Star-Factors with Large Components

Mikio Kano
Department of Computer and Information Sciences
Ibaraki University
Nakanarusawa 4–12–1 Hitachi
Ibaraki 316–8511
JAPAN
E-mail : kano@mx.ibaraki.ac.jp

and

Akira Saito*
Department of Computer Science
Nihon University
Sakurajosui 3–25–40
Setagaya-Ku, Tokyo 156–8550
JAPAN
E-mail : asaito@chs.nihon-u.ac.jp

Abstract

In this note, we prove a theorem on component factors. For a set of connected graphs $\mathcal{H}$, a spanning subgraph H of a graph G is said to be an $\mathcal{H}$-factor if every component of H is isomorphic to some member of $\mathcal{H}$. Amahashi and Kano [Discrete Math. **42** (1982), 1–6] have proved that a graph G which satisfies $i(G-S) \leq m|S|$ for every $S \subset V(G)$ has a $\{K_{1,l} \colon 1 \leq l \leq m\}$-factor, where $i(G)$ is the number of isolated vertices in G. Here we exclude small stars from the set and prove that a graph G which satisfies $i(G - S) \leq \frac{1}{m}|S|$ for every $S \subset V(G)$ has a $\{K_{1,l} \colon m \leq l \leq 2m\}$-factor.

keywords : star, factor, isolated vertex, component

1 Introduction

A spanning subgraph of a graph is called a factor. In particular, for a positive integer k, a k-factor of a graph G is a k-regular spanning subgraph of G. Thus, a 1-factor

*Partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research (C), 22500018, 2010

coincides with a perfect matching. The notion of a k-factor is generalized into a (g, f)-factor, which is defined as a spanning subgraph H that satisfies $g(v) \leq \deg_H v \leq f(v)$ for every $v \in V(H) = V(G)$, where $\deg_H x$ is the degree of x in H, and g and f are integer-valued functions defined on $V(G)$. Since all these notions look at the degrees of vertices, they are often referred as "degree factors". The degree factors have been studied actively over the years.

On the other hand, when we focus on components of a factor, we are led to the notion of "component factors". Let $\mathcal{H}$ be a set of connected graphs. Then a spanning subgraph H of a graph G is called an $\mathcal{H}$-factor if each component of H is isomorphic to some member of $\mathcal{H}$. In particular, if every component of $\mathcal{H}$ is a star, H is called a star-factor. According to these definitions, a 1-factor is a $\{K_2\}$-factor, which is also a star-factor with an additional condition that each component has order two.

In [1], Amahashi and Kano proved the following theorem. For a graph G, let $i(G)$ denote the number of isolated vertices in G.

Theorem A ([1]) *Let m be an integer with $m \geq 2$. If a graph G satisfies $i(G - S) \leq m|S|$ for every $S \subset V(G)$, then G has a $\{K_{1,k} \colon 1 \leq k \leq m\}$-factor.*

Though the above theorem is a result on component factors, it can be deduced from the theory of degree factors. If g and f are constant functions taking values a and b, respectively, a (g, f)-factor is also called an $[a, b]$-factor. Clearly, a $\{K_{1,l} \colon 1 \leq l \leq m\}$-factor is a $[1, m]$-factor. On the other hand, it is easy to see that a $[1, m]$-factor with the smallest number of edges is a $\{K_{1,l} \colon 1 \leq l \leq m\}$-factor. Therefore, the existence of a $\{K_{1,l} \colon 1 \leq l \leq m\}$-factor is equivalent with the existence of a $[1, m]$-factor, and Theorem A can be deduced from Lovász's (g, f)-Factor Theorem [4].

In [3], Kano, Lu and Yu have proved the following theorem.

Theorem B ([3]) *If a graph G satisfies $i(G - S) \leq \frac{1}{2}|S|$ for every $S \subset V(G)$, then G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor.*

The combination of $K_{1,2}$, $K_{1,3}$ and K_5 looks strange, and we do not know why this apparently peculiar combination admits the above simple sufficient condition based on the number of isolated vertices. But we observe that under the same assumption, Theorem A also guarantees the existence of a $\{K_{1,2}, K_{1,3}, K_{1,4}\}$-factor. We also observe that since $K_{1,1}$ is excluded from the set, the results on degree factors are unlikely to deduce Theorem A.

