Spanning trees with a bounded number of leaves in a claw-free graph

Mikio Kano[1], Aung Kyaw[2,*], Haruhide Matsuda[3], Kenta Ozeki[4], Akira Saito[5] and Tomoki Yamashita[6]

[1] Department of Computer and Information Sciences
Ibaraki University, Hitachi, Ibaraki, 316-8511, Japan

[2] Department of Mathematics
East Yangon University, Yangon, Myanmar

[3] Department of Mathematics,
Shibaura Institute of Technology,
Fukasaku, Minuma-ku, Saitama 337-8570, Japan

[4] National Institute of Informatics,
Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan

[5] Department of Computer Science and System Analysis
Nihon University,
Sakurajosui, Setagaya-Ku, Tokyo, 156-8550, Japan

[6] College of Liberal Arts and Sciences,
Kitasato University,
Kitasato, Minami-ku, Sagamihara 252-0373, Japan

*This paper was written while the second author is on a Matsumae International Foundation research fellowship at the Department of Computer and Information Sciences, Ibaraki University. Hospitality and financial support are gratefully acknowledged.

Abstract

For a graph H and an integer $k \geq 2$, let $\sigma_k(H)$ denote the minimum degree sum of k independent vertices of H. We prove that if a connected claw-free graph G satisfies $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning tree with at most k leaves. We also show that the bound $|G| - k$ is sharp and discuss the maximum degree of the required spanning trees.

Keywords: spanning tree, leaf, degree sum, claw-free graphs

1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, we consider only simple graphs, which have neither loops nor multiple edges. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. A vertex of degree one is called an *end-vertex*, and an end-vertex of a tree is usually called a *leaf*. A vertex set S of G is called *independent* if no two vertices of S are adjacent in G. The minimum degree sum of k independent vertices of G is denoted by $\sigma_k(G)$, that is, if G has k independent vertices, let

$$\sigma_k(G) = \min_S \left\{ \sum_{x \in S} \deg_G(x) \ : S \text{ is an independent set of } G \text{ with } k \text{ vertices} \right\}.$$

If G does not have k independent vertices, we define $\sigma_k(G) = +\infty$. The connectivity, the independence number and the minimum degree of G are denoted by $\kappa(G)$, $\alpha(G)$ and $\delta(G)$, respectively. The complete graph of order n is denoted by K_n. The complete bipartite graph with bipartition (X, Y), where $|X| = m$ and $|Y| = n$, is denote by $K_{m,n}$. A graph G is said to be *claw-free* if it contains no $K_{1,3}$ as an induced subgraph.

By Dirac's Theorem, every graph G of order at least three with $\delta(G) \geq \frac{1}{2}|G|$ has a hamiltonian cycle. As an immediate corollary, we can prove that every graph G with $\delta(G) \geq \frac{1}{2}(|G| - 1)$ has a hamiltonian path. For general graphs, the bound $\frac{1}{2}(|G| - 1)$ is sharp. For example, for a positive integer m, the complete bipartite graph $G = K_{m,m+2}$ satisfies $\delta(G) =$

$m = \frac{1}{2}(|G| - 2)$, but G has no hamiltonian path. However, Matthews and Sumner [5] proved that if we restrict ourselves to the class of claw-free graphs, a considerably smaller bound on minimum degree guarantees the existence of a hamiltonian path.

Theorem 1 (Matthews and Sumner [5]) *Let G be a connected claw-free graph. If $\delta(G) \geq \big(|G| - 2\big)/3$, then G has a hamiltonian path.*

Ore's Theorem states that every graph of order at least three with $\sigma_2(G) \geq |G|$ has a hamiltonian cycle. It extends Dirac's Theorem, and implies as a corollary that every graph G with $\sigma_2(G) \geq |G| - 1$ has a hamiltonian path.

A path of order at least two can be interpreted as a tree having exactly two leaves. From this point of view, a hamiltonian path of a graph of order at least two is a spanning tree with exactly two leaves. This interpretation may lead us to consider a spanning tree with a bounded number of leaves. Actually, Broersma and Tuinstra [1] gave a sufficient condition for a connected graph to have such a spanning tree.

