Discrete Geometry on Red and Blue Points in the Plane Lattice

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Abstract

We consider some problems on red and blue points in the plane lattice. An L-line segment in the plane lattice consists of a vertical line segment and a horizontal line segment having a common endpoint. There are some results on geometric graphs on a set of red and blue points in the plane. We show that some similar results also hold for a set of red and blue points in the plane lattice using L-line segments instead of line segments. For example, we show that if n red points and n blue points are given in the plane lattice in general position, then there exists a non-crossing geometric perfect matching covering them each of whose edges is an L-line segment and connects a red point and a blue point.

1 Introduction

We consider some problems on red points and blue points in the plane lattice $\mathbb{Z}^2$ motivated by some results in the plane $\mathbb{R}^2$, where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively. For a point x in the plane, an L-shaped line consisting of a vertical ray and a horizontal ray emanating from x is called an *L-line* with *corner* x. A vertical line and a horizontal line passing through x are also considered as special L-lines with corner x. So for every point x, there are exactly six L-lines with corner x, and two of them are usual lines (see in Fig. 1).

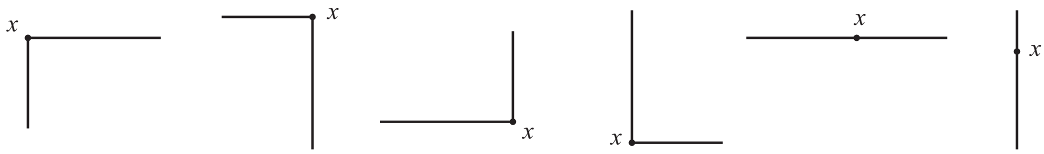


Figure 1: L-lines with corner x.

We regard L-lines as "lines" in the plane lattice, and consider some problems from this point of view. For two points in the plane lattice which are not on the same vertical or horizontal line, there are two L-lines passing through them (see (1) in Fig. 2). On the other hand, for any two points in the plane, there exists exactly one line that passes through them. So there is a difference between lines in the plane and L-lines in the plane lattice. However, as we shall show, they have some nice common properties.

Remark Let S be a set of points in the plane lattice. Usually, S is defined to be in general position if every vertical line or horizontal line contains at most one point of S. On the other hand, by using L-lines, S can be defined to be in general position if no three points of S lie on the same L-line (see (2) in Fig. 2). If S is in general position by means of the new definition, then the highest point and the lowest point of S may lie on the same vertical line, however for any other point x of S, the vertical line passing through x does not pass through any point of $S - \{x\}$. Similarly, the rightmost point and the leftmost point of S may lie on the same horizontal line, but for any other point y of S, the horizontal line passing through y does pass through any point of $S - \{y\}$. Therefore, the above two definitions of general position are slightly different, but they require the same condition for most points in S, and thus the difference is small.

Hereafter, to avoid confusion and for simplicity, we say that S is *in general position* if every vertical line and horizontal line passes through at most one point of S. Namely, we use a standard definition of general position[1].

[1] In the plane, no three points lie on the same line if and only if every three points make a triangle. Similarly, in the plane lattice, every vertical line and horizontal line passes through at most one point if and only if every two points make a digon with two L-line segments.

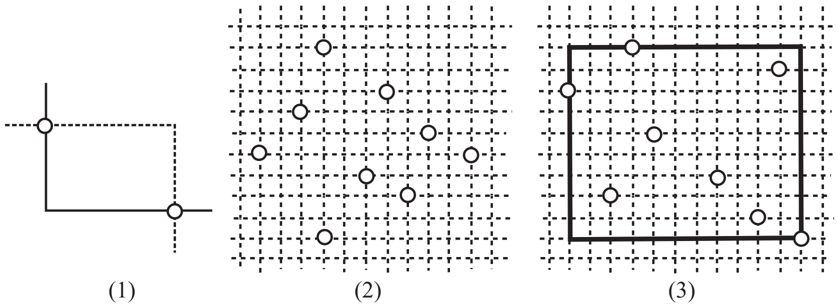


Figure 2: (1) Two L-lines passing through two given points. (2) A set of points no three of which lie on the same L-line. (3) A rectangular hull of a set of points in the plan lattice in general position.

2 Geometric Alternating Matchings

A set X of points in the plane is called *in general position* if no three points of X lie on the same line. It is well-known that if n red points and n blue points are given in the plane in general position, then there exists a non-crossing geometric perfect matching joining the red points and the blue points, where a geometric matching is a matching consisting of line segments. We start with a result on perfect matchings with L-line segments in the plane lattice.

Theorem 1. *Suppose that n red points and n blue points are given in the plane lattice in general position, where $n \geq 1$ is an integer. Then there exists a non-crossing perfect matching with L-line segments joining the red points and the blue points (Fig. 3).*

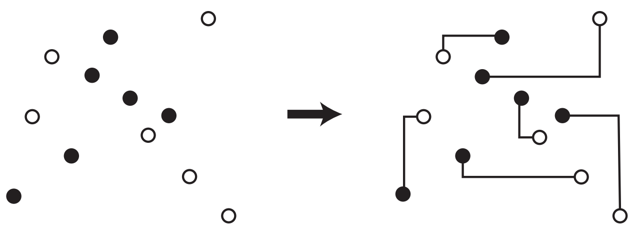


Figure 3: A non-crossing perfect matching with L-line segments joining red points and blue points.