The purpose of this note is to prove the following theorem, which is a generalization of the above observation.

Theorem 1 *Let m be a positive integer and let G be a graph. If $i(G-S) \leq \frac{1}{m}|S|$ holds for every $S \subset V(G)$, then G has a $\{K_{1,l} \colon m \leq l \leq 2m\}$-factor.*

In the next section, we prove the above theorem and discuss its sharpness. In Section 3, we make some concluding remarks.

For standard graph-theoretic notation not explained in this paper, we refer the reader to [2]. For a graph G, let $\alpha(G)$ and $\delta(G)$ denote the independence number and the minimum degree of G, respectively. For $x \in V(G)$, the neighborhood of x in G is denoted by $N_G(x)$. For $S \subset V(G)$, we define $N_G(S)$ by $N_G(S) = \bigcup_{v \in S} N_G(v)$, and let $G[S]$ denote the subgraph of G induced by S.

2 Proof of Theorem 1

In this section, we prove Theorem 1. The proof strategy is similar to the proof of Theorem B in [3].

We use the following corollary of Hall's Theorem, which was also used without a proof in [3]. We give a proof in order to make this note self-contained.

Theorem C ([3]) *Let G be a bipartite graph with partite sets X and Y, and let f be a function from X to the set of positive integers. If $|N_G(S)| \geq \sum_{v \in S} f(v)$ holds for every $S \subset X$, then G has a subgraph H such that $X \subset V(H)$, $\deg_H u = f(u)$ for each $u \in X$ and $\deg_H v \leq 1$ for each $v \in Y$. In particular, if G further satisfies $\sum_{v \in X} f(v) = |Y|$, then G has a star-factor H with $\deg_H u = f(u)$ for each $u \in X$ and $\deg_H v = 1$ for each $v \in Y$.*

Proof. For each $u \in X$, we prepare $f(u)$ new vertices $u_1, u_2, \ldots, u_{f(u)}$, and let $X^f = \{u_i \colon u \in X, 1 \leq i \leq f(u)\}$. Then join u_i and v by an edge if $uv \in E(G)$. Let G^f be the resulting bipartite graph with partite sets X^f and Y. Then G has a required subgraph if and only if G^f has a matching which saturates all the vertices of X^f, and the conclusion follows from Hall's Theorem. $\square$

Now we prove Theorem 1.

Proof of Theorem 1. Let

$$\beta = \min\left\{\frac{1}{m}|X| - i(G - X) \colon X \subset V(G), i(G - X) \geq 1\right\}.$$

Note $\beta \geq 0$ by the assumption.

We proceed by induction on $|G|$. First, we claim the following.

Claim 1 $|G| \geq (m+1)\alpha(G) + m\beta$

Proof. Let A be a largest independent set of G and let $X = V(G) - A$. Then $i(G-X) = |A| = \alpha(G) \geq 1$, and hence $\beta \leq \frac{1}{m}|X| - i(G - X) = \frac{1}{m}|X| - \alpha(G)$, which implies $|X| \geq m(\alpha(G) + \beta)$. Since $|X| = |G| - \alpha$, we obtain the required inequality. $\square$

By Claim 1, $|G| \geq m + 1$. Suppose $m + 1 \leq |G| \leq 2m + 1$. If G has a pair of non-adjacent vertices x and y, then $\alpha(G) \geq 2$ and hence $|G| \geq 2(m + 1)$ by Claim 1. This is a contradiction. Therefore, G is complete and hence has a spanning subgraph which is isomorphic to $K_{1,|G|-1}$. Since $m + 1 \leq |G| \leq 2m + 1$, the theorem follows in this case.

Now suppose $|G| \geq 2m+2$. Let S be a set of vertices in G which satisfies $i(G-S) \geq 1$ and $\frac{1}{m}|S| - i(G-S) = \beta$.