Theorem 2 (Broersma and Tuinstra [1]) *Let $k \geq 2$ be an integer and let G be a connected graph of order at least two. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning tree with at most k leaves.*

The previous corollary of Ore's Theorem corresponds to the case $k = 2$ of the above theorem.

Broersma and Tuinstra also proved that the bound $|G| - k + 1$ of $\sigma_2(G)$ is sharp. However, in view of Theorem 1, for claw-free graphs, a much weaker condition may yield the same conclusion as in Theorem 2. Motivated by this observation, we study a degree sum condition for a claw-free graph to have a spanning tree with a bounded number of leaves, and give the following theorem.

Theorem 3 *Let $k \geq 2$ be an integer and let G be a connected claw-free graph. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning tree with at most k leaves.*

Note that Theorem 1 is a corollary of the case $k = 2$ of the above theorem.

In the next section, we prove the above theorem. In Section 3, we investigate the maximum degree of a spanning tree and prove that under the same assumption as in Theorem 3, G has a spanning tree of maximum degree at most three with at most k leaves. In Section 4, we give concluding remarks.

Before proving Theorem 3, we first show that the bound $|G| - k$ of $\sigma_{k+1}(G)$ is sharp. Consider a graph G constructed from one complete graph K_{k+1} and $k+1$ complete graphs K_m, $m \geq 2$, by identifying one vertex of each K_m with one distinct vertex of K_{k+1} (see Figure 1). Then G is claw-free and satisfies $\sigma_{k+1}(G) = |G| - k - 1$, but G has no spanning tree with at most k leaves.

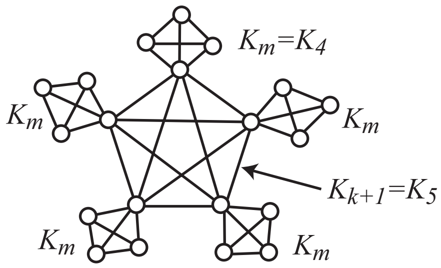

Figure 1: A connected claw-free graph G that has no spanning tree with at most k leaves and satisfies $\sigma_{k+1}(G) = |G| - k - 1$.

Some other results on spanning trees having at most k leaves can be found in [2] and [8].

2 Proof of Theorem 3

We begin with some additional notation. For a vertex v of a graph G, the neighborhood of v in G is denoted by $N_G(v)$. For a vertex set X of G, we write $N_G(X) = \bigcup_{x \in X} N_G(x)$, and the subgraph of G induced by X is denoted by $\langle X\rangle_G$. We write $G - X$ for $\langle V(G) - X\rangle_G$, and for a vertex v, $G - \{v\}$ is briefly denoted by $G - v$.

The graph constructed from two complete graphs K_m and K_n by identifying one vertex of K_m with one vertex of K_n is called a *double complete graph* and denoted by $DC(m, n)$, where $m, n \geq 2$. The common vertex of K_m and K_n is called the *center*, and the other vertices are called *non-central vertices* (See Figure 2). Note that the order of $DC(m, n)$ is $m + n - 1$, and the path of order three is a double complete graph $DC(2, 2)$. Let $\mathcal{D}$ denote the set of all double complete graphs.

When we consider a path or a cycle, we always assign an orientation. Let W be a path or a cycle, and let $v \in V(W)$. Then we denote by $v^{-(W)}$ and $v^{+(W)}$ the predecessor and the successor of W, respectively. We write $v^{--(W)}$ instead of $\big(v^{-(W)}\big)^{-(W)}$. For $A \subset V(W)$, let $A^{-(W)} = \{v^{-(W)} \colon v \in A\}$. If W is clear from the context, we often omit "(W)" and write v^-, v^+, v^{--} and A^- instead of $v^{-(W)}$, $v^{+(W)}$, $v^{--(W)}$ and $A^{-(W)}$, respectively. A path which starts at a vertex u and ends at a vertex v is called a uv-path. For a path P and vertices $u, v \in V(P)$, a subpath of P with ends u and v is denoted by $P(u, v)$. For subgraphs H_1 and H_2 of a graph G, we define $H_1 + H_2$ by $H_1 + H_2 = \big(V(H_1) \cup V(H_2), E(H_1) \cup E(H_2)\big)$. When we consider this operation, an edge is often considered as a subgraph isomorphic to K_2. For example, for $uv \in E(G)$, $H_1 + uv = \big(V(H_1) \cup \{u, v\}, E(H_1) \cup \{uv\}\big)$. For further explanation of terminologies and notation, we refer the reader to [9].

