The Chromatic Index of a Graph Whose Core has Maximum Degree 2 *†

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Abstract

Let G be a graph. The core of G, denoted by G_Δ, is the subgraph of G induced by the vertices of degree $\Delta(G)$, where $\Delta(G)$ denotes the maximum degree of G. A *k-edge coloring* of G is a function $f : E(G) \to L$ such that $|L| = k$ and $f(e_1) \neq f(e_2)$ for all two adjacent edges e_1 and e_2 of G. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number k for which G has a k-edge coloring. A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$ and *Class* 2 if $\chi'(G) = \Delta(G) + 1$. In this paper it is shown that every connected graph G of even order and $\Delta(G_\Delta) \leq 2$ is Class 1 if $|G_\Delta| \leq 9$ or G_Δ is a cycle of order 10.

1 Introduction

All graphs considered in this paper are finite, undirected, with no loops or multiple edges. Let G be a graph. Then $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The number of vertices of G is called the *order* of G and denoted by $|G|$. Also, $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of G, respectively. The *core* of G, denoted by G_Δ, is the subgraph of G induced by all vertices of degree $\Delta(G)$. We denote the cycle of order n by C_n. Let H be a subgraph of G. For a vertex

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u of H, $d_H(u)$ denotes the degree of u in H, and for every vertex v of G, $N_H(v)$ denotes $N_G(v) \cap V(H)$, where $N_G(v)$ is the neighborhood of v in G.

A *matching* in a graph G is a set of pairwise non-adjacent edges and a 1-*factor* is a matching which covers $V(G)$. Let $S \subseteq V(G)$ and H be a component of $G - S$. We call H an *odd component* if H has odd order. The number of odd components of G is denoted by $odd(G)$. For a subset $X \subseteq V(G)$ ($Y \subseteq E(G)$), $G - X$ ($G - Y$) denotes the graph obtained from G by deleting all vertices (edges) of X (Y), respectively. Moreover, we mean $G - H$, the induced subgraph on $V(G) - V(H)$.

A *k-edge coloring* of a graph G is a function $f : E(G) \longrightarrow L$ such that $|L| = k$ and $f(e_1) \neq f(e_2)$ for all two adjacent edges e_1 and e_2 of G. A graph G is *k-edge colorable* if G has a k-edge coloring. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum number k for which G has a k-edge coloring. For a general introduction to the edge coloring, the interested reader is referred to [10].

A celebrated result due to Vizing [21] states that for every graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. A graph G is said to be *Class* 1 if $\chi'(G) = \Delta(G)$ and *Class* 2 if $\chi'(G) = \Delta(G)+1$. Moreover, a connected graph G is called *critical* if it is Class 2 and $G - e$ is Class 1 for every edge $e \in E(G)$. A graph G is called *overfull* if $|E(G)| > \lfloor \frac{|V(G)|}{2} \rfloor \Delta(G)$. It is easy to see that, if G is overfull, then G is Class 2. For more information about overfull graphs see [12]. In [19] it was proved that there is no critical connected graph G of even order with $|G_\Delta| \leq 5$.

Let H, Q and R be subgraphs of G. We denote the number of edges of H with one end point in Q and another end point in R by $e_H(Q, R)$. For a subset $S \subseteq V(G)$, we denote the induced subgraph of G on S by $\langle S \rangle_G$.

Classifying a graph into Class 1 and Class 2 is a difficult problem in general (indeed, NP hard), even when restricted to the class of graphs with maximum degree 3 (see [17]). As a consequence, this problem is usually considered on classes of graphs with particular classes of cores. One possibility is to consider a graph whose core has a simple structure (see [4, 7, 9, 11, 13, 14, 15, 16, 22]). Vizing [22] proved that, if G_Δ has no edge, then G is Class 1. Fournier [11] generalized Vizing's result by proving that, if G_Δ contains no cycle, then G is Class 1. Thus a necessary condition for a graph to be Class 2 is to have a core containing cycles. Hilton and Zhao [14, 15] considered the problem of classifying graphs whose cores are a disjoint union of cycles. Only a few such graphs are known to be Class 2. These include the overfull graphs and the graph P^*, which is obtained from the Petersen graph by removing one vertex and has order 9. Furthermore, they posed the following conjecture.

Conjecture 1. *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$. Then G is Class 2 if and only if G is overfull, unless $G \neq P^*$.*

In [3], the following theorem was proved:

Theorem 1. *Let G be a connected graph such that $\Delta(G_\Delta) \leq 2$, $\Delta(G) = 3$ and $G \neq P^*$. Then G is Class 1.*

In [6] the following result was proved.

Theorem 2. *Let G be a connected graph with $|G_\Delta| = 3$. Then G is Class 2 if and only if for some integer n, G is obtained from K_{2n+1} by removing $n-1$ independent edges.*

An *edge cut* is a set of edges whose removal produces a subgraph with more components than the original graph. So a *k-edge-connected* graph has no edge cut of size $k-1$.

Two following results provide some conditions under which a graph G with $|G_\Delta| = 4$ is Class 1.