Claim 2 $\delta(G) \geq m(\beta+1)$.

Proof. Let x be a vertex of G with $\deg_G x = \delta(G)$. Then $i(G - N_G(x)) \geq 1$ and hence $\frac{1}{m}|N_G(x)| - i(G - N_G(x)) \geq \beta$, or $|N_G(x)| \geq m\big(\beta + i(G - N_G(x))\big) \geq m(\beta+1)$, which implies $\delta(G) \geq m(\beta+1)$. $\square$

Claim 3 *Every component D of $G-S$ with $|D| \geq 2$ has a $\{K_{1,l} \colon m \leq l \leq 2m\}$-factor.*

Proof. Let $T \subset V(D)$. Note $i(G - (S \cup T)) = i(G-S) + i(D-T) \geq 1$. Therefore, $\frac{1}{m}|S \cup T| - i(G - (S \cup T)) \geq \beta$, or $\beta \leq \frac{1}{m}|S| + \frac{1}{m}|T| - i(G-S) - i(D-T)$. Since $\frac{1}{m}|S| - i(G-S) = \beta$, we have $i(D-T) \leq \frac{1}{m}|T|$. Then D has a $\{K_{1,l} \colon m \leq l \leq 2m\}$-factor by the induction hypothesis. $\square$

Let U be the set of isolated vertices in $G-S$ and let $D_1, \dots, D_t$ be the components of $G-S$ of order at least two.

Claim 4 *For every $Y \subset U$ with $Y \neq \emptyset$, $|N_G(Y)| \geq m|Y| + m\beta$, and $|N_G(U)| = m|U| + m\beta$.*

Proof. Since $i(G - N_G(Y)) \geq |Y| \geq 1$, $\frac{1}{m}|N_G(Y)| - i(G - N_G(Y)) \geq \beta$. Then $|N_G(Y)| \geq m\big(\beta + i(G - N_G(Y))\big) \geq m(\beta + |Y|)$.

If $N_G(U) \neq S$, then there exists a vertex $s \in S$ with $N_G(s) \cap U = \emptyset$. Let $S' = S - \{s\}$. Then $i(G-S') \geq |U| = i(G-S) \geq 1$ and $\frac{1}{m}|S'| - i(G-S') \leq \frac{1}{m}|S| - \frac{1}{m} - i(G-S) = \beta - \frac{1}{m} < \beta$. This contradicts the definition of β. Therefore, $N_G(U) = S$ and $\frac{1}{m}|S| - i(G-S) = \frac{1}{m}|N_G(U)| - |U| = \beta$, which implies $|N_G(U)| = m(\beta + |U|)$. $\square$

Suppose $\beta > 1$. Then since $m\beta$ is an integer, $\beta \geq 1 + \frac{1}{m} = \frac{m+1}{m}$. Let $x \in V(G)$. Since $\delta(G) \geq m(\beta+1) \geq 2m+1$ by Claim 2, we can take m distinct neighbors $y_1, y_2, \dots, y_m$ of x in G. Let $G' = G - \{x, y_1, \dots, y_m\}$. Let $T \subset V(G')$. If $i(G'-T) = i(G - (T \cup \{x, y_1, \dots, y_m\})) \geq 1$, then

$$\frac{1}{m}|T \cup \{x, y_1, \dots, y_m\}| - i(G - (T \cup \{x, y_1, \dots, y_m\})) \geq \beta,$$

which implies $\frac{1}{m}|T| + \frac{m+1}{m} - i(G'-T) \geq \beta \geq \frac{m+1}{m}$, or $i(G'-T) \leq \frac{1}{m}|T|$. If $i(G'-T) = 0$, then $i(G'-T) \leq \frac{1}{m}|T|$ trivially holds. Thus, by the induction hypothesis, G' has a $\{K_{1,l} \colon m \leq l \leq 2m\}$-factor F'. Since $G[\{x, y_1, \dots, y_m\}]$ has a spanning subgraph isomorphic to $K_{1,m}$, this subgraph and F' form a $\{K_{1,l} \colon m \leq l \leq 2m\}$-factor of G. Therefore, we may assume $\beta \leq 1$.