We need some new notation. In this paper only axis-parallel rectangles will be used, and so a *rectangle* always means such a rectangle, and thus each edge of an rectangle is a vertical or horizontal line segment. For a set S

of points in the plane lattice, the *rectangular hull* of S, denoted by $rect(S)$, is the smallest closed rectangular enclosing S ((3) in Fig. 2). In particular, every edge of $rect(S)$ contains at least one point of S. For a set X, the cardinality of X is denoted by $|X|$ or $\#X$.

Proof of Theorem 1. We prove Theorem 1 by induction on n. If $n = 1$, then the theorem holds. So we assume $n \geq 2$. Let X be the set of points of S on the boundary of the rectangular hull $rect(S)$. Then $2 \leq |X| \leq 4$. Suppose that X contains both a red point and a blue point. Then there exists an L-line segment L_1 that is on the boundary of $rect(S)$ and joins a red point x to a blue point y of X. By applying the induction hypothesis to $S - \{x, y\}$, we obtain a non-crossing perfect matching with L-line segments joining the red points and the blue points of $S - \{x, y\}$. By adding L_1 to this matching, we can get the desired non-crossing perfect matching.

Next assume that all the points of X have the same color. By symmetry, we may assume that all the points of X are red. For every vertical line ℓ in the plane passing through no points of S, define a function $f(l)$ by

$$\begin{aligned} f(\ell) \quad = \quad & \#\{\text{the red points of } S \text{ to the left of } \ell\} \\ & -\#\{\text{the blue points of } S \text{ to the left of } \ell\}. \end{aligned}$$

Then $f(\ell_1) = 1$ for a vertical line ℓ_1 immediately to the right of the left vertical edge of $rect(S)$, and $f(\ell_2) = -1$ for a vertical line ℓ_2 immediately to the left of the right vertical edge of $rect(S)$. Moreover, we continuously move a vertical line ℓ from ℓ_1 to ℓ_2. Then $f(\ell)$ changes by ± 1 when ℓ crosses a point of S. Hence there exists a vertical line ℓ_3 such that $f(\ell_3) = 0$ and ℓ_3 passes through no point of S. By applying the induction hypothesis to the points of S to the left of ℓ_3 and those of S to the right of ℓ_3, respectively, we can obtain the desired non-crossing perfect matching (See Fig. 4). $\square$

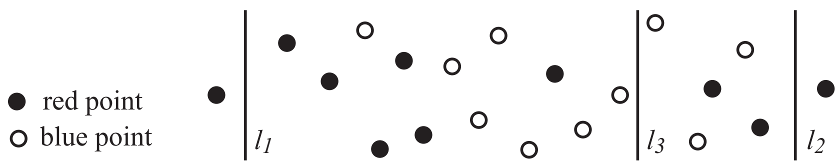


Figure 4: Moving a vertical line ℓ from ℓ_1 to ℓ_2 to find ℓ_3 such that $f(\ell_3) = 0$.

3 Balanced subdivisions

We now turn our attention to another well-known theorem so called Ham-sandwich Theorem, which says that if $2m$ red points and $2n$ blue points are given in the plane in general position, then there exists a line that bisects both red points and blue points. A similar result related to this theorem also holds by using L-lines as in the following theorem.

Theorem 2 ([4], [7]). *Let $m \geq 1$ and $n \geq 1$ be integers. If $2m$ red points and $2n$ blue points are given in the plane lattice in general position, then there exists an L-line that bisects both red points and blue points (Fig. 5).*

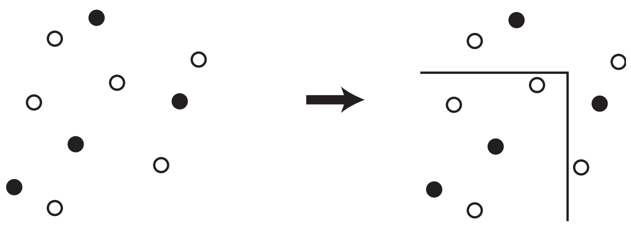


Figure 5: An L-line that bisects both red points and blue points.

The above Theorem 2 was generalized as follows.

Theorem 3 (Bereg [2]). *Suppose that km red points and kn blue points are given in the plane lattice in general position, where $m \geq 1$, $n \geq 1$ and $k \geq 2$ are integers. Then there exists a subdivision of the plane into k regions with at most $k-1$ horizontal line segments and at most $k-1$ vertical line segments such that every region contains precisely n red points and m blue points.*

Bárány and Matoušek obtained the following theorem about another bisection.

Theorem 4 (Bárány and Matoušek [1]). *Suppose that $2m$ red and $2n$ blue points are given in the plane in general position, where $m \geq 1$ and $n \geq 1$ are integers. Let p be a point in the plane such that the red points, the blue points and p are in general position. Then there exist two rays emanating from p that bisect both red points and blue points.*

A ray in the plane is a half-line emanating from a point. Similarly, an *L-ray* in the plane is defined to be a half L-line emanating from a point, and so an L-ray has a corner and an end-point (see Fig. 6). We show that a similar result holds in the plane lattice using L-rays.

Theorem 5. *Suppose that $2m$ red points and $2n$ blue points are given in the plane lattice in general position, where $m \geq 1$ and $n \geq 1$ are integers. Let p be a point in the plane each of whose coordinates is not an integer. Then there exist two L-rays emanating from p that bisect both red points and blue points (see Fig. 6).*

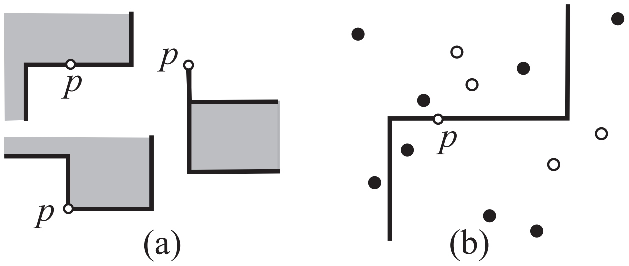


Figure 6: (1) Three types of two L-rays emanating from p, which subdivide the plane into two regions. (2) Two L-rays emanating from p that bisect both red points and blue points.