Theorem 3.[5] *Let G be a 2-edge-connected graph of even order with $|G_\Delta| = 4$. Then G is Class 1.*

Theorem 4.[5] *Let $3 \leq r \leq 4$ be an integer and G be an $(r-2)$-edge-connected graph of order $2n+1$ with $|G_\Delta| \leq r$. Then G is Class 2 if and only if $|E(G)| \geq n\Delta(G) + 1$.*

Theorem 5.[20] *Let G be a critical connected graph with $\Delta(G) \geq 3$. Further suppose that G has $2n+1 \geq 7$ vertices and $|G_\Delta| = 5$. Then $|E(G)| = n\Delta(G) + 1$.*

The following useful result, which follows from Vizing's Adjacency Lemma [8], is given in Schrijver's homepage [18, p.1765].

Theorem 6. *Suppose k is a natural number. Let v be a vertex of a graph G such that v and all its neighbors have degree at most k, while at most one neighbor has degree precisely k. Then G is k-edge colorable if $G - \{v\}$ is k-edge colorable.*

The previous theorem implies the following well-known result which is due to Fournier.

Theorem 7.[11] *If G_Δ is a forest, then G is Class 1.*

Theorem 8.[15] *Let G be a connected graph of Class 2 and $\Delta(G_\Delta) \leq 2$. Then the following statements hold.*

(i) *G is critical;*

(ii) *$\delta(G_\Delta) = 2$;*

(iii) *$\delta(G) = \Delta(G) - 1$, unless G is an odd cycle.*

Theorem 9.[15] *Let G be a critical connected graph. Then every vertex of G is adjacent to at least two vertices of G_Δ.*

Theorem 10.[1] *Let G be a connected graph with $\Delta(G_\Delta) \leq 2$. Suppose that G has an edge cut of size at most $\Delta(G) - 2$ which is a matching or a star. Then G is Class 1.*

A connected graph is called *unicyclic* if it contains precisely one cycle.

Theorem 11.[1] *Let G be a connected graph. If every component of G_Δ is a unicyclic graph or a tree and G_Δ is not a disjoint union of cycles, then G is Class 1.*

Theorem 12.[1] *Let G be a connected graph of even order. If $\Delta(G_\Delta) \leq 2$ and $|G_\Delta|$ is odd, then G is Class 1.*

Now, we are in a position to prove our main theorem.

Theorem 13. *Let G be a connected graph of even order and $\Delta(G_\Delta) \leq 2$. If $|G_\Delta| \leq 9$ or $G_\Delta = C_{10}$, then G is Class 1.*

Proof. For simplicity, let $\Delta = \Delta(G)$. The proof is by induction on $\Delta + |G|$. First note that if $\delta(G_\Delta) \leq 1$ or $\delta(G) < \Delta - 1$ or there exists a vertex $x \in V(G)$ such that $|N_{G_\Delta}(x)| \leq 1$, then by Theorems 8 and 9, G is Class 1 and we are done. Thus, one can easily assume that G_Δ is a disjoint union of cycles, $\delta(G) = \Delta - 1$ and

$$|N_{G_\Delta}(x)| \geq 2 \qquad \text{for every} \quad x \in V(G). \tag{1}$$

By (1), we find that $2(|G|-|G_\Delta|) \le e_G(G_\Delta, G-G_\Delta) = (\Delta-2)|G_\Delta|$, and so

$$|G| \le \frac{\Delta|G_\Delta|}{2} \le 5\Delta. \tag{2}$$

Moreover, if $|G_\Delta|$ is odd, then by Theorem 12, G is Class 1. Thus we can assume that

$$|G_\Delta| \text{ is even, } G_\Delta \text{ is a disjoint union of cycles and } |G_\Delta| \le 8 \text{ or } G_\Delta = C_{10}. \tag{3}$$

Note that since G_Δ is a disjoint union of cycles, $\Delta \ge 2$. If $\Delta = 2$, then by the connectivity of G, G is a cycle of even order and so G is Class 1. If $\Delta = 3$, then since $|G|$ is even, by Theorem 1, the assertion is proved. So we may assume that $\Delta \ge 4$. If G has an edge cut of size at most 2, then by Theorem 10, G is Class 1 and we are done. Thus we can suppose that G is 3-edge connected. First we prove the following claim.

Claim 1. *G has a 1-factor.*

To the contrary, by Tutte's 1-factor Theorem [2, p.44] and by the assumption that G is of even order, there exists a non-empty subset $T \subseteq V(G)$ such that $odd(G-T) > |T|$. Let $m = odd(G-T)$. Since $|G|$ is even, we have $m \equiv |T| \pmod 2$, which implies that $m \ge |T|+2$. First assume $T = \{u\}$. Then there exists a component D of $G-T$ such that $e_G(u, D) \le \Delta - 2$ by $m \ge 3$. So by Theorem 10, G is Class 1 and we are done. Thus we may assume $|T| \ge 2$.