Since $i(G-S) \geq 1$, $U \neq \emptyset$. Take $u_0 \in U$. Define $f \colon U \to \boldsymbol{Z}$ by

$$f(u) = \begin{cases} m + m\beta & \text{if } u = u_0 \\ m & \text{if } u \neq u_0 \end{cases}$$

Then $\sum_{u\in U} f(u) = m|U| + m\beta = |N_G(U)|$ by Claim 4. Again by Claim 4, for $Y \subset U$ with $Y \neq \emptyset$, $\sum_{u\in Y} f(u) \leq m|Y| + m\beta \leq |N_G(Y)|$. Therefore, by Theorem C, $G[U \cup S]$ has a $\{K_{1,m}, K_{1,m+m\beta}\}$-factor. Note that since $\beta \leq 1$, $m + m\beta \leq 2m$. Then this factor and a $\{K_{1,l}\colon m \leq l \leq 2m\}$-factor in each D_i $(1 \leq i \leq t)$ form a $\{K_{1,l}\colon m \leq l \leq 2m\}$-factor of G. $\Box$

Next, we prove that Theorem 1 is best-possible if $m \geq 2$. Note that $i(G-S) > \frac{1}{m}|S|$ is equivalent to $i(G-S) \geq \frac{1}{m}|S| + \frac{1}{m}$.

Theorem 2 *Let m be an integer with $m \geq 2$. Then there exist infinitely many graphs G such that*

(1) $i(G-S) \leq \frac{1}{m}|S| + \frac{1}{m}$ holds for every $S \subset V(G)$, but

(2) G does not have a $\{K_{1,l}\colon m \leq l \leq 2m\}$-factor.

Proof. Let r be an integer with $r \geq 2$. Let $H_1, H_2, \ldots, H_r$ be disjoint copies of K_{m+1}, and let H_0 be a copy of K_m. Choose one vertex x_i from each H_i $(0 \leq i \leq r)$, and join x_0 and x_j by an edge for each j with $1 \leq j \leq r$. Let G_r be the resulting graph.

Assume G_r has a $\{K_{1,l}\colon m \leq l \leq 2m\}$-factor F. Since $|H_0| = m$, a component F_0 of F which intersects H_0 also contains a vertex v in H_{i_0} for some i_0, $1 \leq i_0 \leq r$. This is possible only if x_0 is the center of F_0. Then $H_{i_0} - v$ contains a component of F which is different from F_0. However, this is impossible since $|H_{i_0} - v| = m$.

Next, we prove that G_r satisfies $i(G_r - S) \leq \frac{1}{m}|S| + \frac{1}{m}$ for each $S \subset V(G_r)$. Let S_0 be a set of vertices of G_r with $\frac{1}{m}|S_0| - i(G_r - S_0) = \min\big\{\frac{1}{m}|S| - i(G_r - S)\colon S \subset V(G)\big\}$. If $V(H_i) \cap S_0 \neq \emptyset$ and $|V(H_i) - S_0| \geq 2$ for some i with $0 \leq i \leq r$, take $x \in V(H_i) - \big(S_0 \cup \{x_i\}\big)$ and let $S' = S_0 \cup (V(H_i) - \{x\})$. Then $i(G_r - S') = i(G_r - S_0) + 1$ and $|S'| \leq |S_0| + m - 1$, and hence $\frac{1}{m}|S'| - i(G_r - S') \leq \frac{1}{m}|S_0| - i(G_r - S_0) - \frac{1}{m}$. This contradicts the choice of S_0. On the other hand, if $V(H_i) \subset S_0$ for some i with $0 \leq i \leq r$, then since $|H_i| \geq m \geq 2$, we can take $x \in V(H_i) - \{x_i\}$. Let $S'' = S_0 - \{x\}$. Then $i(G_r - S'') = i(G_r - S_0) + 1 > i(G_r - S_0)$ and $|S''| = |S_0| - 1 < |S_0|$. Therefore, $\frac{1}{m}|S''| - i(G_r - S'') < \frac{1}{m}|S_0| - i(G_r - S_0)$. This again contradicts the choice of S_0. Therefore, we may assume that for each i, $0 \leq i \leq r$, either $V(H_i) \cap S_0 = \emptyset$ or $|V(H_i) - S_0| = 1$. By symmetry of $H_1, \ldots, H_r$, we may assume $|V(H_i) - S_0| = 1$ for $1 \leq i \leq s$ and $V(H_i) \cap S_0 = \emptyset$ for $s+1 \leq i \leq r$.

