# A Note on Leaf-constrained Spanning Trees in a Graph

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**Abstract**

An independent set $S$ of a connected graph $G$ is called a *frame* if $G - S$ is connected. If $|S| = k$, then $S$ is called a *k-frame*. We prove the following theorem. Let $k \geq 2$ be an integer, $G$ be a connected graph with $V(G) = \{v_1, v_2, \dots, v_n\}$, and $\deg_G(u)$ denote the degree of a vertex $u$. Suppose that for every 3-frame $S = \{v_a, v_b, v_c\}$ such that $1 \leq a < b < c \leq n$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$, it holds that $\deg_G(v_a) + \deg_G(v_b) + \deg_G(v_c) - |N_G(v_a) \cap N_G(v_b) \cap N_G(v_c)| \geq |G| - k + 1$. Then $G$ has a spanning tree with at most $k$-leaves. Moreover, the condition is sharp. This theorem is a generalization of the results of E. Flandrin, H.A. Jung and H. Li (*Discrete Math.* 90 (1991), 41–52) and of A. Kyaw (*Australasian Journal of Combinatorics.* 37 (2007), 3–10) for traceability.

## 1 Introduction

Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. In this paper, we consider only simple graphs, which has neither loops nor

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multiple edges. We write $|G|$ for the order of $G$, that is, $|G| = |V(G)|$. For a vertex $v$ of $G$, we denote by $\deg_G(v)$ the degree of $v$ in $G$, and by $N_G(v)$ the neighborhood of $v$ in $G$. A vertex of degree one is called an *end-vertex*, and an end-vertex of a tree is usually called a *leaf*. Let $X$ be a nonempty subset of $V(G)$. We write

$$N_G(X) = \bigcup_{x \in X} N_G(x) \qquad \text{and} \qquad \deg_G(X) = \sum_{x \in X} \deg_G(x).$$

The subgraph of $G$ induced by $X$ is denoted by $\langle X \rangle_G$. We write $G - X$ for $\langle V(G) - X \rangle_G$, and for a vertex $v$ of $G$, write $G - v$ for $G - \{v\}$. For an integer $i \geq 1$, define

$$N_G(X; i) = \{x \in V(G);\ |N_G(x) \cap X| = i\}.$$

In particular,

$$N_G(\{u, v, w\}; 3) = N_G(u) \cap N_G(v) \cap N_G(w).$$

Let $H$ be a sugraph of a graph $G$. If $xy$ is an edge of $G$ not contained in $H$, then $H + xy$ denotes the subgraph of $G$ obtained from $H$ by adding $xy$. For an edge $uv$ of $H$, $H - uv$ is defined analogously. A subset $S \subseteq V(G)$ is called *independent* if no two vertices of $S$ are adjacent in $G$. An independent set $S$ of $G$ is called a *frame* if $G - S$ is connected. A frame $S$ with $|S| = k$ is called a *$k$-frame*. For sets $X$ and $Y$, the cardinality of $X$ is denote by $|X|$, and $X \setminus Y$ is denoted by $X - Y$ if $Y \subseteq X$. For further explanation of terminology and notation, we refer to [2].

In [4], E. Flandrin, H.A. Jung and H. Li obtained the following theorem for a graph to have a hamiltonian path.

**Theorem 1 ([4])** *Let $G$ be a connected graph. If $\deg_G(\{u, v, w\}) - |N_G(\{u, v, w\}; 3)| \geq |G| - 1$ for every independent set $\{u, v, w\}$ of $G$, then $G$ has a hamiltonian path.*

A. Kyaw [5] improved the previous result in the following way.

**Theorem 2 ([5])** *Let $G$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that for every 3-frame $S = \{v_a, v_b, v_c\}$ of $G$ such that $1 \leq a < b < c \leq n$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$, it holds that $\deg_G(\{v_a, v_b, v_c\}) - |N_G(\{v_a, v_b, v_c\}; 3)| \geq |G| - 1$. Then $G$ has a hamiltonian path.*

Generalizing Theorem 1 and Theorem 2, we prove the following result.

**Theorem 3** *Let $k \geq 2$ be an integer, and $G$ be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that for every 3-frame $S = \{v_a, v_b, v_c\}$ of $G$ such that $1 \leq a < b < c \leq n$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b-1$ and $\deg_G(v_c) \leq c-2$, it holds that*

$$\deg_G(\{v_a, v_b, v_c\}) - |N_G(\{v_a, v_b, v_c\}; 3)| \geq |G| - k + 1. \tag{1}$$

*Then $G$ has a spanning tree with at most $k$ leaves.*

We first show that the condition $|G| - k + 1$ in (1) is sharp. Consider a complete bipartite graph $H = K_{m,m+k}$. It has no spanning tree with at most $k$ leaves, and for any numbering of vertices of $H$, $H$ satisfies $\deg_H(\{v_a, v_b, v_c\}) - |N_H(\{v_a, v_b, v_c\}; 3)| \geq |H| - k$. Hence the condition is sharp.