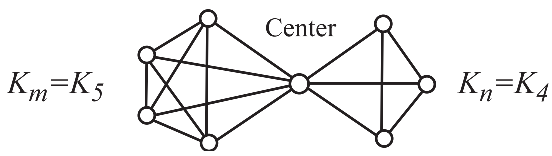

Figure 2: The double complete graph $DC(m, n)$, whose order is $m + n - 1$.

Enomoto [3], Jung [4] and Nara [6] implicitly characterized the connected graphs G such that G satisfies $\deg_G(x) + \deg_G(y) \geq |G| - 1$ for every pair of vertices x and y of G which are end-vertices of some hamiltonian path of G, but G has no hamiltonian cycle. The next lemma is a corollary of this characterization. We give its proof for the self-containedness of the paper.

Lemma 4 *Let G be a claw-free graph having a hamiltonian path. Suppose that* $\deg_G(x)+\deg_G(y) \geq |G|-1$ *for every pair of vertices x and y which are end-vertices of some hamiltonian path. Then G has a hamiltonian cycle, or G is a double complete graph.*

Proof. Assume G has no hamiltonian cycle. Let P be a hamiltonian path and let x and y be the first and the last vertices of P, respectively. By the assumption, $xy \notin E(G)$. If $N_G(x)^- \cap N_G(y) \neq \emptyset$, then $P(x,v) + vy + P(y,v^+) + v^+x$, where $v \in N_G(x)^- \cap N_G(y)$, is a hamiltonian cycle, a contradiction. Thus, $N_G(x)^- \cap N_G(y) = \emptyset$. Since $N_G(x)^- \cup N_G(y) \subset V(G) - \{y\}$ and $\left|N_G(x)^- \cup N_G(y)\right| = |N_G(x)^-| + |N_G(y)| = |N_G(x)| + |N_G(y)| = \deg_G(x) + \deg_G(y) \geq |G| - 1$, we have $N_G(x)^- \cup N_G(y) = V(G) - \{y\}$ and $\deg_G(x) + \deg_G(y) = |G| - 1$. On the other hand, since $N_G(x) \cup N_G(y) \subset V(G) - \{x,y\}$ and $\deg_G(x) + \deg_G(y) \geq |G| - 1$, we have $N_G(x) \cap N_G(y) \neq \emptyset$. We consider two cases.

Case 1. $|N_G(x) \cap N_G(y)| = 1$.

In this case, $N_G(x) \cup N_G(y) = V(G) - \{x,y\}$. Let $N_G(x) \cap N_G(y) = \{z\}$. Since $N_G(x)^- \cap N_G(y) = \emptyset$ and $N_G(x) \cup N_G(y) = V(G) - \{x,y\}$, $v \in N_G(x) - \{x^+\}$ implies $v^- \in N_G(x)$. This implies $P(x^+,z) \subset N_G(x)$. Similarly, $P(z,y^-) \subset N_G(y)$. Since $N_G(x) \cap N_G(y) = \{z\}$, we have $N_G(x) = P(x^+,z)$ and $N_G(y) = P(z,y^-)$.

Let $x_1 \in P(x^+,z^-)$. Then $x_1^+ \in N_G(x)$ and $P(x_1,x) + xx_1^+ + P(x_1^+,y)$ is a hamiltonian path of G. If $N_G(x_1) \cap P(z^+,y) \neq \emptyset$, then $P(x,x_1) + x_1y_1 + P(y_1,y) + yy_1^- + P(y_1^-,x_1^+) + x_1^+x$, where $y_1 \in N_G(x_1) \cap P(z^+,y)$, is a hamiltonian cycle of G, a contradiction. Therefore, $N_G(x_1) \subset P(x,z) - \{x_1\}$. Since $\deg_G(x_1) + \deg_G(y) \geq |G| - 1$ by the assumption, we have $N_G(x_1) \cap N_G(y) = \{z\}$. The we can apply the same argument as in the previous paragraph to x_1 and y, and obtain $N_G(x_1) = P(x,z) - \{x_1\}$. This implies that z is a cutvertex of G and $P(x,z)$ induces a complete graph. By symmetry, $P(z,y)$ also induces a complete graph. Therefore, G is a double complete graph.

