Geometric Graphs in the Plane Lattice

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Abstract. An L-line segment in the plane consists of a vertical line segment and a horizontal line segment having a common end-point. In this paper, we consider some problems on non-crossing geometric embeddings of graphs in the plane lattice, whose vertices are given points of the plane lattice in general position and whose edges are suitable L-line segments.

1 Introduction

A geometric graph in the plane is a graph drawn in the plane whose edges are straight line segments. There is a large amount of research on geometric graph drawings or geometric graph embeddings in the plane without crossings (for example, see [6], [3] and [5]). In this paper we consider graph drawings or graph embeddings in the plane lattice without crossings, whose edges are L-line segments defined below instead of straight line segments.

For a point x in the plane lattice, an L-shaped line consisting of a vertical ray and a horizontal ray emanating from x is called an *L-line*. Similarly, an L-shaped line segment consisting of a vertical line segment and a horizontal line segment with a common end-point is called an *L-line segment* (see (1) of Fig. 1). We consider some problems on geometric graphs in the plane lattice, whose edges are L-line segments. A set X of points in the plane is *in general position* if no three points of X lie on the same line. On the other hand, a set S of points in the plane lattice is said to be *in general position* if every vertical line and every horizontal line passes through at most one point of S. It is shown that L-line segments play a similar role in the plane lattice as line segments in the plane ([2]).

In this paper we consider some problems on geometric graph drawings and embeddings in the plane lattice, in which L-line segments plays a similar role as line segments in geometric graph drawings and embeddings in the plane. For example, for a 3-tree T of order $|T|$ and for any given set S of $|T|$ points in the plane lattice in general position, we consider a non-crossing geometric embedding of T on S, where a 3-tree is a tree with maximum degree at most three. In [7] and [4], some problems of non-crossing geometric embeddings or drawings of

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matchings and cycles are considered, and it is shown that they are $\mathcal{NP}$-hard. Some other results in the plane lattice related to this paper can be found in [1] and [6].

2 Geometric Hamilton Cycles

For a set X of points in the plane lattice in general position, a cycle (path) which passes through all the points of X and whose edges are L-line segments is called a *geometric Hamilton cycle (path) on* X. The *rectangular hull* of X, denoted by $rect(X)$, is the smallest closed rectangle enclosing X, each of whose edges is a vertical or horizontal line segment ((2) of Fig. 1). In particular, every edge of $rect(X)$ contains exactly one point of X. The cardinality of X is denoted by $|X|$.

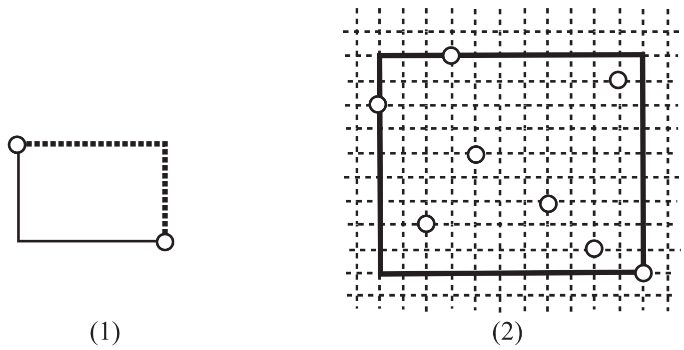

Fig. 1. (1) Two L-line segments joining two points; (2) A rectangular hull $rect(X)$ of a set X of points in the plan lattice in general position.

We begin with the following result.

Theorem 1. *Let S be a set of points in the plane lattice in general position containing at least two points. Then there exists a non-crossing geometric Hamilton cycle on S.*

Proof. We may assume that $|S| \geq 3$. We first consider the case where the top edge, the left edge and the right edge of the rectangular hull $rect(S)$ contain one distinct point of S each. Let x be the point of S lying on the top edge of $rect(S)$. By symmetry, we may assume that the second top point y of S is to the left of x. Take the point z of S lying on the right edge of $rect(S)$. Then there exists a non-crossing geometric Hamilton path on $S - \{z\}$ starting at x and ending at the point w lying on the bottom edge of $rect(S)$. By adding two L-line segments joining w to z and z to x to this Hamilton path, we obtain the desired non-crossing geometric Hamilton cycle on S (see (1) of Fig. 2).