Proof. First take a big rectangle $\mathcal{R}$ that contains all the red points, the blue points, and p. We subdivide $\mathcal{R}$ into four regions by the horizontal line and the vertical line passing through p. Then for each given red or blue point x contained in the right-lower region, we assign a new point y with the same color as x on the right edge of $\mathcal{R}$ such that x and y lie on the same horizontal line (see Fig. 7). Every given point x' in other regions, we assign a point y' with the same color as x' on the boundary of $\mathcal{R}$ as shown in Fig. 7.

Since given points are in general position, the assignment defined above is a bijection. It is easy to see that the boundary of $\mathcal{R}$ can be divided into two continuous parts so that each part contains precisely m red points and n blue points (see two bold marks on the boundary of $\mathcal{R}$ in Fig. 7). Notice that the reader is refereed to [3], for example, about the proof of this fact. Then we can obtain the desired two L-rays, which emanates from the point p and passes through the partitioning marks on the boundary. $\square$

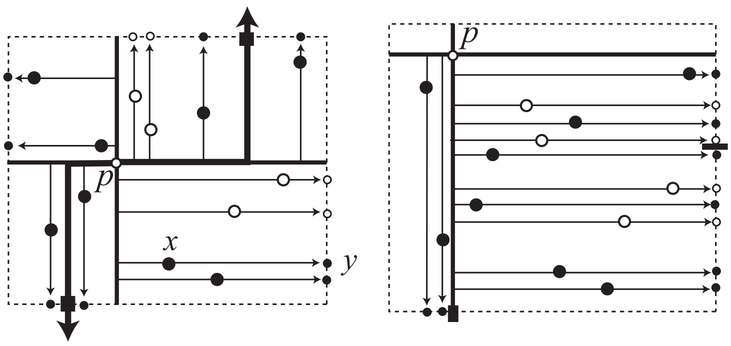


Figure 7: The given red and blue points are assigned to red and blue points, respectively, on the boundary of $\mathcal{R}$.

4 Geometric Spanning Trees

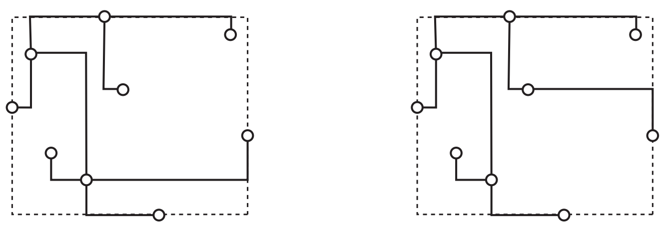
For a set X of points in the plane, we can draw non-crossing geometric spanning trees on X each of whose edges is a line segment joining two points of X. In this paper, we call such a spanning tree *X-tree*. Given a set R of red points and a set B of blue points in the plane in general position, the minimum number of crossings of R-tree and B-tree is determined in the next theorem, where $conv(X)$ denotes the convex hull of X.

Theorem 6 (Tokunaga [6])**.** *Let R and B be two disjoint sets of red points and blue points such that $R \cup B$ is in general position. Let $\tau(R, B)$ denote the number of edges xy of the boundary of $conv(R \cup B)$ such that one of $\{x, y\}$ is red and the other is blue. Then $\tau(R, B)$ is even, and the minimum number of crossings in $T_R \cup T_B$ among all pairs $\{R$-tree T_R, B-tree $T_B\}$ is equal to*

$$\max\left\{\frac{\tau(R,B)-2}{2},\ 0\right\}.$$

In particular, we can draw an R-tree and a B-tree without crossings if and only if $\tau(R, B) \leq 2$.

We consider a similar problem on the plane lattice, and prove a similar result as shown in the following Theorem 7. For a set X of points in the plane lattice in general position, we can draw non-crossing spanning trees on X each of whose edges is an L-line segment connecting two points of X, which is called a *spanning tree on X with L-line segments* or simply *X-tree with L-line segments.* Hereafter, we consider only non-crossing spanning trees with L-line segments, and so X-tree means X-tree with L-line segments (see Fig. 8). For a tree T and a vertex $v \in V(T)$, the degree of v in T is denoted by $\deg_T(v)$. The maximum degree of T is denoted by $\Delta(T)$.



(1) X-tree (2) X-tree of degree at most 3

Figure 8: Examples of X-trees in the plane lattice.

Theorem 7. *Let R and B be two disjoint sets of red points and blue points in the plane lattice such that $R \cup B$ is in general position. Let $\tau^*(R, B)$ denote the number of L-line segments xy on the boundary of $rect(R \cup B)$ such that one of $\{x, y\}$ is red and the other is blue. Then $\tau^*(R, B)$ is 0, 2, or 4, and the minimum number of crossings in $T_R \cup T_B$ among all pairs $\{R$-tree T_R, B-tree $T_B\}$ is equal to 1 if $\tau^*(R, B) = 4$. Moreover, if $\tau^*(R, B) \leq 2$, then we can draw an R-tree T_R and a B-tree T_B without crossings such that $\Delta(T_R) \leq 3$ and $\Delta(T_B) \leq 3$ (see Fig. 9).*

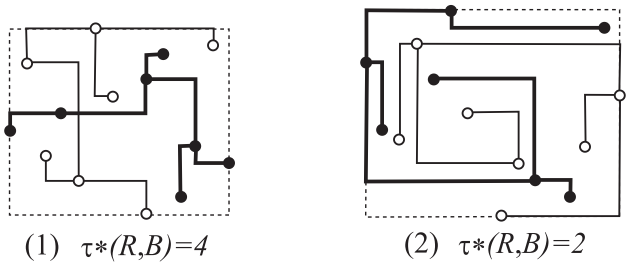


Figure 9: Two spanning trees R-tree and B-tree with minimum number of crossings.