Let $B_1, \ldots, B_c$ (big) and $S_1, \ldots, S_d$ (small) be the odd components of $G-T$ such that $|B_i| \ge \Delta$ for every $1 \le i \le c$ and $|S_j| \le \Delta - 1$ for every $1 \le j \le d$, where $m = c+d$. Since $|T| \le m-2$,

$$|T| \le c+d-2. \tag{4}$$

Also, since G is 3-edge connected,

$$e_G(T, B_i) \ge 3 \quad \text{for every} \quad 1 \le i \le c.$$

For every $1 \le j \le d$, since $1 \le |S_j| \le \Delta - 1 = \delta(G)$, the following hold:

$$\begin{aligned} e_G(T, S_j) &= \sum_{x \in V(S_j)} e_G(T, x) \\ &\ge (\delta(G) - (|S_j|-1))|S_j| \\ &\ge (\Delta - |S_j|)|S_j| \qquad (5) \\ &\ge \Delta - 1. \qquad (6) \end{aligned}$$

Let $q = |T \cap V(G_\Delta)|$ and $r = |E(\langle T \rangle_G) \cap E(G_\Delta)|$. Since G_Δ is a 2-regular graph of order at most 10, the number of edges of G_Δ joining T to $V(G) - T$ satisfies

$$2q - 2r = e_{G_\Delta}(T, G - T) \le 2(|G_\Delta| - q) \le 2(10 - q).$$

Hence

$$q \le 5 + \frac{r}{2}. \tag{7}$$

Since $|N_{G_\Delta}(x)| \ge 2$ for every $x \in V(G)$, $|B_j| \ge \Delta$ and since G is 3-edge connected, we obtain that

$$e_G(T, B_j) \ge \begin{cases} 3 & \text{if } |V(B_j) \cap V(G_\Delta)| \ge 2, \\ \Delta + 1 & \text{if } |V(B_j) \cap V(G_\Delta)| = 1, \\ 2\Delta & \text{otherwise.} \end{cases} \tag{8}$$

Let c_0, c_1 and c_2 be the number of components B_j's such that $|V(B_j) \cap V(G_\Delta)| = 0$, $|V(B_j) \cap V(G_\Delta)| = 1$ and $|V(B_j) \cap V(G_\Delta)| \ge 2$, respectively. It is easy to see that $c_2 \le 3$ by $|G_\Delta| \le 10$. Moreover, $c = c_0 + c_1 + c_2$ and

$$\begin{aligned} e_G(T, B_1 \cup \cdots \cup B_c) &\ge 3c_2 + (\Delta + 1)c_1 + 2\Delta c_0 \\ &= (\Delta - 1)c - (\Delta - 4)c_2 + 2c_1 + (\Delta + 1)c_0. \end{aligned} \tag{9}$$

Obviously, using (6) and (9), we have

$$\begin{aligned} & q\Delta - 2r + (|T| - q)(\Delta - 1) \\ \ge\ & e_G(T, B_1 \cup \cdots \cup B_c \cup S_1 \cup \cdots \cup S_d) \qquad (10) \\ \ge\ & (\Delta - 1)c - (\Delta - 4)c_2 + 2c_1 + (\Delta + 1)c_0 + (\Delta - 1)d. \qquad (11) \end{aligned}$$

This implies that

$$(|T| - c - d)(\Delta - 1) + q - 2r + (\Delta - 4)c_2 - 2c_1 - (\Delta + 1)c_0 \ge 0. \tag{12}$$

On the other hand, by (4) and (7), we obtain that

$$\begin{aligned} & (|T| - c - d)(\Delta - 1) + q - 2r + (\Delta - 4)c_2 - 2c_1 - (\Delta + 1)c_0 \\ \le\ & -2(\Delta - 1) + 5 - \frac{3r}{2} + (\Delta - 4)c_2 - 2c_1 - (\Delta + 1)c_0. \end{aligned} \tag{13}$$

Hence, if $c_2 \le 2$, then

$$(|T| - c - d)(\Delta - 1) + q - 2r + (\Delta - 4)c_2 - c_1 - (\Delta + 1)c_0 < 0. \tag{14}$$

This contradicts (12). Thus, one can assume that $c_2 = 3$ by $c_2 \leq 3$. If $c_0 \geq 1$, then similarly (14) holds by (13), and we get a contradiction. So, $c_0 = 0$. We show that

$$c = c_2 = 3 \quad \text{and} \quad V(G_\Delta) \subseteq T \cup (\cup_{i=1}^{3} B_i). \tag{15}$$

To the contrary, let D be a component of $G-(T\cup(\cup_{i=1}^{3} B_i))$ such that $|V(D)\cap V(G_\Delta)| \geq 1$. Now, since $c_2 = 3$ and $|G_\Delta| \leq 10$, we have $q \leq 3$. Note that if $q \leq 1$, then G_Δ is a disjoint union of at least four cycles, a contradiction. If $q = 2$, then G_Δ consists of at least three cycles and $|G_\Delta| \geq 11$, a contradiction. If $q = 3$, then G_Δ consists of at least two cycles and $|G_\Delta| \geq 11$, a contradiction. Therefore (15) holds.

By (15) G_Δ passes through exactly three components of $G - T$. By (11) and (15),

$$q - 2r + |T|(\Delta - 1) \geq 9 + (\Delta - 1)d. \tag{16}$$

Now, if $d \geq |T|$, then by $\Delta \geq 4$,

$$q - 2r \geq 9 + (d - |T|)(\Delta - 1) \geq 9,$$

which contradicts (7). Thus, we can suppose that $d \leq |T| - 1$. Now, by $c = 3$ and (4),

$$d = |T| - 1. \tag{17}$$

By (5), (8), (10) and (15), we obtain that

$$q - 2r + |T|(\Delta - 1) \geq 9 + \sum_{j=1}^{d} (\Delta - |S_j|)|S_j|.$$

Thus

$$(|T| - d)(\Delta - 1) + q - 2r - 9 - \sum_{j=1}^{d} \big((\Delta - |S_j|)|S_j| - (\Delta - 1)\big) \geq 0. \tag{18}$$