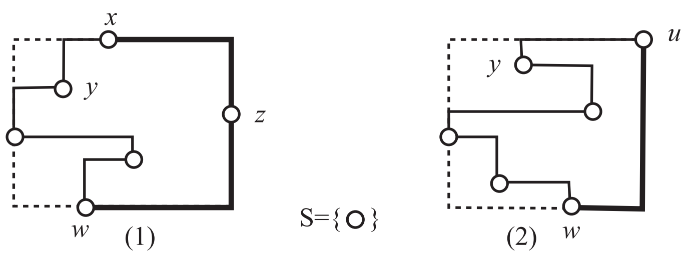

Fig. 2. (1),(2) Non-crossing geometric Hamilton cycles on S.

We next consider the case where the top-right corner of $rect(S)$ contains a point of S, say u. There exists a non-crossing geometric Hamilton path on S, which starts at u and ends at the point w lying on the bottom edge of $rect(S)$ (see (2) of Fig. 2). Then by adding an L-line segment connecting u and w, we obtain the desired non-crossing geometric Hamilton cycle on S. □

3 Geometric Embeddings of Spiders

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex v of G is denoted by $\deg_G(v)$. The number of vertices of G is called the *order* of G, and denoted by $|G|$. Let S be a set of $|G|$ points in the plane lattice in general position. Then we say that G is *geometrically embedded* on S if there exist a bijection $\psi : V(G) \to S$ and a set of L-line segments joining two points of S such that two points $\psi(x)$ and $\psi(y)$ of S are joined by an L-line segment if and only if x and y are joined by an edge of G. We often call the above embedding a *geometric embedding* $\psi : G \to S$. If no two L-line segments intersect, such an geometric embedding is said to be *non-crossing*. A *spider* is a tree that has exactly one vertex with degree at least three, and the vertex of degree at least three is called its *trunk*.

Theorem 2. *Let T be a spider such that the trunk has degree three or four, and let S be a set of $|T|$ points in the plane lattice in general position. Then T can be geometrically embedded on S without crossings.*

Proof. Let u be the trunk of the spider T. We first consider the case where u has degree three. Then $T - u$ consists of three paths, and their orders (i.e., the numbers of their vertices) are denoted by a, b, c, respectively. Sweep the points of S from top to bottom by horizontal line, and denote the $(a+1)$-th point by x (see (1) of Fig. 3). We remove these $a+1$ points above x from S, and divide the rectangle containing the remaining points of S into two rectangles by a vertical line so that the left rectangle contains b points and the right rectangle contains c points. Let y be the left most point in the left rectangle and z be the right most point in the right rectangle (see (1) of Fig. 3). Then there exist three non-crossing geometric Hamilton paths that start at x, y and z, respectively, and have order $a+1$, b and c, respectively. Moreover there are two L-line segments joining x to y and x to z, respectively as shown in (1) or (2) of Fig. 3 according to the situation of x and y. Hence the spider T is embedded on S without crossings.

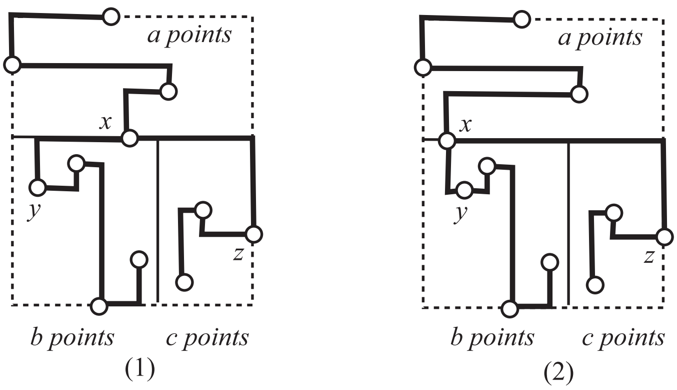

Fig. 3. (1) A non-crossing geometric embedding of a spider T with trunk degree 3 on S, where y is to the left of x. (2) A non-crossing geometric embedding of a spider T on S, where y is to the right of x.

We next consider the case where the trunk u has degree four. Then $T-u$ consists of four paths, and their orders are denoted by a, b, c, d, respectively. Sweep the points of S from left to right by vertical line, and denote the a-th point by p and the vertical line passing p by l_1. Next sweep the points of S from right to left by vertical line, and denote the b-th point by q and the vertical line passing q by l_2 (see Fig. 4). We remove these $a+b$ points from S, and sweep the remaining points of S between l_1 and l_2 from top to bottom by horizontal line, and denote the c-th point, the $(c+1)$-th point and the $(c+2)$-th point by y, x, and z, respectively (see Fig. 4). We divide the points between l_1 and l_2 into two sets and one point $\{x\}$ by two horizontal lines passing through y and z, respectively. Then each of the four regions contains a geometric Hamilton path starting at p, q, y or z. By joining x to p, q, y and z by L-line segments, we can obtain the desired non-crossing geometric embedding of T. $\square$

4 Geometric Embedding of 3-trees

In this section we give a partial solution to the following our conjecture.