Note that, if $\tau^*(R, B) = 4$ then any T_R and T_B cross at least once. In fact, let $x_r \in R$, $y_r \in R$, $x_b \in B$, and $y_b \in B$ be the left, right, top, bottom points in $rect(R \cup B)$, respectively. The path in T_R starting from x_r to y_r and the path in T_B starting from x_b to y_b cross at least once.

In order to prove Theorem 7, we need some definitions and a lemma. An *orthogonal spiral polygon* is a polygon whose boundary consists of two chains of edges, which are called *outer chain* and *inner chain*, respectively. Every internal angle of the outer chain is $\pi/2$ and every internal angle of the inner chain is $3\pi/2$ (see (1) in Fig. 10). Notice that we allow that an edge of the inner chain is included in an edge of the outer chain, namely, some part of polygon may consist of only edges (no inner points) and be flattened.

Lemma 8. *Let $\mathcal{P}$ be an orthogonal spiral polygon in the plane lattice and S be a set of points in the plane lattice in general position contained in $\mathcal{P}$. Assume that every edge of the outer chain of $\mathcal{P}$ contains exactly one point of S, and every edge of the inner chain contains exactly one point of S or is included in another edge of the outer chain. Then there exists S-tree T such that (i) $\Delta(T) \leq 3$ and (ii) T is included in $\mathcal{P}$ (See (2) in Fig. 10).*

Proof. An edge of the inner chain of $\mathcal{P}$ included in another edge of the outer chain is called *flattened rectangle*. Note that a flattened rectangle may have

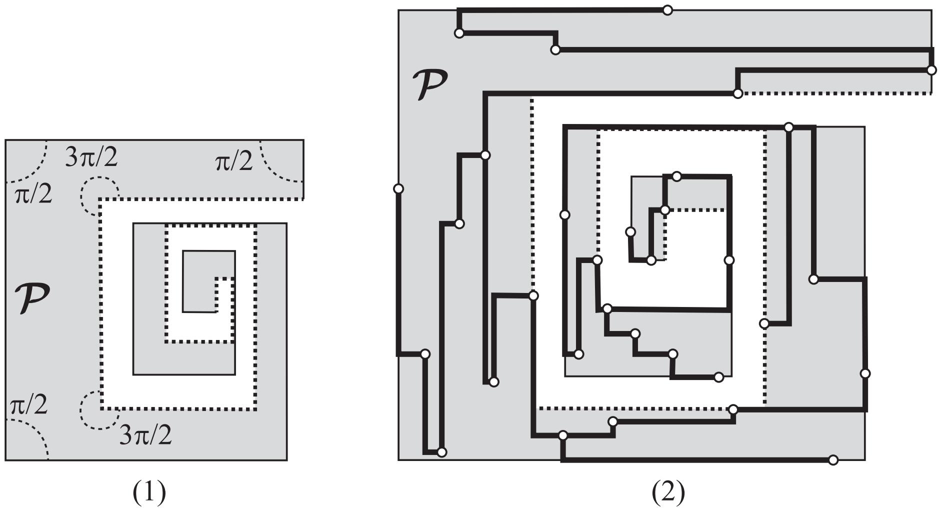


Figure 10: (1) An orthogonal spiral polygon $\mathcal{P}$ with an outer chain and an inner chain consisting of bold and broken edges, respectively. (2) An orthogonal spiral polygon $\mathcal{P}$ with a point set S and an S-tree T such that $\Delta(T) \leq 3$.

one point of S by our assumption for $\mathcal{P}$. In (2) of Fig. 10, we can find two flattened rectangles such that one of them has one point of S and another one has no points of S. We may assume that no point of S is at a corner of the inner chain of $\mathcal{P}$ if the corner is not contained in an edge of the outer chain. Otherwise, we may move one of the edges incident to the corner so that the resulting orthogonal spiral polygon $\mathcal{P}'$ is included in $\mathcal{P}$. In the other words, we consider a minimal orthogonal spiral polygon included in $\mathcal{P}$ on S.

In preparation for our construction of an S-tree, we decompose the orthogonal spiral polygon $\mathcal{P}$ into closed rectangles as shown in Fig. 11. If $\mathcal{P}$ has no flattened rectangles, then we decompose $\mathcal{P}$ as shown (1) in Fig. 11, where X, Y, and Z denote closed rectangles. The top edge of Y is included in the bottom edge of X, and the left edges of X and Y forms an edge of the outer chain of $\mathcal{P}$. Two consecutive rectangles Y and Z have the same properties, and so on.

If $\mathcal{P}$ has some flattened rectangles, then we decompose $\mathcal{P}$ as shown (2) in Fig. 11, where each X_i denotes each closed rectangle. Some rectangles (X_3, X_5, X_6, X_8 in Fig. 11) are flattened. Remove these flattened rectangle from $\mathcal{P}$. Then, the remaining parts of $\mathcal{P}$ consists of some orthogonal spiral polygons, so we decompose each of these as shown (1) in Fig. 11.

We denote these decomposed rectangles by X_1, X_2, ..., X_k in spiral order.