Case 2. $|N_G(x) \cap N_G(y)| \geq 2$.

In this case, there exist $x_0 \in N_G(x)$ and $y_0 \in N_G(y)$ such that $x_0 \in P(y_0^+, y)$. Choose such x_0 and y_0 so that $P(y_0, x_0)$ is as short as possible. Since $N_G(x)^- \cap N_G(y) = \emptyset$, $y_0^+ \neq x_0$.

Since $xy \notin E(G)$ and $N_G(x)^- \cup N_G(y) = V(G) - \{y\}$, x_0^{--} exists and $x_0^{--} \in N_G(x)^- \cup N_G(y)$. Since $x_0^- \notin N_G(x)$ by the choice of (x_0, y_0), $x_0^{--} \in N_G(y)$. Again by the choice of (x_0, y_0), we have $y_0 = x_0^{--}$. Since $P(y_0^+, x) + xx_0 + P(x_0, y)$ and $P(y_0^+, y) + yy_0 + P(y_0, x)$ are both hamiltonian paths, we can apply the same argument as that for P to these paths, and obtain $\deg_G(y_0^+) + \deg_G(y) = \deg_G(y_0^+) + \deg_G(x) = \deg_G(y) + \deg_G(x) = |G| - 1$, which yields $\deg_G(x) = \deg_G(y) = \deg_G(y_0^+) = \frac{1}{2}(|G| - 1)$.

Let $C = P(x, y_0) + y_0 y + P(y, x_0) + x_0 x$. Then $V(C) = V(G) - \{y_0^+\}$. Let $C = v_0 v_1 \dots v_{|G|-2} v_0$. If y_0^+ is adjacent to a consecutive vertices of C, then we can insert y_0^+ to this cycle to obtain a hamiltonian cycle of G, contradicting the assumption. Since $\deg_G(y_0^+) = \frac{1}{2}\big(|G| - 1\big)$, y_0^+ is adjacent to every other vertex of C. Let $v_i \in N_G(y_0^+)$. Then $v_{i-2} \in N_G(y_0^+)$. Since $\{v_{i-1}, v_{i+1}, y_0^+\} \subset N_G(v_i)$ and G is claw-free, we have $v_{i-1}v_{i+1} \in E(G)$. Then by replacing $v_{i-2}v_{i-1}v_i v_{i+1}$ in C with $v_{i-2}y_0^+ v_i v_{i-1} v_{i+1}$, we have a hamiltonian cycle of G. This is a contradiction, and the lemma follows. □

Win [10] introduced a *k-ended system* to prove the existence of a spanning tree with at most k leaves. In this paper, we modify the definition of a k-ended system and define a *k-extended system*. It plays an important role in the proof of our main theorem.

Let G be a connected claw-free graph, and F be a subgraph of G. The set of components of F is denoted by $\mathcal{C}(F)$. We call F an *extended system* if each component of F is a path, a cycle or a double complete graph. For an extended system F, we define a mapping f from $\mathcal{C}(F)$ to $\{1, 2\}$ as follows.

For every $C \in \mathcal{C}(F)$,

$$f(C) = \begin{cases} 1 & \text{if } C \text{ is } K_1,\ K_2, \text{ a cycle or a double complete graph,} \\ 2 & \text{otherwise (i.e., a path of order at least four),} \end{cases}$$

and define

$$f(F) = \sum_{C \in \mathcal{C}(F)} f(C).$$

Let $\mathcal{C}_i(F) = \{C \in \mathcal{C}(F) : f(C) = i\}$ for $i = 1, 2$. An extended system F is called a *k-extended system* if $f(F) \leq k$.

The following lemma is an easy but important observation.