On the other hand, by (7) and (17) , we find that

$$\begin{aligned} &(|T| - d)(\Delta - 1) + q - 2r - 9 - \sum_{j=1}^{d} \big((\Delta - |S_j|)|S_j| - (\Delta - 1)\big) \\ \leq\ & \Delta - 10 + 5 - \frac{3}{2}r - \sum_{j=1}^{d} \big((\Delta - |S_j|)|S_j| - (\Delta - 1)\big). \end{aligned}$$

If $\Delta = 4$, then $|S_j| = 1$ or 3 and so $(\Delta - |S_j|)|S_j| - (\Delta - 1) = 0$ for all j. Thus

$$\begin{aligned} &\Delta - 10 + 5 - \frac{3}{2}r - \sum_{j=1}^{d} \big((\Delta - |S_j|)|S_j| - (\Delta - 1)\big) \\ =\ & 4 - 10 + 5 - \frac{3}{2}r \\ =\ & -1 - \frac{3}{2}r < 0. \end{aligned}$$

This contradicts (18). Hence $\Delta \geq 5$. If $3 \leq |S_k| \leq \Delta - 2$ for some k, then $-\big((\Delta - |S_k|)|S_k| - (\Delta - 1)\big) \leq -\Delta + 3$. So,

$$\begin{aligned} & \Delta - 10 + 5 - \frac{3}{2}r - \sum_{j=1}^{d} \big((\Delta - |S_j|)|S_j| - (\Delta - 1)\big) \\ \leq\ & \Delta - 10 + 5 - \frac{3}{2}r - \Delta + 3 \\ =\ & -2 - \frac{3}{2}r < 0. \end{aligned}$$

This contradicts (18). Therefore, since $|S_j|$ is odd, we conclude that

$$\Delta \geq 5, \quad \text{and} \quad |S_j| = 1 \ \text{ or } \ \Delta - 1 \quad \text{for every } 1 \leq j \leq d. \tag{19}$$

By (6), (15), (17) and by the fact that every vertex u of T is adjacent to at least two vertices of G_Δ, we find that

$$|T|(\Delta - 2) \geq e_G(T, \cup_{j=1}^{d} S_j) \geq d(\Delta - 1) = (|T| - 1)(\Delta - 1). \tag{20}$$

This concludes that $|T| \leq \Delta - 1$.

First assume that $|S_k| = 1$ for some k, $1 \leq k \leq d$. Let $V(S_k) = \{w\}$. Then since $d_G(w) = \Delta - 1$, $|T| \geq \Delta - 1$. Thus $|T| = \Delta - 1$ and $d = \Delta - 2$ by (17). It follows from (2) that

$$4\Delta - 1 + \sum_{j=1}^{\Delta-2} |S_j| \leq |T| + |B_1| + |B_2| + |B_3| + \sum_{j=1}^{d} |S_j| \leq |G| \leq 5\Delta.$$

Hence $|S_j| = 1$ for all $1 \leq j \leq d$ by (19). Let $S_j = \{x_j\}$, $1 \leq j \leq d$. Then $N_G(x_j) = T$ for every j, and so for every vertex $u \in T$, $e_G(u, \cup_{j=1}^{d} S_j) = d = \Delta - 2$, which implies $d_G(u) = \Delta$ as $|N_{G_\Delta}(u)| \geq 2$. So $T \subset V(G_\Delta)$ and $e_G(u, \cup_{i=1}^{3} B_i) \leq 2$ for every $u \in T$. Now, since $c_2 = 3$, $q \leq 4$ and $e_G(T, B_i) \geq 3$, we obtain

$$3 \times 3 \leq e_G(T, B_1 \cup B_2 \cup B_3) \leq |T| \times 2 = q \times 2 \leq 8.$$

This is a contradiction.

Next, suppose that $|S_j| = \Delta - 1$ for every $1 \leq j \leq d$. Then it follows from (1) and (15) that $e_G(T, S_j) \geq 2|S_j| = 2\Delta - 2$ for every $1 \leq j \leq d$ and $e_G(u, \cup_{j=1}^{d} S_j) \leq \Delta - 2$ for every $u \in T$. Then similar to (20), we have

$$|T|(\Delta - 2) \geq e_G(T, \cup_{j=1}^{d} S_j) \geq (|T| - 1)(2\Delta - 2),$$

and so $|T| = 1$. This is a contradiction with $|T| \geq 2$. Consequently the proof of the claim is complete.

Now, let M be a 1-factor of G, and $H = G - M$. Then $\Delta(H) = \Delta - 1$, $\delta(H) = \delta(G) - 1$, $V(H_\Delta) = V(G_\Delta)$, $H_\Delta \subseteq G_\Delta$, $\delta(H_\Delta) \geq \delta(G_\Delta) - 1 = 1$, and by (1),

$$|N_H(v) \cap V(H_\Delta)| \geq 1 \quad \text{for every} \quad v \in V(H). \tag{21}$$

It is obvious that if H is Class 1, then so is G. Thus we can assume that H is Class 2. In particular, H is not connected since otherwise by induction hypothesis, H is Class 1.

Claim 2. *G_Δ consists of exactly two disjoint cycles.*

By (3), G_Δ is a disjoint union of cycles. Now, suppose that G_Δ is a cycle. If $\delta(H_\Delta) = 1$, then by Theorem 7, every component of H is Class 1, and so is H, a contradiction. Hence we may assume that H_Δ is a cycle. By (21), H is connected, a contradiction. Thus G_Δ is a disjoint union of at least two cycles. By (3), G_Δ is a disjoint union of two cycles. Therefore the claim is proved.