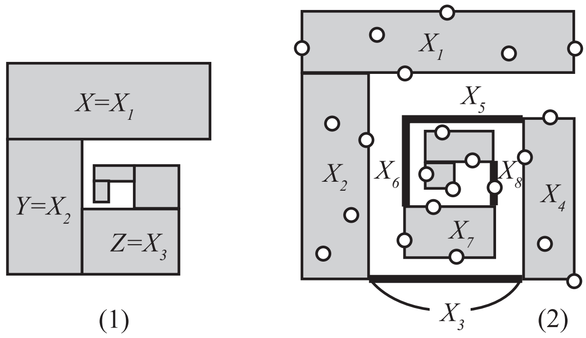


Figure 11: (1) A decomposition of an orthogonal spiral polygon without flattened rectangles. (2) A decomposition of an orthogonal spiral polygon with flattened rectangles, where X_3, X_5, X_6, X_8 are flattened rectangles and only X_8 contains one point of S.

Claim 1. *No three flattened rectangles without points of S are consecutive in $\mathcal{P}$.*

Proof. Otherwise, some edge of the outer chain has no points of S, which contradicts our assumption. □

Fig. 12 shows an outline of our construction of an S-tree. First, construct a path in each non-flattened rectangle X_i from "outer" point to "inner" point as shown (1). Then, we connect those paths as shown (2).

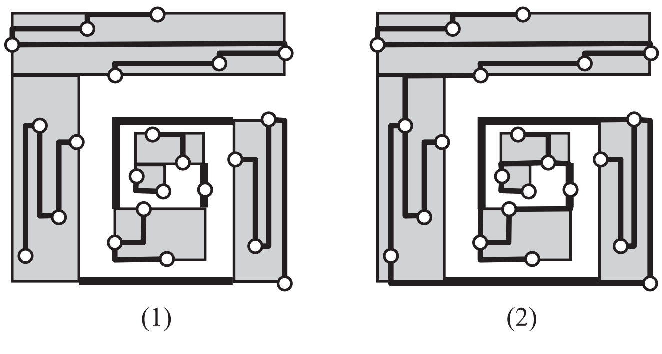


Figure 12: An outline of our construction of an S-tree.

We shall show more precisely how to do it. Each non-flattened rectangle X_i consists of four edges. Exactly one of these edges includes an edge of the inner chain of $\mathcal{P}$. We regard this edge as the *bottom* edge of the rectangle. Then, in accordance with the bottom edge, we can define *top*, *right*, and *left* edge of the rectangle. Let p_i, q_i, r_i, t_i be the most top, bottom, right, left points of $X_i \cap S$, respectively. We construct a path P_i with L-line segments that satisfies the following three properties: (i) P_i starts at p_i and ends at q_i. (ii) P_i passes through all the points in $X_i \cap S$ from top to bottom. (iii) Each L-line segment xy such that x is upper than y, starts at x to the right or left and ends at y from above. (See Fig. 13).

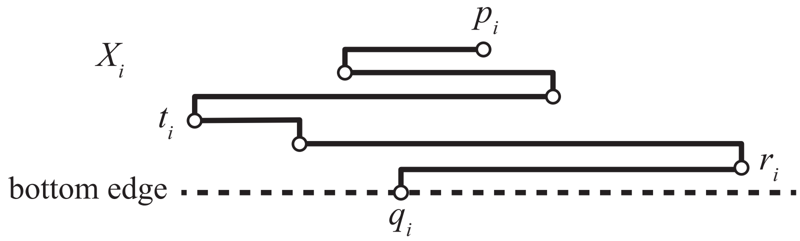


Figure 13: A path P_i on X_i.

For each flattened rectangle X_i with one point $x_i \in S$, let $p_i = q_i = r_i = t_i = x_i$ and $P_i = \{x_i\}$. For each flattened rectangle X_i without points of S, let a_i be the center point on X_i, $p_i = q_i = r_i = t_i = a_i$, and $P_i = \emptyset$. We will use the points a_i as dummy points of P_i.

Next, we connect each two paths P_i and P_{i+1} as follows.

Case 1. $P_{i+1} \neq \emptyset$.

In this case, we can always connect the bottom point q_i and the right point r_{i+1} with an L-line segment or a line segment without crossings as follows. If $P_i = \emptyset$ or $|P_i| = 1$ then X_i is flattened rectangle. Thus, by the definition of decomposition of $\mathcal{P}$, the bottom point q_i is on an edge of the inner chain of $\mathcal{P}$. Therefore, we can connect q_i and r_{i+1} without crossings as shown in Fig. 14 (1),(2), and (3).

If $|P_i| \geq 2$ then similarly the bottom point q_i of P_i is on an edge of the inner chain of $\mathcal{P}$, since otherwise some edge of the inner chain has no points of S and is not included in an edge of the outer chain, which contradicts our assumption. Hence, in X_i, a horizontal line segment from q_i to the left can be added to the path P_i for connecting P_i and P_{i+1}, and this segment does not overlap with another segment from X_{i-1}, namely from the right. Therefore, we can connect q_i and r_{i+1} without crossings as shown in Fig. 14 (4). Note that if $|P_{i+1}| \geq 2$ and $q_i = r_{i+1}$, that is q_i is on a corner of the inner chain, then the degree $\deg_T(q_i)$ may be four. However, we assumed that no point of S is at a corner of the inner chain of $\mathcal{P}$ if the corner is not contained in an edge of the outer chain.