Lemma 5 *Let G be a claw-free graph and D be an induced double complete subgraph of G. If a vertex $v \in V(G) - V(D)$ is adjacent to the center of D, then v is also adjacent to a non-central vertex of D.*

Proof. Let D_1 and D_2 be the two blocks of D. Then both D_1 and D_2 are complete graphs. Let x be the center of D and let $x_i \in D_i - \{x\}$ $(i = 1,\ 2)$. Since D is an induced subgraph of G, $x_1x_2 \notin E(G)$. Since $\{x_1, x_2, v\} \subset N_G(x)$ and G is claw-free, $\{x_1v, x_2v\} \cap E(G) \neq \emptyset$. $\square$

The next lemma shows a relationship between a k-extended system and a spanning tree with at most k leaves in a claw-free graph.

Lemma 6 *Let $k \geq 2$ be an integer and G be a connected claw-free graph. If G has a spanning extended system F_0, then G has a spanning tree with at most $f(F_0)$ leaves. In particular, if G has a spanning k-extended system, then G has a spanning tree with at most k leaves.*

Proof. Take a spanning extended system F with $f(F) \leq f(F_0)$ so that the number of double complete graphs is as small as possible. Then every double complete graph of F is an induced subgraph of G since if two non-central vertices of a double complete graph D of F are joined by an edge e of G, then $D + e$ has a hamiltonian cycle, and so D should be replaced by this hamiltonian cycle.

Since G is connected, there exists a minimal set X of edges such that F together with X forms a connected spanning subgraph of G. We shall construct a spanning tree with at most k leaves consisting of F and X. By Lemma 5, we may assume that no edge in X is incident with the center of a double complete graph. For any double complete graph D of F, there exists an edge $e_D \in X$ incident with a vertex v_D of D, where v_D is not the center of D. Then D has a hamiltonian path starting at v_D, and we replace D with this hamiltonian path.

For any cycle C of F, there exists an edge $e_C \in X$ incident with a vertex v_C of C. Delete an edge of C incident with v_C. By repeating the above procedure for every double complete graph and every cycle of F, we obtain a spanning tree T. By the construction, for each $C \in \mathcal{C}(F)$, the number of leaves of T contained in C is at most $f(C)$.

Hence T has at most $f(F) \leq f(F_0)$ leaves. $\Box$

We call a k-extended system F of G a *maximal k-extended system* if G has no k-extended system F' such that $V(F)$ is a proper subset of $V(F')$. In order to prove our theorem, we need the following lemma.

Lemma 7 *Suppose that a graph G does not have a spanning k-extended system. Let F be a maximal $(k+1)$-extended system of G. Then G does not have a k-extended system F' with $V(F') = V(F)$. In particular, F is not a k-extended system, and so $f(F) = k+1$.*

Proof. Let F be a maximal $(k+1)$-extended system of G. Assume that G has a k-extended system F' with $V(F') = V(F)$. Since G does not have a spanning k-extended system, there exists a vertex $v \in V(G) - V(F')$, and thus G has a $(k+1)$-extended system $F' \cup \{v\}$, which contradicts the maximality of F. $\Box$

By Lemma 6, in order to prove our Theorem 3, it suffices to prove the following theorem.

Theorem 8 *Let $k \geq 2$ be an integer and G be a claw-free graph. If $\sigma_{k+1}(G) \geq |G| - k$, then G has a spanning k-extended system.*

Proof. Suppose that G has no spanning k-extended system. Take a maximal $(k+1)$-extended system F so that

(F1) $\sum_{P\in\mathcal{C}_2(F)}|P|$ is as large as possible,

(F2) The number of cycles in $\mathcal{C}_1(F)$ is as large as possible subject to (F1), and

(F3) $\sum_{P\in\mathcal{C}_2(F)}\big(\deg_{\langle V(P)\rangle_G}(x_P)+\deg_{\langle V(P)\rangle_G}(y_P)\big)$ is as small as possible, subject to (F1) and (F2), where x_P and y_P are the end-vertices of P.

By Lemma 7, $f(F)=k+1$. We begin with a simple but important observation.

Claim 1 *For each $D\in\mathcal{C}_1(F)$ and for each $v\in V(D)$ that is not the center of D if D is a double complete graph, D has a hamiltonian path containing v as one of its end-vertices.*

The next claim follows from the condition (F2) and the same argument as in the first paragraph of the proof of Lemma 6.

