Balanced line for a 3-colored point set in the plane

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Abstract

In this note we prove the following theorem. For any three sets of points in the plane, each of $n \geq 2$ points such that any three points (from the union of three sets) are not collinear and the convex hull of $3n$ points is monochromatic, there exists an integer $k \in \{1, 2, \ldots, n-1\}$ and an open half-plane containing exactly k points from each set.

1 Introduction

Bisecting two finite sets of points in the plane by a line is a simple exercise. The existence of such a line follows from the discrete version of the classical ham-sandwich theorem [2] that states that, for any d finite point sets $S_1, S_2, \ldots, S_d$ in $\mathbb{R}^d$, there exists a hyperplane h such that each open half-space defined by h contains at most half of points of each set S_i.

A short survey related to this paper is found in [1]. Another variation of the problem is about balanced lines [3, 4]. A set of points in the plane is *in general position* if any three points are not collinear. Given a set of n black and n white points in general position in the plane, a line l is said to be *balanced* if each open half-plane bounded by l contains precisely the same number of black points as white points. Our definition of balanced line is slightly different from [3] since we do require the line to pass through two points of the sets. Pach and Pinchasi [3] proved that the number of balanced lines is at least n answering the question of George Baloglou.

Sharir and Welzl [4] found that balanced lines in the plane are related to halving triangles in $\mathbb{R}^3$. Let P be a set of $2n+1$ points in $\mathbb{R}^3$ in general position, i.e. no four points are coplanar. A *halving triangle of* P is a triangle

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spanned by three points in P such that the plane containing the three points bisects the remaining points of P (i.e. an open half-space bounded by the plane contains exactly $n-1$ points of P). They proved that the number of halving triangles is at least n^2. This bound is tight since points in convex position have exactly n^2 halving triangles.

In this note we study balanced lines for three point sets. Let $S = R \cup B \cup G$ be a set of $3n$ points in the plane in general position such that $|R| = |B| = |G| = n \geq 2$ (red, blue and green points). A line l is called *balanced* if an open half-plane bounded by l contains exactly k red, k blue and k green points for some $k \in \{1, 2, \ldots, n-1\}$. Unfortunately, a balanced line does not always exist, see an example in Figure 1 (b). To develop an intuition we check points on the line first.

It is known that if n red points and n blue points lie on a line in general position (i.e., no two points lie on the same position) and if the two end points have the same color, then there exists a balanced point.

Proposition 1 *Assume that n red points and n blue points are given on the line and no two points lie on the same position, where n is a positive integer. If both endpoints are red, then the line can be divided into two parts, the right part I_1 and the left part I_2, by a point so that I_1 contains k red points and k blue points for some $1 \leq k \leq n-1$.*

Remark. Notice that the condition of Proposition 1 that both endpoints are the same color is necessary. For example, a configuration *rrrbrbrbbb*, where r and b denote a red point and a blue point, respectively, has no balanced point given in Proposition 1.

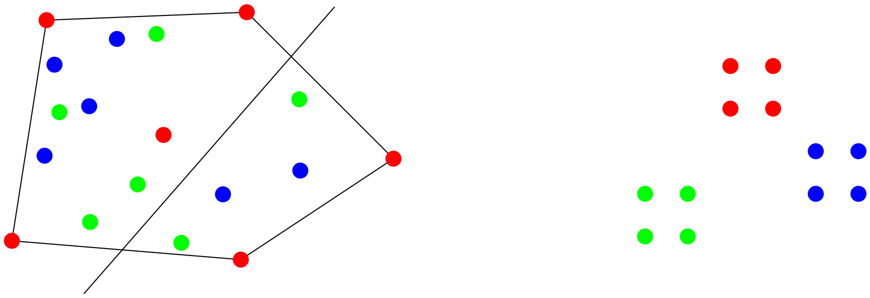
We will prove that a balanced line for points in the plane exists if the convex hull of S is monochromatic.

Theorem 2 *Let S be a set of $3n \geq 6$ points in the plane in general position colored in red/blue/green such that*
(i) the number of points of each color is n, and
(ii) the vertices of the convex hull have the same color.
Then there exists a balanced line of S.

2 Existence of a Balanced Line

In this Section we prove Theorem 2.

Proof. Let d be a direction such that any two points of S have different projections on a line with slope d. Let $p_1, \ldots, p_{3n}$ be the order of points in



(a) (b)

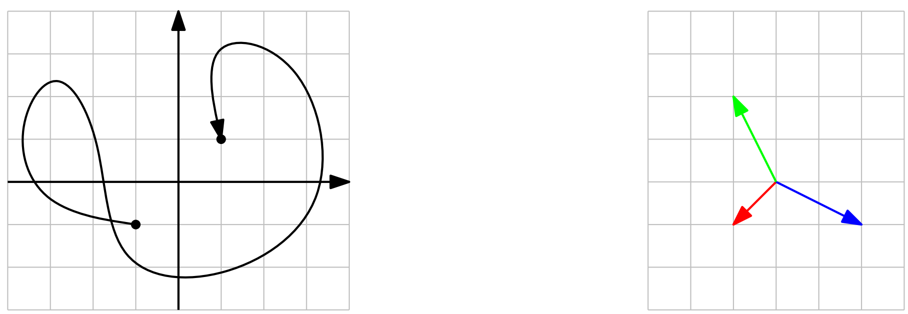
Figure 1: (a) Balanced line in a set of 18 points such that the convex hull is monochromatic. (b) A set of 12 points with non-monochromatic convex hull such that a balanced line does not exist.