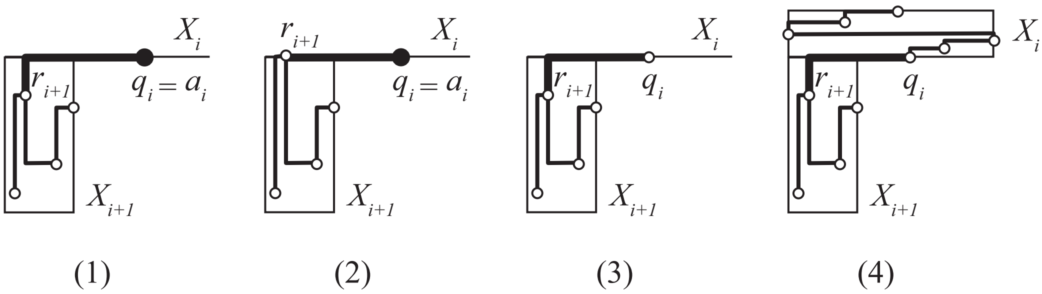


Figure 14: How to connect each two paths P_i and P_{i+1} if $P_{i+1} \neq \emptyset$. The black points are dummy points.

Case 2. $P_{i+1} = \emptyset$.

In this case, we can always connect the left point t_i and the dummy point a_{i+1} with an L-line segment or a line segment without crossings as shown in Fig. 15.

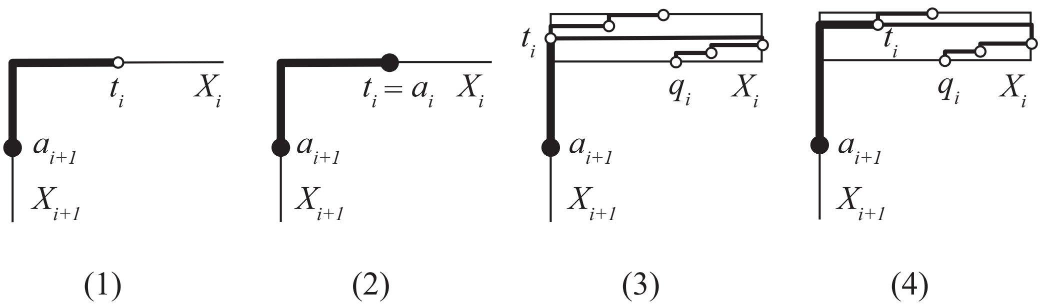


Figure 15: How to connect each two paths P_i and P_{i+1} if $P_{i+1} = \emptyset$.

Consequently, by ignoring the dummy points a_i, we get a tree T on S without crossings such that (i) $\Delta(T) \leq 3$ and (ii) T is included in $\mathcal{P}$. We shall show that T is S-tree, namely, every edge of T is L-line segment.

The edges of T not through dummy points are L-line segments. Thus, we consider edges of T through dummy points. By Claim 1, there are just seven cases as shown in Fig. 16, where the bold edges are edges of T through one or two dummy points.

The cases (1), (4), (5), and (6) in Fig. 16 contradicts our assumption that every edge of the outer chain of $\mathcal{P}$ contains exactly one point of S. In the other cases, each bold edge is an L-line segment. Therefore, the tree T is a desired S-tree. Consequently the lemma is proved. $\square$

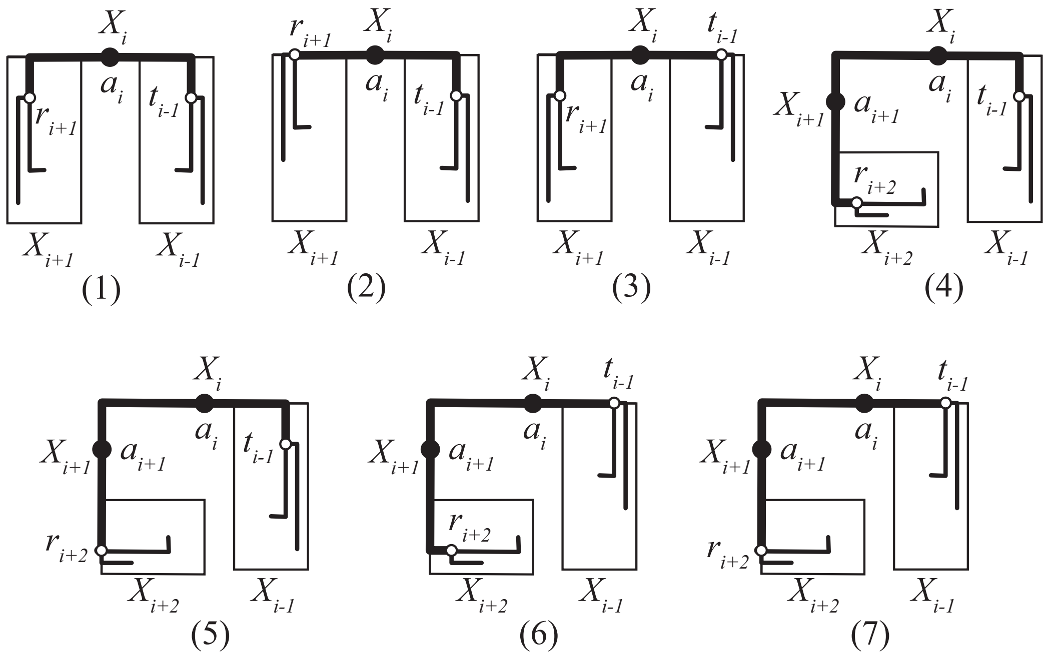


Figure 16: Seven cases of edges of T through dummy points.

Proof of Theorem 7. We first prove the theorem in the case where the top point and the leftmost point in $rect(R \cup B)$ are red and the bottom point and the rightmost point in $rect(R \cup B)$ are blue, in particular, $\tau^*(R, B) = 2$.