Conjecture 1. Let T be a 3-tree and S be a set of $|T|$ points in the plane lattice in general position. Then T can be geometrically embedded on S without crossings.

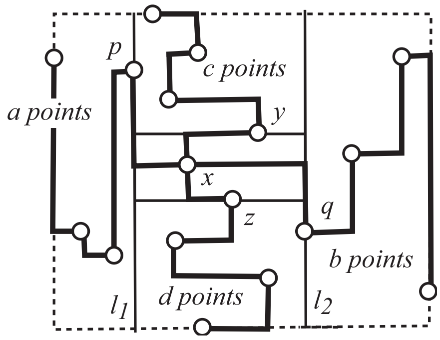

Fig. 4. A non-crossing geometric embedding of a spider T with trunk degree 4.

Our result is the following. Namely, we shall prove that if the vertices of T with degree 3 is contained in a path of T, then the conjecture holds.

Theorem 3. *Let T be a 3-tree such that all the vertices of degree 3 is contained in a path of T, and let S be a set of $|T|$ points in the plane lattice in general position. Then T can be geometrically embedded on S without crossings.*

In order to prove Theorem 3, we need some definitions and notation. We denote by $V_3(T)$ the set of vertices of T with degree 3. Let T be a 3-tree such that $V_3(T)$ contains at least two vertices and is contained in a path of T, and let Q be the shortest path of T that passes through all the vertices of $V_3(T)$. Let w_1 and w_2 be the two end-vertices of Q, (see Fig. 5). A vertex v of T is said to be *good* if $v \notin V(Q)$ and $\{v\} \cup V_3(G)$ is included in a path of T, in particular, $\deg_T(v) \leq 2$ (see Fig. 5). If T has exactly one vertex with degree 3, then all the other vertices are good.

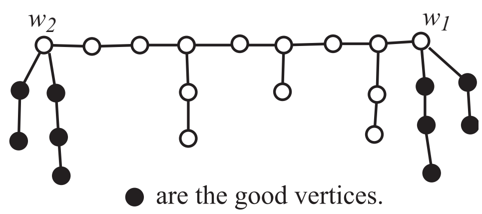

Fig. 5. A 3-tree of Theorem 3 and its good vertices.

Theorem 3 follows from Theorem 1 if $V_3(T)$ is empty, and otherwise it follows form the following proposition.

Proposition 1. *Let T be a 3-tree such that $V_3(T)$ is non-empty and is contained in a path of T, and let S be a set of $|T|$ points in the plane lattice in general position. Then for every good vertex v of T and for every point x of S on the edge of $rect(S)$, there exists a non-crossing geometric embedding $\psi : T \to S$ that satisfies $\psi(v) = x$.*

Proof. We shall prove the proposition by induction on $|T|$. By symmetry, we may assume that the point x lies on the top edge of $rect(S)$. We consider some cases.

Case 1. $|V_3(T)| = 1$.

Let w be the unique vertex of T with degree 3, and denote the orders of paths of $T - w$ by a, b and $c + d + 1$, where the path Q of $T - w$ containing v has order $c+d+1$, and the two paths of $Q - v$ have orders c and d, respectively. Here we assume that x is not a corner point of $rect(S)$ and $c, d \geq 1$. Notice that if x lies on the corner of $rect(S)$, $c = 0$ or $d = 0$, then by the almost the same argument given below, we can prove the the proposition.

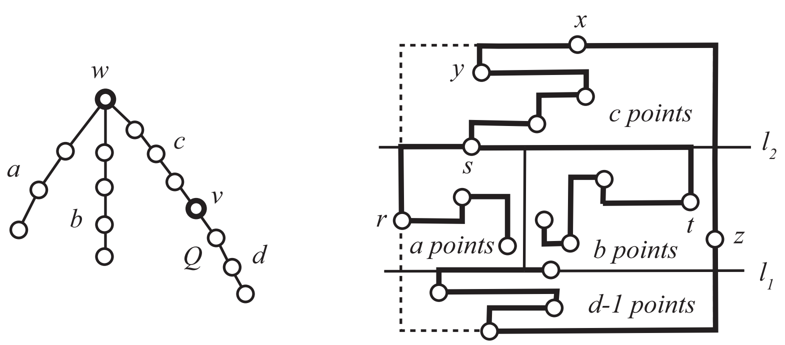

Fig. 6. A 3-tree having one vertex of degree 3, and its non-crossing geometric embeddings.