Claim 2 *Every double complete graph D of F is an induced subgraph of G.*

Next, we investigate the adjacency between the components of F.

Claim 3 *The following three statements hold.*
(i) *No two components of $\mathcal{C}_1(F)$ are connected by an edge of G.*
(ii) *No end-vertex of a path in $\mathcal{C}_2(F)$ is connected to a component of $\mathcal{C}_1(F)$ by an edge of G.*
(iii) *No two end-vertices of two distinct paths or of the same path in $\mathcal{C}_2(F)$ are joined by an edge of G*

Proof. (i) Assume that two components Q_1 and Q_2 of $\mathcal{C}_1(F)$ are joined by an edge e of G. By Lemma 5, we may assume that no end-vertex of e is the center of a double complete graph. So Q_1+e+Q_2 contains a hamiltonian path P_0. By replacing Q_1 and Q_2 of F by P_0, we obtain another maximal

$(k+1)$-extended system F' on $V(F)$. If $|P_0| \geq 4$ this contradicts the condition (F1). If $|P_0| \leq 3$, then $f(P_0) = 1$ and hence F' is a k-extended system, which contradicts Lemma 7.
(ii) If an end-vertex of a path $P \in \mathcal{C}_2(F)$ is joined to a component $Q \in \mathcal{C}_1(F)$ by an edge e of G, then by an argument similar to the one in (i), we see that $P + e + Q$ has a hamiltonian path. Thus, we can derive a contradiction by Lemma 7.
(iii) If two end-vertices of two paths or of the same path in $\mathcal{C}_2(F)$ are joined by an edge of G, then we can obtain a k-extended system with vertex set $V(F)$, which contradicts Lemma 7. $\square$

For every component $Q \in \mathcal{C}_1(F)$, we take one vertex x_Q from Q so that x_Q is a non-central vertex of Q if Q is a double complete graph. For every path $P \in \mathcal{C}_2(F)$, let x_P and y_P be the two end-vertices of P. Define $\mathrm{End}(F)$ by

$$\mathrm{End}(F) = \bigcup_{Q \in \mathcal{C}_1(F)} \{x_Q\} \ \cup \bigcup_{P \in \mathcal{C}_2(F)} \{x_P, y_P\}.$$

Then $|\mathrm{End}(F)| = f(F) = k+1$ by Lemma 7. Claim 3 and Lemma 5 yield the next two claims.

Claim 4 $\mathrm{End}(F)$ *is an independent set of* G.

Claim 5 *For every component* $Q \in \mathcal{C}_1(F)$ *of* F *and the vertex* $\{x_Q\} = \mathrm{End}(F) \cap V(Q)$, *it follows that*

$$\sum_{x \in \mathrm{End}(F)} \bigl|N_G(x) \cap V(Q)\bigr| = \bigl|N_G(x_Q) \cap V(Q)\bigr| \leq |Q| - 1 = |Q| - f(Q).$$

Now we measure the neighborhood of $\mathrm{End}(F)$ in a path of $\mathcal{C}_2(F)$.

Claim 6 *Let* P *be a path in* $\mathcal{C}_2(F)$. *Then for each distinct pair of vertices* z, w *in* $\mathrm{End}(F) - \{x_P, y_P\}$, *the following statements hold.*
(i) $N_G(z) \cap N_G(w) \cap V(P) = \emptyset$.
(ii) $N_G(x_P)^- \cap N_G(y_P) \cap V(P) = \emptyset$.
(iii) $N_G(z)^- \cap N_G(y_P) \cap V(P) = \emptyset$ *and* $N_G(z)^+ \cap N_G(x_P) \cap V(P) = \emptyset$.
(iv) $N_G(z) \cap N_G(x_P) \cap V(P) = \emptyset$.

Proof. Let Q and R be the components of F containing z and w, respectively.