We take some rectangles containing only red points or blue points, and obtain two disjoint orthogonal spiral polygons that contain all the red points and all the blue points, respectively. First take the largest rectangle X_1 which contains no blue points, whose top edge is the top edge of $rect(R \cup B)$ and whose bottom edge contains a red point (see Fig. 17). Next take the largest rectangle Y_1 which contains no red points, whose bottom edge is the bottom edge of $rect(R \cup B)$ and whose top edge contains a blue point. Then remove open region $X_1 \cup Y_1$ together with the red and blue points in $X_1 \cup Y_1$ from $rect(R \cup B)$, and denote the resulting rectangle by $Rect_2$, whose red point set is R_2 and blue point set is B_2.

Hereafter we assume that rectangle X_i contain no blue points and rectangles Y_i contains no red points. Take the largest rectangle X_2 whose left edge is the left edge of $Rect_2$ and whose right edge contains a red point if any. Namely, if the leftmost point of $Rect_2$ is blue (i.e., this case may occur if the leftmost point in $rect(R \cup B)$ lies on the left edge of X_1), then X_2 consist of only one edge and contains no inner points and no red points. Similarly, take

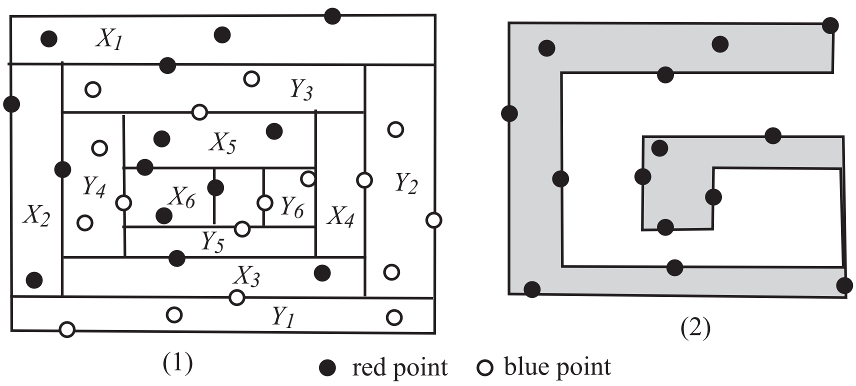


Figure 17: (1) Rectangles $X_1, X_2, \ldots, X_6$ and $Y_1, Y_2, \ldots, Y_6$. (2) An orthogonal spiral polygon containing all the red points, which is included in $X_1 \cup \ldots \cup X_6$.

the largest rectangle Y_2 whose right edge is the right edge of $Rect_2$ and whose left edge contains a blue point if any. So it may occur that Y_2 consists of one edge and contain no blue points. Then remove open region $X_2 \cup Y_2$ and the red and blue points in $X_2 \cup Y_2$ from $Rect_2$, and denote the resulting rectangle by $Rect_3$, whose point set is $R_3 \cup B_3$. Note that if X_2 contains no red points, then $Rect_3$ is obtained from $Rect_2$ only by removing Y_2. Moreover, if X_2 contains no red points and Y_2 contains no red blue points, then $Rect_3$ is equal to $Rect_2$, but we next take the largest rectangle X_3 whose bottom edge is the bottom edge of $Rect_3$ and whose top edge contains a red point, and X_3 contains at least one red point.

We repeat the same procedure until $rect(R_k \cup B_k)$ contains neither red points nor blue points (see (1) of Fig. 17). Then $X_1 \cup X_2 \cup \ldots \cup X_k$ is an orthogonal spiral polygon containing all the red points. If an edge does not contain a red point, we move the edge to inside until it contains a red point or is included in another edge. By repeating this procedure, we can obtain the desired orthogonal spiral polygon, which contains all the point of R and each of whose edges either contains one red point or is included in another edge. By lemma 8, we can obtain a spanning tree with L-line segments on R with maximum degree at most 3. Similarly, we can obtain a blue spanning tree with maximum degree at most 3, and it is clear that these two spanning trees do not cross.

We next consider the case where $\tau^*(R, B) = 2$ and the top point, the leftmost point and the bottom point in $rect(R \cup B)$ are red and the rightmost point in $rect(R \cup B)$ is blue (see Fig. 18).

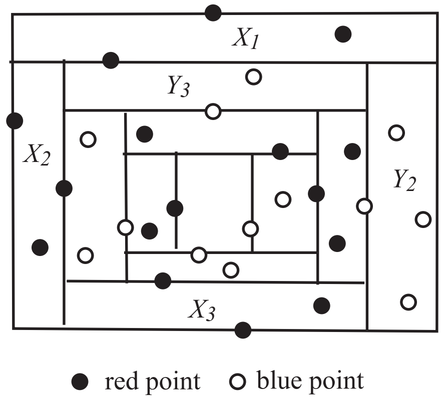


Figure 18: The top point, the leftmost point and the bottom point in $rect(R \cup B)$ are red and the rightmost point in $rect(R \cup B)$ is blue.

We first take a rectangle X_1 as above. Then $rect(R \cup B) - X_1$ satisfies the condition of the case discussed above, and so we can apply the procedure to take $X_2, Y_2, X_3, Y_3, \ldots, X_k, Y_k$. Then we can obtain two disjoint orthogonal spiral polygons from $X_1 \cup \cdots \cup X_k$ and $Y_2 \cup \cdots \cup Y_k$, respectively, and thus we can get the desired two spanning trees. In the other cases of $\tau^*(R, B) \leq 2$, we can similarly obtain the desired two spanning trees.