If $x_0 \notin S_0$, take $x \in V(H_0) - \{x_0\}$ and let $S_2 = \big(S \cup V(H_0)\big) - \{x\}$. Then either $|S_2| = |S_0|$ or $|S_2| = |S_0| + m - 1$, and $i(G_r - S_2) \geq i(G_r - S_0) + 1$. Therefore, $\frac{1}{m}|S_2| - i(G_r - S_2) \leq \frac{1}{m} - i(G_r - S_0) - \frac{1}{m}$, which again contradicts the choice of S_0. Therefore, $x_0 \in S_0$ and $|V(H_0) \cap S_0| = 1$.

Now we have $|S_0| = sm+m-1$ and $i(G_r - S_0) = s+1$, and hence $\frac{1}{m}|S_0| - i(G_r - S_0) = -\frac{1}{m}$. This implies $i(G_r - S) \leq \frac{1}{m}|S| + \frac{1}{m}$ for each $S \subset V(G)$. $\Box$

Theorem 1 is not sharp for $m = 1$. Theorem 1 says that a graph G with $i(G-S) \leq |S|$ for every $S \subset V(G)$ has a $\{K_{1,1}, K_{1,2}\}$-factor. But according to Theorem A, if G satisfies $i(G-S) \leq 2|S|$ for every $S \subset V(G)$, then G already has a $\{K_{1,1}, K_{1,2}\}$-factor.

3 Concluding Remarks

In this note, we have proved a sufficient condition for a graph to have a star-factor in which the order of each component falls in a certain interval. It is described in terms of the number of isolated vertices in vertex-deleted subgraphs.

This note only deals with a star-factor in which the order of each component falls in the interval $[m+1, 2m+1]$. Though it looks special, it is actually a natural interval in the following sense.

Suppose we try to obtain a sufficient condition for a graph to have a star-factor in which the order of each component falls in the interval $[m+1, 2m]$ instead of $[m+1, 2m+1]$. Since $2m+1$ is not expressed as a sum of integers in $\{m+1, m+2, \ldots, 2m\}$, K_{2m+1} does not have a $\{K_{1,l} \colon m \leq l \leq 2m-1\}$-factor. Therefore, if we try to obtain a sufficient condition for a graph to have a $\{K_{1,l} \colon m \leq l \leq 2m-1\}$-factor, then this condition fails to hold for K_{2m+1}. Such a condition would be much stronger than the one we have obtained in this note.

Comparing Theorem 1 with Theorem B, we may conjecture that a graph satisfying the same assumption as in Theorem 1 actually has a $(\{K_{1,l} \colon m \leq l \leq 2m-1\}) \cup \{K_{2m+1}\})$-factor. However, until we obtain more insight into the meaning of this apparently peculiar combination, we do not intend to pursue this direction.

References

[1] A. Amahashi and M. Kano, On factors with given components. *Discrete Math.* **42** (1982), 1–6.

[2] G. Chartrand and L. Lesniak, *Graphs & Digraphs* (4th ed.), Chapman and Hall/CRC, Boca Raton, Florida, U.S.A. (2005).

[3] M. Kano, H. Lu and Q. Yu, Component factors with large components in graphs, *Appl. Math. Lett.* **23** (2010), 385–389.

[4] L. Lovász, Subgraphs with prescribed valencies, *J. Combinatorial Theory* **8** (1970), 391–416.