every leaf of T_1 is adjacent to at least one leaf of T, the number of leaves of T is greater than or equal to the number of leaves of T_1. It is easy to see that all the leaves of T_2 are red. Hence by induction hypothesis, the number of red vertices of T_2 is at least the number of blue vertices of T_2. Therefore the lemma holds.

We now prove Theorem 3.

Proof of Theorem 3. Let G be a connected bipartite graph with bipartition (A, B) that satisfies all the conditions in Theorem 3. Suppose that G has no spanning k-ended tree. We choose a spanning tree T of G so that

(T1) the number of leaves of T is as small as possible;

(T2) the length of a longest path in T is as large as possible subject to (T1).

Then the number of leaves of T is $|Leaf(T)| = \ell \geq k + 1 \geq 3$ since G has no spanning k-ended tree. In particular, T is not a Hamilton path and has at least one vertex of degree at least three.

For convenience, we call a vertex of B a *red vertex* and a vertex of A a *blue vertex*. We shall consider two cases.

Case 1. *T contains both a red leaf and a blue leaf.*

Let v be a red leaf and w a blue leaf of T. It is easy to see that no two leaves of T are adjacent in G since otherwise we can get a spanning tree having fewer leaves than T. For a vertex $x \in V(T) - \{v, w\}$, x_v denotes the vertex which is adjacent to x and lies on the path $P_T(x, v)$, and x_w is defined analogously. In other words, x_v (x_w) is the parent of x in a rooted tree T with root v (w), respectively. So if x does not lie on $P_T(v, w)$, then $x_v = x_w$, and if x lies on $P_T(v, w)$, then $x_v \neq x_w$.

Suppose that v is adjacent to a vertex x in G but not in T, and that w is adjacent to x_v in G. Then x is not a leaf of T, and $T_1 = T - xx_v + vx + wx_v$ contains a unique cycle, which has an edge e_1 incident with a vertex of degree at least three in T_1. Then $T_1 - e_1$ is a spanning tree of G with $\ell - 1$ leaves. This contradicts the choice (T1) of T. Hence w is not adjacent to x_v in G. If v is adjacent to a vertex x in T, then $x_v = v$ and so w is not adjacent to x_v in G.

Moreover, if v is adjacent to two distinct vertices y and z in G such that $y_v = z_v$, then y is not a leaf of T and y_v has degree at lest three in T, and thus $T + vy - yy_v$ is a spanning tree of G with $\ell - 1$ leaves, which contradicts the choice (T1) of T. Hence, if v is adjacent to two distinct vertices y and z in G, then $y_v \neq z_v$.

By the symmetry of v and w, we obtain the following statements:

(i) If v is adjacent to x in G, then w is not adjacent to x_v in G.
(ii) $y_v \neq z_v$ for all two distinct $y, z \in N_G(v)$.
(iii) If w is adjacent to x in G, then v is not adjacent to x_w in G.
(iv) $y_w \neq z_w$ for all two distinct $y, z \in N_G(w)$.

Therefore the following four vertex sets are pairwise disjoint since $N_G(v)$ and $N_G(w)$ consist of blue vertices and red vertices, respectively.

$$N_G(v),\ \{x_v : x \in N_G(v)\},\ N_G(w),\ \{x_w : x \in N_G(w)\}.$$

It is clear that $Leaf(T) - \{v, w\}$ is disjoint from the above four subsets. Thus

$$\begin{aligned}|G| &\geq |N_G(v)| + |\{x_v : x \in N_G(v)\}| + |N_G(w)| + |\{x_w : x \in N_G(w)\}| \\ &\quad + |Leaf(T) - \{v, w\}| \\ &= 2\deg_G(v) + 2\deg_G(w) + \ell - 2 \\ &\geq 2\sigma_2(G) + k - 1.\end{aligned}$$

This contradicts the assumption (1), and hence this case is proved.

Case 2. *All the leaves of T have the same color.*

If $|A| = |B|$, then we may assume that all the leaves are red by symmetry of A and B. If $|A| < |B|$, then all the leaves are red by Lemma 2.1.