Now, we want to show that H has a component whose core is a cycle. First note that by (21), every component of H contains at least one vertex of H_Δ. If the core of each component of H has a vertex of degree 1, then by Theorem 8, each component of H is Class 1 and so H is Class 1, a contradiction. Thus H contains at least one component, say Q, whose core is a disjoint union of cycles. If Q_Δ contains exactly two cycles, then by (21) $Q = H$. Thus H is connected, a contradiction. Therefore Q_Δ is a cycle.

Let $R = H - Q$. Clearly, since $|G|$ is even, $|Q| \equiv |R| \pmod 2$. First assume that Q has even order. Then by induction hypothesis Q is Class 1. Moreover, if the core of R is not a cycle, then by Theorem 7, R is Class 1. If the core of R is a cycle, then R is connected, and since $|R|$ is even, by induction hypothesis R is Class 1, and so is H, a contradiction. Therefore we may assume that both Q and R have odd orders. Since H is Class 2 and by the fact that if the core of R is not a cycle, then R is Class 1, we may assume that Q is Class 2.

Let $C_k = Q_\Delta$ be a cycle of order $k \in \{3, 4, 5\}$. We need four following claims.

Claim 3. $|Q| = \Delta - 3 + k$.

Let $|Q| = 2h+1$. Since Q is Class 2 and $\Delta(Q) = \Delta - 1 \geq 3$, by Theorems 8 and 10, Q is critical and 2-edge connected. Moreover, if $Q_\Delta = C_5$, then $|Q| \geq 7$. Since $Q_\Delta = C_k$, $k \in \{3, 4, 5\}$, it follows from Theorems 4, 5 and 8 that

$$\frac{k(\Delta - 1) + (2h + 1 - k)(\Delta - 2)}{2} = |E(Q)| \geq h(\Delta - 1) + 1.$$

Thus $|Q| = 2h + 1 \leq \Delta - 3 + k$. On the other hand,

$$|Q| \geq |C_k| + |N_Q(x) \cap V(Q - C_k)| = k + \Delta - 3 \quad \text{for every} \;\; x \in V(C_k)$$

since $Q_\Delta = C_k$ and $\Delta(Q) = \Delta - 1$. Thus $|Q| = \Delta - 3 + k$ and $N_Q(x) \supseteq V(Q) - V(C_k)$ for every $x \in V(C_k)$. Therefore the claim is proved, and the following (22) holds.

$$xy \in E(Q) \quad \text{for every} \;\; x \in V(Q_\Delta) \;\; \text{and} \;\; y \in V(Q) - V(Q_\Delta). \tag{22}$$

Let $F = \{u_1v_1, \ldots, u_tv_t\}$ be the set of those edges of M such that $u_i \in V(Q)$ and $v_i \in V(R)$ for every $1 \leq i \leq t$. We show that $V(Q_\Delta) \subseteq \{u_1, \ldots, u_t\}$. To the contrary, let $x \in V(Q_\Delta) \setminus \{u_1, \ldots, u_t\}$. Since M covers all vertices of G, there exists a vertex $y \in V(Q) - \{u_1, \ldots, u_t\}$ such that $xy \in M$. If $y \in V(Q_\Delta)$, then since $x \in V(Q_\Delta)$, Q_Δ is not a cycle, a contradiction. If $y \notin V(Q_\Delta)$, then $xy \in M$ contradicts (22). Since $Q_\Delta = C_k$, without loss of generality, we may assume that

$$V(Q_\Delta) = \{u_1, \ldots, u_k\} \subseteq \{u_1, \ldots, u_t\},$$

$$\text{where} \;\; u_iu_{i+1} \in E(Q_\Delta) \;\; \text{for all} \;\; 1 \leq i \leq k-1 \;\; \text{and} \;\; u_ku_1 \in E(Q_\Delta). \tag{23}$$

Moreover, since G_Δ is an induced subgraph of G and $Q_\Delta = C_k$, we have

$$u_iv_i \notin E(G_\Delta) \quad \text{for} \;\; i = 1, \ldots, t, \tag{24}$$

and

$$V(R_\Delta) \cap \{v_1, \ldots, v_k\} = \emptyset. \tag{25}$$

Now, we want to give a lower bound for $t = |F|$. First note that if $|F| \leq \Delta - 2$, then by Theorem 10, G is Class 1. Now, suppose that $|F| = \Delta - 1$. Let $Q' = G - R$ and $R' = G - Q$. Add a new vertex w_1 and join w_1 to each u_i, $1 \leq i \leq t$, and denote the resultant graph by Q''. Also, do the same thing for R' with a new vertex w_2, and denote the resultant graph by R''. Since $|G| > |R''|, |Q''|$ and $\Delta(G) \geq \Delta(R''), \Delta(Q'')$, by the induction hypothesis both Q'' and R'' have a Δ-edge coloring with colors $\{1, \ldots, \Delta\}$. By a suitable permutation of colors, one may assume that $c(w_1u_i) = c(w_2v_i) = i$ for $i = 1, \ldots, \Delta - 1$, where $c(e)$

denotes the color of e. Then by assigning color i to each edge u_iv_i, $i = 1, \ldots, \Delta - 1$, we obtain a Δ-edge coloring of G and so G is Class 1.