Since

$$\begin{aligned} & \deg_G(\{x, y, z\}) - |N_G(\{x, y, z\}; 3)| \\ = \; & \deg_G(\{x, y\}) + |N_G(z) - N_G(\{x, y, z\}; 3)|, \end{aligned}$$

Theorem 3 includes the following theorem of H. Broersma and H. Tuinstra [1].

**Theorem 4 ([1])** *Let $G$ be a connected graph. If $\deg_G(\{u, v\}) \geq |G| - k + 1$ for every independent set $\{u, v\}$ of $G$, then $G$ has a spanning tree with at most $k$ leaves.*

Some other results related to our theorem can be found in [3], [6], [7] and others.

## 2 Proof of Theorem 3

Let $G$ be a connected graph. We call a tree $T$ of $G$ a *maximum tree with 3 leaves* if there exists no tree $T'$ with 3 leaves in $G$ such that $|T| < |T'|$. To prove Theorem 3, we need the following lemmas.

**Lemma 5** *Suppose that a connected graph $G$ has no hamiltonian path. Let $T$ be a maximum tree with 3 leaves of $G$, which might be spanning, $r$ be the unique vertex of $T$ with $\deg_T(r) = 3$, and $V_1, V_2, V_3$ be the vertex sets of components of $T - r$. For every $1 \leq i \leq 3$, let $u_i$ be the leaf of $T$ contained in $V_i$, $w_i$ be the vertex of $V_i$ adjacent to $r$ in $T$, and $U = \{u_1, u_2, u_3\}$. For each vertex $x \in V_i$, the vertex that precedes $x$ on the path from $r$ to $x$ is denoted by $x^-$. Then the following holds:*

*(i)* $U$ *is an independent set of* $G$.
*(ii) For all two distinct* $i, j \in \{1, 2, 3\}$, *if* $x \in V_i \cap N_G(u_j)$, *then* $x \neq w_i$ *and* $x^- \notin N_G(U - \{u_j\})$.
*(iii) For every* $1 \leq i \leq 3$, $|V_i| \geq 1 + \sum_{j=1}^{3} |N_G(u_j) \cap V_i| - |N_G(U; 3) \cap V_i|$.
*(iv)* $|T| \geq 2 + \deg_G(U) - |N_G(U; 3)|$.

*Proof.* (i) Suppose $u_i u_j \in E(G)$ for some $1 \leq i < j \leq 3$. Then $T' = T + u_i u_j - r w_i$ is a path of $G$. Since $G$ does not have a hamiltonian path, there exist two adjacent vertices $x \in V(T')$ and $y \in V(G) - V(T')$. Then $T' + xy$ is a tree with 3 leaves, which contradicts the maximality of $T$. Hence (i) is proved.

(ii) Suppose that a vertex $x \in V_i$ is adjacent to $u_j$ for some $j \neq i$. If $x = w_i$, then $T + w_i u_j - r w_i$ is a path of $G$, and as in the proof of (i) we derive a contradiction. Hence we may assume $x \neq w_i$. Then $T + x u_j - x x^-$ is a maximum tree with 3 leaves, whose leaf set is $U - u_j + x^-$. Thus by (i) $x^-$ and $u_\ell \in U - \{u_j\}$ are not adjacent in $G$. Hence (ii) holds.

(iii) Let $\{i, j, \ell\} = \{1, 2, 3\}$ and $(N_G(\{u_j, u_\ell\}))^- = \{x^- : x \in N_G(\{u_j, u_\ell\})\}$. By (i) and (ii), it follows that $\{u_i\}$, $N_G(u_i) \cap V_i$, $(N_G(\{u_j, u_\ell\}))^- \cap V_i$ and $(N_G(u_j) \cap N_G(u_\ell) - N_G(U; 3)) \cap V_i$ are pair-wise disjoint. So we have

$$\begin{aligned} |V_i| &\geq |\{u_i\}| + |N_G(u_i) \cap V_i| + |(N_G(\{u_j, u_\ell\}))^- \cap V_i| \\ &\quad + |(N_G(u_j) \cap N_G(u_\ell) - N_G(U; 3)) \cap V_i| \\ &= 1 + |N_G(u_i) \cap V_i| + |(N_G(\{u_j, u_\ell\})) \cap V_i| \\ &\quad + |(N_G(u_j) \cap N_G(u_\ell) - N_G(U; 3)) \cap V_i| \\ &= 1 + \sum_{j=1}^{3} |N_G(u_j) \cap V_i| - |(N_G(U; 3) \cap V_i)|. \end{aligned}$$