Let $P_T(v, w)$ be a longest path in T, which connects two red leaves v and w. First suppose that $T - V(P_T(v, w))$ consists of only isolated vertices. In other words, assume that every vertex of $T - V(P_T(v, w))$ is a leaf of T. Since all the leaves of T are red, the number of red vertices of T equals the number of leaves in $Leaf(T) - \{v, w\}$ plus the number of red vertices in $P_T(v, w)$. Since $\ell \geq k + 1$, we have

$$\begin{aligned}|B| &= \ell - 2 + \frac{|P_T(v, w)| + 1}{2} \\ &\geq k + \frac{|P_T(v, w)| - 1}{2} \\ &= k + |A|.\end{aligned}$$

This contradicts the assumption that $|B| \leq |A| + k - 1$.

Hence $T - V(P_T(v, w))$ contains a path of order at least two. We denote a path including such a path and connecting to $P_T(v, w)$ by $Q(s, t)$, where s is a vertex on $P_T(v, w)$ and t is a leaf of T. So $|Q(s, t)| = |V(Q(s, t))| \geq 3$ and $V(Q(s, t)) \cap V(P_T(v, w)) = \{s\}$.

Suppose that v is adjacent to a vertex $x \in V(T) - V(P_T(v, w))$ in G. Then $T - xx_v + vx$ is a spanning tree that has at most ℓ leaves and contains a longer path than $P_T(v, w)$. This contradicts the choice (T2) of T. Hence $N_G(v) \subseteq V(P_T(v, w)) - \{v, w\}$. Similarly $N_G(w) \subseteq V(P_T(v, w)) - \{v, w\}$.

Assume that v is adjacent to a vertex x of $P_T(v, w) - \{v, w\}$ in G but not in T. Then x is a blue vertex. Let x^* denote the vertex of $P_T(v, w)$ such that the distance between x and x^* is two and x^* is closer to v than x. In brief, x^* is the next blue vertex of x on $P_T(v, w)$ closer to v. If x_v, which lies between x and x^*, is s, then $T - xx_v + vx$ is a spanning tree with $\ell - 1$ leaves, a contradiction. Thus $x_v \neq s$.

Assume that w is adjacent to x^* in G. If $x = s$, then $T_1 = T + wx^* - xx_v$ is a spanning tree of G that has at most ℓ leaves and includes a longer path $P_{T_1}(t, v)$ than $P_T(v, w)$, a contradiction. Similarly, we can derive a contradiction in the case $x^* = s$. If $s \notin \{x, x_v, x^*\}$, then $T_2 = T + vx + wx^* - xx_v - ss_w$ is a spanning tree of G that has at most ℓ leaves and includes a longer path $P_{T_2}(t, s_w)$ than $P_T(v, w)$, a contradiction. Hence w is not adjacent to x^* in G.

Therefore, we obtain

$$\begin{aligned}
\sigma_2(G) &\le \deg_G(v) + \deg_G(w) \\
&\le |N_G(v)| + |\{\text{the blue vertices on } P_T(v, w)\} \\
&\qquad\qquad - \{x^* : x \in N_{G-E(T)}(v)\}| \\
&= |\{\text{the blue vertices on } P_T(v, w)\}| + 1 \\
&= \frac{|P_T(v, w)| - 1}{2} + 1 \\
&\le \frac{(|G| - \ell + 1) - 1}{2} + 1 \qquad (\text{by } |Q(s, t)| \ge 3) \\
&\le \frac{|G| - k + 1}{2}. \qquad (\text{by } \ell \ge k + 1)
\end{aligned}$$

This is a final contradiction. Consequently the theorem is proved. □

3 Proof of Theorem 4

The proof of Theorem 4 is quite similar to that of Theorem 3.

Let G be a connected bipartite graph with bipartition (A, B) which satisfies all the conditions in Theorem 4. For convenience, we call a vertex of B a red vertex and a vertex of A a blue vertex.

Suppose that G has no spanning k-ended tree. We choose a spanning tree T of G so that

(T1) the number of leaves of T is as small as possible;

(T2) the length of a longest path in T is as large as possible subject to (T1).

Then the number of leaves of T is $|Leaf(T)| = \ell \ge k + 1$ since G has no spanning k-ended tree. We shall consider two cases.