Hence we can assume that $|F| \geq \Delta$. Now, since $|Q| = \Delta - 3 + k$ and $k \leq 5$, we have $|Q| \leq \Delta + 2$. This implies that

$$\Delta \leq |F| \leq \Delta + 2. \tag{26}$$

By (22) and since $\delta(Q) = \Delta - 2$, for every $y \in V(Q) - V(Q_\Delta)$, we have $\Delta - 2 \geq d_Q(y) \geq k$, which implies

$$\Delta \geq k + 2. \tag{27}$$

Now, we want to prove the following claim.

Claim 4. *If $\{u_iu_j, v_iv_j\} \subseteq E(G)$ for some $i, j \in \{1, \ldots, t\}$, then G is Class 1.*

Consider $M' = (M - \{u_iv_i, u_jv_j\}) \cup \{u_iu_j, v_iv_j\}$. Let $Q' = Q - \{u_iu_j\}$ and $R' = R - \{v_iv_j\}$. We claim that $G' = G - M'$ is Class 1. We show that there exists a path which joins a vertex of Q'_Δ to a vertex of R'_Δ in G'. First note that since Q is Class 2, by Theorems 8 and 9, every $v \in V(Q)$ satisfies $|N_{Q_\Delta}(v)| \geq 2$. Thus, $|N_{Q'_\Delta}(u_i)| \geq 1$ and $|N_{Q'_\Delta}(u_j)| \geq 1$. Moreover, by (21), $|N_{R_\Delta}(v)| \geq 1$ for every $v \in V(R)$. Now, if $v_j \notin V(R_\Delta)$, then since $|N_{R_\Delta}(v_i)| \geq 1$, $|N_{R'_\Delta}(v_i)| \geq 1$ which implies that there exists a path which joins a vertex of Q'_Δ to a vertex of R'_Δ in G'. If $v_j \in V(R_\Delta)$, then there exists a path which joins v_j to a vertex of Q'_Δ in G'.

If R_Δ is a cycle, then G' is connected and by induction hypothesis, G' is Class 1 and so G is Class 1. Otherwise, for every component K of G', $\delta(K_\Delta) = 1$ and $\Delta(K_\Delta) \leq 2$. Thus by Theorem 8, G' is Class 1, so is G and the claim is proved.

Now, two cases may be occurred. First suppose that Q and R are Class 2. Then by (3) and since Q_Δ is a cycle, we can suppose that $R_\Delta = C_r$, for $r = 3, 4, 5$. So, similar to the proof of Claim 3, $|R| = \Delta - 3 + r$. Now, similar to (23) and with no loss of generality, one can assume that $v_t \in V(R_\Delta)$ and so by (24), $u_t \notin V(Q_\Delta)$ and $v_1 \notin V(R_\Delta)$ and so $u_1u_t \in E(Q)$ and $v_1v_t \in E(R)$, by (22). By Claim 4, G is Class 1 and we are done.

Next, assume that Q is Class 2 and R is Class 1. First we prove the following claim.

Claim 5. *If $|N_{R_\Delta}(v_i)| + |N_{R_\Delta}(v_{i+1})| \leq 3$ for some $1 \leq i \leq k \pmod k$, then G is Class 1.*

Without loss of generality, suppose that $|N_{R_\Delta}(v_1)| + |N_{R_\Delta}(v_2)| \leq 3$. First note that if $v_1v_2 \in E(G)$, then by Claim 4, G is Class 1 and we are done. So, suppose that $v_1v_2 \notin E(G)$. By (1) and assumptions, we can assume that $|N_{R_\Delta}(v_1)| = 1$ and $|N_{R_\Delta}(v_2)| \leq 2$. Let $N_{R_\Delta}(v_1) = \{x\}$. Now, consider $Q - \{u_1u_2\}$, add a new vertex w_1 and join w_1 to u_1 and u_2. Then call the resultant graph by Q'. Clearly, $\Delta(Q') = \Delta(Q) = \Delta - 1$. Note that since $\Delta \geq 4$, Q'_Δ is a path and by Theorem 7, Q' has a $(\Delta - 1)$-edge coloring with colors $\{1, \ldots, \Delta - 1\}$. Moreover, we can assume that $c(w_1u_1) = 1$ and $c(w_1u_2) = 2$.

Now, add a new vertex w_2 to R, join w_2 to v_1 and v_2 and call the resultant graph by R'. By (25), $V(R_\Delta) \cap \{v_1, v_2\} = \emptyset$ and so $\Delta(R') = \Delta(R) = \Delta - 1$. We claim that R' is Class 1. Let $R'' = R' - \{v_1\}$. Thus $d_{R''}(w_2) = 1$ and $d_{R''}(x) = \Delta - 2$ which implies that $x \notin V(R''_\Delta)$. We claim that every component K of R'' is Class 1 and so is R''. If $\delta(K_\Delta) \leq 1$, then by Theorem 11, K is Class 1. If K_Δ is a cycle, then clearly $w_2 \in V(K)$. Now, by Theorem 8 and since $1 = \delta(K) < \Delta(K) - 1$, K is Class 1. This implies that R'' is Class 1. Now, by Theorem 6, since $d_R(v_1) = \Delta - 1$ and $d_R(x) = \Delta - 1$ and R'' is Class 1, R' has a $(\Delta - 1)$-edge coloring with colors $\{1, \ldots, \Delta - 1\}$. Moreover, we can assume that $c(w_2v_1) = 1$ and $c(w_2v_2) = 2$. Now, color u_1v_1 and u_2v_2 by 1 and 2, respectively and then color every edge $f \in (F - \{u_1v_1, u_2v_2\}) \cup \{u_1u_2\}$ by Δ to obtain a Δ-edge coloring of G and the claim is proved.