direction d. For every k, let r_k, b_k, g_k be the number of red/blue/green points in $\{p_1, \dots, p_k\}$, respectively. Consider point $q_k = (3b_k - k, 3g_k - k)$. Note that $q_k \neq (0,0)$ if k is not multiple of 3. The theorem follows if $q_k = (0,0)$ for some $k = 3, 6, \dots, 3(n-1)$. Suppose to the contrary that $q_k \neq (0,0)$ for any k and any direction d.

Consider path $\phi_d = q_1 q_2 \dots q_{3n-1}$. By the definition $q_1 = (-1,-1)$ and $q_{3n-1} = (1,1)$, see Figure 2 (a). There are three types of vectors $\overrightarrow{q_{k-1}q_k}$ depending on the color of p_k, see Figure 2 (b). Note that the segments $q_{k-1}q_k$ do not contain grid points except the endpoints. Therefore path ϕ_d does not contain the origin. If we trace vector $\overrightarrow{0a}$ where a traverses path ϕ_d the *turning angle* of a, defined as $\sum_{i=1}^{3n-2} \angle q_i O q_{i+1}$, will be $t\pi$ where t is an odd integer.

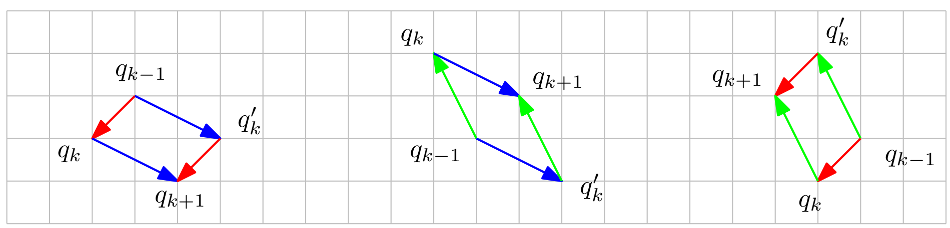
We show that the turning angle of ϕ_d does not change with d. It suffices to consider a flip of two points p_k and p_{k+1} when d changes. Suppose that p_k is red and p_{k+1} is blue. Then path $q_{k-1}q_kq_{k+1}$ changes to $q_{k-1}q'_kq_{k+1}$ as shown in Figure 3 (a). We show that parallelogram $q_{k-1}q_kq_{k+1}q'_k$ does not contain the origin. Suppose to the contrary that it contains the origin. Then $y(q_k) = 0$ and $3g_k = k$ and $k \equiv 0 \mod 3$. On the other hand $x(q_k) = 3b_k - k \in \{-1, -2\}$ contradicting $k \equiv 0 \mod 3$. The case, where p_k is blue and p_{k+1} is red, is symmetric.

Similarly, we can show that parallelogram $q_{k-1}q_kq_{k+1}q'_k$ does not contain the origin if p_k and p_{k+1} have different colors, see Figure 2 (b) and (c). Note that ϕ_{-d} is symmetric to ϕ_d and its turning angle is $-t\pi$. This contradicts



(a) (b)

Figure 2: (a) Path ϕ_d with turning angle π. (b) Vectors $q_{k-1}q_k$ depending on the color of p_k.



(a) (b) (c)

Figure 3: Flipping p_k and p_{k+1}. Path $q_{k-1}q_kq_{k+1}$ changes to $q_{k-1}q'_kq_{k+1}$. (a) p_k is red and p_{k+1} is blue. (b) p_k is green and p_{k+1} is blue. (c) p_k is red and p_{k+1} is green.

the fact that the turning angle ϕ_d does not change under rotation of d. ■

We finally note that the condition that the numbers of red, blue and green points are equal in Theorem 2 is also necessary. It is easy to make an example with distinct number of points of each color that does not admit a balanced line. It is also natural to change the definition of balanced line in this case. For an red points, bn blue points and cn green points are given in the plane in general position, a line l is called *balanced* if an open half-plane bounded by l contains exactly ak red points and bk blue points and ck green points for some $k \in \{1, 2, \ldots, n-1\}$. For example, the configuration of points shown in Figure 4 has no such balanced line.

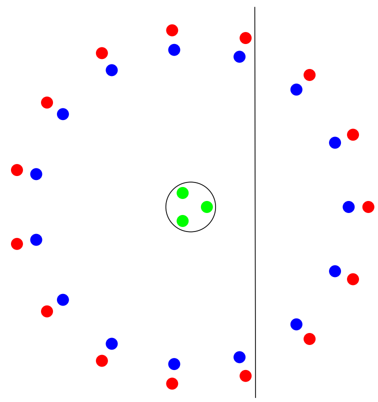


Figure 4: Example of 15 red, 15 blue and 3 green points without balanced line. Any line cutting off 5 red points does not intersect the circle enclosing green points.

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