Case 1. *T contains both a red leaf and a blue leaf.*

Let v be a red leaf of T and w a blue leaf of T. By the same argument of Theorem 3, we obtain $|G| \ge 2\deg_G(v) + 2\deg_G(w) + \ell - 2 \ge 2\sigma_{1,1}(G) + k - 1$, and thus $\sigma_{1,1}(G) \le (|G| - k + 1)/2$. On the other hand, by the condition of Theorem 4 and by $|G| = |A| + |B| \le 2|B|$, we have $|G|/2 \le |B| \le \sigma_{1,1}(G)$. This contradicts the above inequality, and hence this case is proved.

Case 2. *All the leaves of T have the same color.*

By the same argument in Case 2 of the proof of Theorem 3, we may assume that all leaves of T are red. Let $P_T(v,w)$ be a longest path in T. By the same argument as in Case 2 of Theorem 3, we can show that $T - V(P_T(v,w))$ contains a path of length at least one. We take a path $Q(s,t)$ as in the proof of Theorem 3, where s is a vertex on $P_T(v,w)$, t is a leaf of T and $|Q(s,t)| = |V(Q(s,t))| \geq 3$.

Let u denote the vertex adjacent to t in T. Then u is a blue vertex of $Q(s,t)$ since t is red. It follows from the choice (T2) of T that u is adjacent to neither v nor w in G. Moreover, we can similarly show that $N_G(v) \subseteq V(P_T(v,w)) - \{v,w\}$ as in the proof of Theorem 3.

Assume that v is adjacent to a vertex x of $P_T(v,w) - \{v,w\}$ in $G - E(T)$, and that u is adjacent to a vertex x_v in G, where x_v is defined in the proof of Theorem 3. Then $T + vx + ux_v - xx_v - rs$ is a spanning tree that has at most ℓ leaves and contains a longer path than $P_T(v,w)$. This contradicts (T2). Hence u is not adjacent to x_v in G. For a red vertex v and a blue vertex u, we obtain

$$\begin{aligned}\sigma_{1,1}(G) &\leq \deg_G(v) + \deg_G(u) \\ &\leq |N_G(v)| + |\{\text{the red vertices in } G - V(P(v,w))\}| \\ &\quad + |\{\text{the red vertices in } P_T(v,w)\} - \{v,w\} - \{x_v : x \in N_{G-E(T)}(v)\}| \\ &\leq |\{\text{the red vertices in } V(G) - \{v,w\}\}| + 1 \\ &= |B| - 1.\end{aligned}$$

This is a final contradiction. Consequently the theorem is proved. □

4 Sharpness of Theorem 3

Let $m \geq 2$ be an integer and let a and b be two non-negative integers such that $a + b = k - 1$. Construct the bipartite graph G with partite sets A and B as follows: Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where $|A_1| = m + a$, $|A_2| = m$, $|B_1| = m$ and $|B_2| = m + b$. Join each vertex of A_1(resp. A_2) to every vertex of B_1(resp. B_2), and join a vertex x of A_2 to a vertex y of B_1. The resulting graph G is shown in Figure 1.

Then G satisfies $|G| = 4m + a + b$ and $\sigma_2(G) = 2m = (|G| - (a + b))/2 = (|G| - k + 1)/2$. But G has no spanning k-ended tree. To show this, we assume that G has a spanning k-ended tree T. Since T has at most k leaves and $a + b = k - 1$, the number of the edges in T is at least $2(|A_1| + |B_2|) - k + |\{xy\}| = 2(2m + a + b) - k + 1 = 4m + a + b$. This

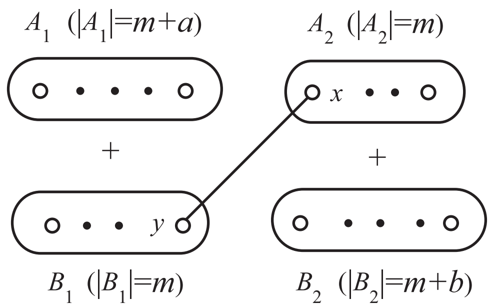

Figure 1: A bipartite graph G with bipartition $(A_1 \cup A_2, B_1 \cup B_2)$

contradicts $|E(T)| = |G| - 1 = 4m + a + b - 1$. Therefore the lower bound on $\sigma_2(G)$ is sharp.

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