So, we can assume that

$$|N_{R_\Delta}(v_i)| + |N_{R_\Delta}(v_{i+1})| \geq 4 \ \text{ for } \ i = 1, \ldots, k \ (\text{mod } k). \tag{28}$$

This implies that

$$\sum_{i=1}^{k} |N_{R_\Delta}(v_i)| \geq 2k.$$

Moreover, since $V(G_\Delta) \cap \{u_{k+1}, \ldots, u_t\} = \emptyset$, (1) yields that $|N_{R_\Delta}(v_i)| \geq 2$ for $i = k + 1, \ldots, t$. This implies that

$$\sum_{i=1}^{t} |N_{R_\Delta}(v_i)| \geq 2t. \tag{29}$$

Now, we want to prove the following claim. Let $L = R - \{v_1, \ldots, v_t\}$.

Claim 6. *Let $u_iu_j \in E(G)$ for some $i, j \in \{1, \ldots, t\}$ and $xy \in M \cap E(L)$. If $v_ix, v_jy \in E(G)$, then G is Class 1.*

Consider $M' = (M - \{u_iv_i, u_jv_j, xy\}) \cup \{u_iu_j, v_ix, v_jy\}$. Let $G' = G - M'$. Now, remove two edges v_ix and v_jy of R and add xy to the edges of R and call the resultant graph by R'. By (28) and with no loss of generality, one can assume that $|N_{R_\Delta}(v_i)| \geq 2$. This implies that v_i is adjacent to at least one vertex of R'_Δ. Also, since Q is Class 2, by Theorems 8 and 9, $|N_{Q'_\Delta}(u_i)| \geq 1$, where $Q' = Q - \{u_iu_j\}$. Thus there exists a path which joins one vertex of Q'_Δ to a vertex of R'_Δ. Now, if G' is connected, then by induction hypothesis, G' is Class 1 and so G is Class 1. Otherwise, since there exists a path which joins one vertex of Q'_Δ to a vertex of R'_Δ, for every component K of G', $\delta(K_\Delta) \leq 1$ and $\Delta(K_\Delta) \leq 2$. Thus by Theorem 8, K is Class 1 and so is G'. This implies that G is Class 1 and the claim is proved.

By (23), $V(Q_\Delta) \cap \{u_1, \dots, u_t\} = \{u_1, \dots, u_k\}$, where $k = 3, 4, 5$. Now, by (22), $u_iu_j \in E(Q)$ for $i = 1, \dots, k$ and $j = k+1, \dots, t$. Note that $d_Q(u_i) = \Delta - 1$ and $d_Q(u_j) = \Delta - 2$ for $i = 1, \dots, k$ and $j = k+1, \dots, t$, respectively. Now, by Claim 3, u_i is not adjacent to exactly $k-3$ vertices in the set $\{u_1, \dots, u_t\}$ for $i = 1, \dots, k$. Moreover, u_j is not adjacent to at most $k-2$ vertices in the set $\{u_1, \dots, u_t\}$ for $j = k+1, \dots, t$. Note that if $\{u_iu_j, v_iv_j\} \subseteq E(G)$, for some $i, j \in \{1, \dots, t\}$, then by Claim 4, G is Class 1 and we are done. Thus, we can suppose that for $k = 3, 4, 5$,

$$|N_R(v_i) \cap \{v_1, \dots, v_t\}| \leq k - 3 \quad \text{for} \quad i = 1, \dots, k,$$

$$|N_R(v_j) \cap \{v_1, \dots, v_t\}| \leq k - 2 \quad \text{for} \quad j = k+1, \dots, t.$$

Since $d_R(v_i) \geq \Delta - 2$ for $i = 1, \dots, t$, we conclude that for $k = 3, 4, 5$,

$$e_R(v_i, L) \geq \Delta - k + 1 \quad \text{for} \quad i = 1, \dots, k. \tag{30}$$

$$e_R(v_j, L) \geq \Delta - k \quad \text{for} \quad j = k+1, \dots, t. \tag{31}$$

Now, two cases may be occurred:

First suppose that $|L| \leq 2\Delta - 2k + 2$. Let $M \cap E(L) = \{x_1y_1, \dots, x_my_m\}$. Thus $m \leq \Delta - k + 1$. With no loss of generality, suppose that

$$N_R(v_1) \cap V(L) = \{x_1, \dots, x_{s+t}, y_1, \dots, y_s\}.$$

Thus by (30),

$$2s + t \geq \Delta - k + 1 \quad \text{for some} \quad s, t. \tag{32}$$

Now, if

$$\{x_1, \dots, x_s, y_1, \dots, y_{s+t}\} \cap (N_R(v_2) \cap V(L)) \neq \emptyset,$$

then since $u_1u_2 \in E(Q)$ by Claim 6, we are done. So, we can suppose that

$$N_R(v_2) \cap V(L) \subseteq \{x_{s+1}, \dots, x_m, y_{s+t+1}, \dots, y_m\}.$$

Thus by (30), (32) and since $|L| \leq 2\Delta - 2k + 2$,

$$\begin{aligned} |N_R(v_2) \cap V(L)| &\leq 2\Delta - 2k + 2 - (2s + t) \\ &\leq 2\Delta - 2k + 2 - (\Delta - k + 1) \\ &= \Delta - k + 1. \end{aligned}$$