Let y be the highest point of $S - x$. By symmetry, we may assume that y is to the left of x (see Fig. 6). Let z be the point of S on the right edge of $rect(S)$, and let p be the point of S on the bottom edge of $rect(S)$. Sweep $S - \{z\}$ from bottom to top by horizontal line, and take $d-1$ points and denote the horizontal line passing through the $(d-1)$-th point by l_1. Sweep $S - \{z\}$ from top to bottom by horizontal line, and take the $(c+1)$-th point, say s, and denote the horizontal line passing through s by l_2. Divide the open rectangle between l_1 and l_2 into two open rectangles by vertical line segment so that the left open rectangle and the right open rectangle except z contain a and b points, respectively. Let us denote the left most point in the left rectangle by r and the right most point in the right rectangle by t. Then there are three Hamilton paths starting at r, t and s as shown in Fig. 6, where a Hamilton path starting at s passes through

y, x, z, and $d-1$ points below l_1 including p. By joining s to r and t by L-line segments, we can obtain the desired non-crossing geometric embedding of T on S.

Case 2. $|V_3(T)| \geq 2$ *and* $\deg_T(v) = 1$.

Let v_1 be the vertex adjacent to v, and let y be the highest point of $S-x$. By induction, $T-v$ can be embedded on $S-x$ so that $\psi(v_1) = y$ since v_1 is a good vertex of $T-v$. Note that even if $v_1 = w_i$ for some $i \in \{1,2\}$, v_1 becomes a good vertex of $T-v$. By adding an L-line segment joining y to x to the embedding ψ of $T-v$ on $S-x$, we obtain the desired embedding of T on S (see (2) if Fig. 7).

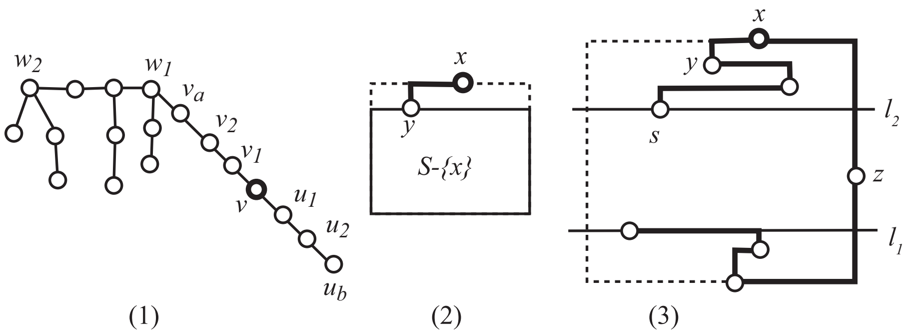

Fig. 7. (1) A tree T; (2) A geometric embedding of T in the case of $\deg_T(v) = 1$; (3) A geometric embedding of T in the case where $\deg_G(v) = 2$ and x is not a corner point.

Case 3. $|V_3(T)| \geq 2$ *and* $\deg_T(v) = 2$.

By symmetry, we may assume that there is a path that connects w_1 and v and does not pass through w_2. Let $(w_1, v_{a-1}, \ldots, v_1, v, u_1, u_2, \ldots, u_b)$ be the unique path in T that starts at w_1, passes through v and ends at end-vertex u_b of T (see (1) of Fig 7). Let y_1 be the highest point of $S-x$. By symmetry we may assume that y_1 is to the left of x.

We first assume that x is not a right-top corner point. Let z be the point of S on the right edge of $rect(S)$. Sweep $S-\{z\}$ from bottom to top by horizontal line, and take $b-1$ points and denote the horizontal line passing through the $(b-1)$-th point by l_1. Sweep $S-\{z\}$ from top to bottom by horizontal line, and take the $(a+1)$-th point, say s, and denote the horizontal line passing through s by l_2.

We embed the path $(w_1, v_{a-1}, \ldots, v_1, v, u_1, u_2, \ldots, u_b)$ on the set of points of S that are above l_2, below l_1 or z (see (3) of Fig 7). By induction and by the fact that w_1 is a good vertex of a tree $T^* = T - \{v_{a-1}, \ldots, v_1, v, u_1, u_2, \ldots, u_b\}$, T^* can be embedded on the set of points of S between l_2 and l_1 except z so that w_1

is mapped to s. By combining these two embeddings, we can obtain the desired embedding of T on S.

We next assume that x is a right-top corner point. In this case, the point of S on the right edge of $rect(S)$ is x. Then, sweep S from bottom to top by horizontal line, and take b points and denote the horizontal line passing through the b-th point by l_1. Latter, by applying the same argument give above, we can obtain the desired geometric embedding of T on S.

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