We finally consider the case where $\tau^*(R, B) = 4$, the rightmost point and the leftmost point of $rect(R \cup B)$ are red, and the top point and the bottom point of $rect(R \cup B)$ are blue (see Fig. 19).

First take the largest rectangle X_1 whose left edge is the left edge of $rect(R \cup B)$ and whose right edge contains a red point. Let r_1 and r_2 be the leftmost and the rightmost red points in X_1, respectively. We construct a path starting at r_1 ending at r_2 from left to right as shown in (1) of Fig. 19. Then remove X_1 and their points from $rect(R \cup B)$, and denote the resulting rectangle by $Rect_1$, whose point set is $R_1 \cup B_1$.

Next take the largest rectangle Y_1 whose bottom edge is the bottom edge of $Rect_1$ and whose top edge contains a blue point. Let b_1 and b_2 be the bottom and the top blue points in Y_1, respectively. We construct a path starting at b_1 from bottom to top as shown in (1) of Fig. 19. Then remove Y_1 and their points from $Rect_1$, and denote the resulting rectangle by $Rect_2$ with point set $R_2 \cup B_2$.

We take the largest rectangle X_2 whose bottom edge is the bottom edge of $Rect_2$ and whose top edge contains a red point. Let b_3 be the bottom point in B_2. We connect b_2 and b_3 by an L-line segment such that a horizontal line segment starts at b_2.

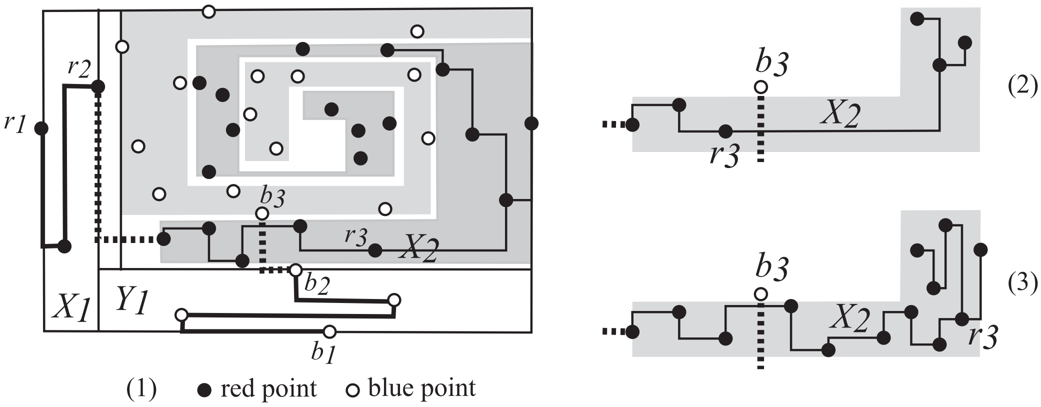


Figure 19: An outline of our construction of an R-tree and a B-tree with exactly one crossing. Three cases on the place of r_3.

Let r_3 be the rightmost red point in X_2. Remove $X_2 \setminus \{r_3\}$ and their points from $Rect_2$, and denote the resulting rectangle by $Rect_3$ with point set $R_3 \cup B_3$. Then $\tau^*(R_3, B_3) = 2$, and so there exist a red R_3-tree T'_R and a blue B_3-tree T'_B. When we construct these two trees, we first take the largest rectangle that contains r_3.

By the construction method for S-tree in the proof of Lemma 8, we can add a horizontal line segment to the left-hand side of r_3. Then, we construct a path starting at r_2 ending at r_3 from left to right as shown in (1) of Fig. 19. Wherever r_3 is, exactly one L-line segment crosses the L-line segment b_2b_3. (See Fig. 19)

Consequently we obtain the desired red panning tree and blue spanning tree with exactly one crossing, and the proof is complete. $\square$.

The degrees of points of our R-tree and B-tree in the proof of Theorem 7 is at most 3 except b_3. We now give a problem concerning Theorem 7.

Problem 9. *In Theorem 7, even if* $\tau^*(R, B) = 4$ *then is it possible to require that* $\Delta(T_R) \leq 3$ *and* $\Delta(T_B) \leq 3$ *? Moreover, is it possible to replace a spanning tree with maximum degree 3 by a Hamilton path (i.e., a spanning tree with maximum degree 2) ?*

We conclude this paper with the following conjecture. Let T be a tree and P be a set of $|T|$ points in the plane lattice in general position, where $|T|$ denotes the order of T. If T can be drawn on P without crossing such that each edge of T is an L-line segment connecting two points of P, then we say that T can be drawn on P with L-line segments (see Fig. 20).

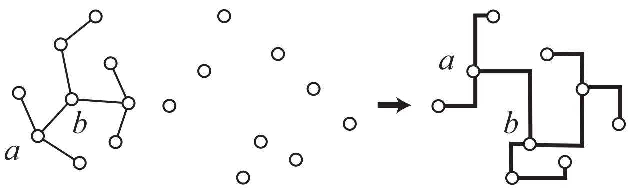


Figure 20: (1) A tree T with maximum degree 3. (2) a set P of $|T|$ points in the plane lattice in general position. (3) T is drawn on P with L-line segments without crossing.

Conjecture 10. *Let T be a tree with maximum degree 3, and let P be a set of $|T|$ points in the plane lattice in general position. Then T can be drawn on P with L-line segments without crossing.*

A partial solution to this conjecture is given in [5], namely it is proved that if a tree T with maximum degree 3 has the property that all the vertices of degree 3 is contained in a path of T, then the conjecture holds.

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