(i) Suppose $N_G(z) \cap N_G(w) \cap V(P) \neq \emptyset$ and take a vertex $v \in N_G(z) \cap N_G(w) \cap V(P)$. Then $v \neq x_P, y_P$ by Claim 4. Since $\{z, w, v^-\} \subset N_G(v)$ and G is claw-free, $zv^- \in E(G)$ or $wv^- \in E(G)$. By symmetry, we may assume that $zv^- \in E(G)$. If $Q \neq R$, then replace P, Q, R of F by two hamiltonian paths Q' and R' in $P(x_P, v^-)+v^-z+Q$ and $P(y_P, v)+vw+R$, respectively. Then we obtain a new $(k+1)$-extended system F' on $V(F)$. If $f(Q') + f(R') < f(P) + f(Q) + f(R)$, then F' is a k-extended system, which contradicts Lemma 7. Thus, $f(Q') + f(R') \geq f(P) + f(Q) + f(R)$. This is possible only if $\{Q', R'\} \subset \mathcal{C}_2(F')$ and $\{Q, R\} \subset \mathcal{C}_1(F)$. However, this contradicts the condition (F1). If $Q = R$, then Q is a path whose end-vertices are z and w and $P(x_P, v^-) + v^-z + Q + wv + P(v, y_P)$ is a hamiltonian path of $\langle V(P) \cup V(Q)\rangle_G$, and by replacing P and Q with this path, we have a k-extended system on $V(F)$, contradicting Lemma 7.

(ii) If $N_G(x_P)^- \cap N_G(y_P) \cap V(P) \neq \emptyset$, then $\langle V(P)\rangle_G$ has a hamiltonian cycle, and so G has a k-extended system with vertex set $V(F)$, which contradicts Lemma 7.

(iii) By symmetry, it suffices to show that $N_G(z)^- \cap N_G(y_P) \cap V(P) = \emptyset$. Assume that there exists a vertex $v \in N_G(z)^- \cap N_G(y_P) \cap V(P)$. Then $P(x_P, v) + vy_P + P(y_P, v^+) + v^+z + Q$ has a hamiltonian path of $\langle V(P) \cup V(Q)\rangle_G$, and so by replacing P and Q of F with this path, we have a k-extended system on $V(F)$. This contradicts Lemma 7.

(iv) Suppose that there exists a vertex v in $N_G(z) \cap N_G(x_P) \cap V(P)$. Then $v \neq y_P$ by Claim 4. Since $\{v^+, x_p, z\} \subset N_G(v)$ and G is claw-free, we have $v^+z \in E(G)$ by (iii) and Claim 4. Suppose that Q is a path of order at least four. If $v \neq x_P^+$, then replace P and Q by the cycle $P(x_P, v) + vx_P$ and a hamiltonian path of $P(y_P, v^+) + v^+z + Q$. If $v = x_P^+$, replace P and Q with x_Pv and a hamiltonian path of $P(y_P, v^+)+v^+z+Q$. In either case, G has a k-extended system on $V(F)$, which contradicts Lemma 7.

Next suppose that Q is a cycle. Let us denote the two vertices of Q adjacent to z by z^- and z^+. Then since $\{v, z^-, z^+\} \subset N_G(z)$ and

G is claw-free, we may assume that $z^-v \in E(G)$ or $z^-z^+ \in E(G)$ by symmetry. If $z^-v \in E(G)$, then $P(x_P, v) + vz^- + Q + zv^+ + P(v^+, y_P)$ has a hamiltonian path, and by replacing P and Q with this path, we again have a k-extended system on $V(F)$, a contradiction. Therefore we may assume that $z^-z^+ \in E(G)$. If the order of Q is at least four, replace P and Q with the path $P' = P(x_P, v) + vz + zv^+ + P(v^+, y_P)$ and the cycle $Q - z + z^-z^+$. If the order of Q is three, replace P and Q with the path P' and z^-z^+. Then in either case, we obtain a maximal $(k+1)$-extended system with $\sum_{P \in \mathcal{C}_2(F')} |P| > \sum_{P \in \mathcal{C}_2(F)} |P|$. This contradicts the condition (F1).