(iv) Since $2 \geq \sum_{j=1}^{3} |N_G(u_j) \cap \{r\}| - |N_G(U; 3) \cap \{r\}|$, and by (iii) we obtain

$$\begin{aligned} \sum_{i=1}^{3} |V_i| + 1 &\geq 2 + \sum_{i=1}^{3} \sum_{j=1}^{3} |N_G(u_j) \cap V_i| - \sum_{i=1}^{3} |N_G(U; 3) \cap V_i| \\ &\quad + \sum_{j=1}^{3} |N_G(u_j) \cap \{r\}| - |N_G(U; 3) \cap \{r\}| \\ &= 2 + \sum_{j=1}^{3} |N_G(u_j) \cap V(T)| - |N_G(U; 3) \cap V(T)|. \end{aligned}$$

Since $N_G(u_j) \subset V(T)$ for every $1 \leq j \leq 3$ by the maximality of $T$, we have

$$|T| = \sum_{i=1}^{3} |V_i| + 1 \geq 2 + \sum_{j=1}^{3} |N_G(u_j)| - |N_G(U;3)|.$$

□

By using Lemma 5, we can measure the order of a tree with at most 3 leaves in $G$.

**Lemma 6** *Let $m \geq 1$ be an integer and $G$ be a connected graph with the vertex set $\{v_1, v_2, \ldots, v_n\}$. Assume that for every 3-frame $S = \{v_a, v_b, v_c\}$ such that $a < b < c$, $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$, it holds that $\deg_G(S) - |N_G(S;3)| \geq m$. Then $G$ has either a hamiltonian path or a tree with 3 leaves and of order at least $m + 2$.*

*Proof.* Assume that $G$ does not have a hamiltonian path. Let $T$ be a maximum tree with 3 leaves of $G$, and denote the leaves of $T$ by $v_a$, $v_b$, $v_c$, where $1 \leq a < b < c \leq n$. Choose such a tree $T$ so that

$$a + b + c \quad \text{is maximum} \tag{2}$$

among all the maximum trees with 3 leaves of $G$. Let $r$ be the unique vertex of $T$ with $\deg_T(r) = 3$, $U = \{v_a, v_b, v_c\}$ be the set of leaves of $T$, and $N_T(r) = \{w_a, w_b, w_c\}$, which lie on the paths from $r$ to $v_a$, $v_b$, $v_c$, respectively. By the maximality of $T$, we have $N_G(U) \subset V(T)$, and so by Lemma 5 (i), $U$ is a 3-frame of $G$.

We now show that $\deg_G(v_a) \leq a$, $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_c) \leq c - 2$. We first consider $v_c$. For every $v_t \in N_G(v_c) - N_T(v_c)$, it follows that $v_t \notin U \cup \{w_a, w_b\}$ from Lemma 5, and choose an edge $v_t v_x$ of $T$ in the cycle of $T + v_c v_t$. Then $T + v_c v_t - v_t v_x$ is a tree with the leaf set $\{v_a, v_b, v_x\}$. By (2) we have $x < c$. Since $a, b < c$, there exist at least $|N_G(v_c) - N_T(v_c)| + 2 = \deg_G(v_c) + 1$ vertices $v_y$ whose indexes $y$ are less than $c$. Hence $\deg_G(v_c) + 1 < c$, which implies $\deg_G(v_c) \leq c - 2$. By the same argument as above, we can show that $\deg_G(v_b) \leq b - 1$ and $\deg_G(v_a) \leq a$.

By Lemma 5 (iv), we obtain

$$|T| \geq 2 + \deg_G(U) - |N_G(U;3)| \geq 2 + m.$$

Consequently the lemma is proved. □

**Proof of Theorem 3.** If $G$ has a hamiltonian path, then this path is the desired tree. So we may assume that $G$ does not have a hamiltonian path. Choose a maximal tree $T$ with 3 leaves as in Lemma 6. Then

$$|T| \geq |G| - k + 1 + 2 = |G| - k + 3.$$

This implies $k \geq 3$, and also the theorem is proved when $k = 2$ or 3. Assume $k \geq 4$. By connecting all the vertices in $V(G) - V(T)$ to $T$ by edges or paths, we can obtain a spanning tree of $G$ with at most $3 + |G| - |T|$ leaves, which is the desired spanning tree of $G$ as $3 + |G| - |T| \leq k$. □

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