So, by (30),

$$N_R(v_2) \cap V(L) = \{x_{s+1}, \ldots, x_m, y_{s+t+1}, \ldots, y_m\}$$

and $|L| = 2\Delta - 2k + 2$. Now, if

$$\{x_{s+t+1}, \ldots, x_m, y_{s+1}, \ldots, y_m\} \cap (N_R(v_3) \cap V(L)) \neq \emptyset,$$

then since $u_2u_3 \in E(Q)$ by Claim 6, we are done. So, by a similar argument as we did for v_2, we conclude that

$$N_R(v_3) \cap V(L) = \{x_1, \ldots, x_{s+t}, y_1, \ldots, y_s\}.$$

Now, we do this procedure for v_i, $i \leq k$ and so

$$\begin{cases} N(v_{k+1}) \subseteq N(v_1) & \text{if } k \text{ is even} \\ N(v_{k+1}) \subseteq N(v_2) & \text{if } k \text{ is odd.} \end{cases}$$

Now, if $s \geq 1$, then with no loss of generality one may assume that there exists an edge x_iy_i for some $i = 1, \ldots, t$ such that

$$\begin{cases} \{v_1x_i, v_{k+1}y_i\} \subseteq E(Q) & \text{if } k \text{ is even} \\ \{v_2x_i, v_{k+1}y_i\} \subseteq E(Q) & \text{if } k \text{ is odd.} \end{cases}$$

Moreover, by (22), $\{u_1u_{k+1}, u_2u_{k+1}\} \subseteq E(Q)$ and so by Claim 6, G is Class 1. Thus we can suppose that $s = 0$ and so

$$N(v_i) \subseteq \{x_1, \ldots, x_m\} \text{ for } i = 1, \ldots, t.$$

Now, by pigeonhole principle, (26), (30) and (31), for some $i = 1, \ldots, t$,

$$d_R(x_i) \geq \frac{k(\Delta - k + 1) + (\Delta - k)^2}{\Delta - k + 1}.$$

Now, by (27), $d_R(x_i) > \Delta - 1$, a contradiction.

Now, suppose that $|L| > 2\Delta - 2k + 2$. Note that since M is a 1-factor, L has even order. Thus we can suppose that

$$|L| \geq 2\Delta - 2k + 4. \tag{33}$$

By (26), let $|F| = \Delta + i$, where $i = 0, 1, 2$. Therefore we find

$$|R| \geq 3\Delta - 2k + 4 + i. \tag{34}$$

Now, we want to determine an upper bound for $|R|$. Suppose that $|R_\Delta| = r$. Let X be the set of those vertices of $L - R_\Delta$ such that $|N_{R_\Delta}(x)| = 1$. So, for every $y \in L - (X \cup R_\Delta)$, $|N_{R_\Delta}(y)| \geq 2$. Note that since G_Δ is a disjoint union of cycles, the minimum degree of the core of every component of H_Δ is at least 1. Thus, for every $w \in V(R_\Delta)$, since $d_R(w) = \Delta - 1$, $e_R(w, R - R_\Delta) \leq \Delta - 2$. Moreover, let $N_{G_\Delta}(x) = \{v_x, w_x\}$ such that $N_{R_\Delta}(x) = \{v_x\}$. Clearly, $|X| = |\{w_x \,|\, x \in X\}|$ and so $e_R(w_x, R - R_\Delta) \leq \Delta - 3$. Let $|V(R_\Delta) \cap \{v_1, \ldots, v_t\}| = d$. Now, since $V(R_\Delta) \cap \{v_1, \ldots, v_k\} = \emptyset$, by (28), (29) we find that

$$\begin{aligned}
& 2(t-d) + |X| + 2(|R| - (t + |X| + r - d)) \\
\leq\ & e_R(R_\Delta, R - R_\Delta) \\
\leq\ & |X|(\Delta - 3) + (r - |X|)(\Delta - 2).
\end{aligned}$$

This implies that

$$|R| \leq \frac{r\Delta}{2}. \tag{35}$$

Now, by (34),

$$3\Delta - 2k + 4 + i \leq \frac{r\Delta}{2}.$$

Since $r \in \{3, 4, 5\}$, this implies that

$$\Delta \leq \frac{4k - 8 - 2i}{6 - r}. \tag{36}$$

Now, three cases can be considered:

(i) $r = 3$. Since G_Δ has even order, $k \in \{3, 5\}$. So, by Claim 3 and since $|Q|$ is odd, Δ is odd. Now, by (36), $\Delta \leq 4$. Thus $\Delta = 4$, a contradiction.

(ii) $r = 4$. Since G_Δ has even order, $k = 4$. Moreover, by Claim 3, $|Q| = \Delta + 1$ and so $i = 1$. Thus, by (36) we conclude that $\Delta \leq 3$, a contradiction.

(iii) $r = 5$. Since G_Δ has even order and $|G_\Delta| \leq 8$, $k = 3$. Moreover, by Claim 3 and since $|Q|$ is odd, Δ is odd. Now, by (36), $\Delta \leq 3$, a contradiction and the proof is complete.

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