We finally consider the case that Q is K_1, K_2 or a double complete graph. In this case, consider $Q - z$ and the path $P' = P(x_P, v) + vz + zv^+ + P(v^+, y_P)$. Note that $Q - z$ is empty, K_1, K_2, a double complete graph or a complete graph of order at least three. In the last case, $Q - z$ has a hamiltonian cycle. Therefore, by replacing P and Q with P' and a certain subgraph of $Q - z$, we obtain a maximal $(k+1)$-extended system F' with $\sum_{P \in \mathcal{C}_2(F')} |P| > \sum_{P \in \mathcal{C}_2(F)} |P|$. This contradicts the choice (F1) of F. $\square$

Claim 7 *For each $P \in \mathcal{C}_2(F)$,*

$$\sum_{x \in \mathrm{End}(F)} \left| N_G(x) \cap V(P) \right| \leq |P| - f(P).$$

Proof. First assume that $N_G(z) \cap V(P) = \emptyset$ for every $z \in \mathrm{End}(F) - \{x_P, y_P\}$. Let $H = \langle V(P) \rangle_G$. By the condition (F3), for each hamiltonian path P^* of H,

$$\sum_{Q \in \mathcal{C}_2(F) - \{P\}} \left(\deg_{\langle V(Q) \rangle_G}(x_Q) + \deg_{\langle V(Q) \rangle_G}(y_Q) \right) + \deg_H(x_{P^*}) + \deg_H(y_{P^*})$$
$$\geq \sum_{Q \in \mathcal{C}_2(F)} \left(\deg_{\langle Q \rangle_G}(x_Q) + \deg_{\langle Q \rangle_G}(y_Q) \right),$$

which implies $\deg_H(x_{P^*}) + \deg_H(y_{P^*}) \geq \deg_H(x_P) + \deg_H(y_P)$. Thus, if $\deg_H(x_P) + \deg_H(y_P) \geq |H| - 1$, then by Lemma 4, either H has a hamiltonian cycle or H is a double complete graph. Then whichever occurs, we can replace P with an appropriate subgraph of H to obtain a k-extended system on $V(F)$, which contradicts Lemma 7. Therefore,

$$\sum_{x\in \mathrm{End}(F)} \big|N_G(x)\cap V(P)\big| = \big|N_G(x_P)\cap V(P)\big| + \big|N_G(y_P)\cap V(P)\big|$$
$$= \deg_H(x_P)+\deg_H(y_P) \le |H|-2 = |P|-f(P).$$

Next we assume that $N_G(z_1)\cap V(P)\neq\emptyset$ for some vertex $z_1\in \mathrm{End}(F)-\{x_P,y_P\}$. Let $v\in N_G(z_1)\cap V(P)$, $P_1=P(x_P,v^-)$ and $P_2=P(v^+,y_P)$. Then $|P|=|P_1|+|P_2|+1$. By Claim 6 (i)–(iv), $\big(N_G(x_P)\cap V(P_1)\big)^-$, $N_G(y_P)\cap V(P_1)$ and

$$\Big(\big(N_G(z)\cap V(P_1)\big)^-\Big)_{z\in \mathrm{End}(F)-\{x_P,y_P\}}$$

are well-defined and these $k+1$ sets are pairwise disjoint. Moreover, they do not contain v^- by Claim 6 (iii). Thus

$$\sum_{z\in \mathrm{End}(F)} \big|N_G(z)\cap V(P_1)\big| \le |P_1|-1.$$

By symmetry of P_1 and P_2, we obtain $\sum_{z\in \mathrm{End}(F)}\big|N_G(z)\cap V(P_2)\big|\le |P_2|-1$. By Claim 6 (i) and (iv), v is not adjacent to any vertex in $\mathrm{End}(F)-\{z_1\}$, and so $\sum_{z\in \mathrm{End}(F)}|N_G(z)\cap\{v\}|=1$. By summing these three inequalities, we have

$$\begin{aligned}\sum_{z\in \mathrm{End}(F)} \big|N_G(z)\cap V(P)\big| &= \sum_{z\in \mathrm{End}(F)} \big|N_G(z)\cap V(P_1)\big| + \sum_{z\in \mathrm{End}(F)} \big|N_G(z)\cap V(P_2)\big| \\ &\quad + \sum_{z\in \mathrm{End}(F)} \big|N_G(z)\cap \{v\}\big| \\ &\le |P_1|-1+|P_2|-1+1 \\ &= |P|-2 = |P|-f(P). \qquad \square\end{aligned}$$

We now prove Theorem 8. Assume that $N_G(z)\cap N_G(w)-V(F)\neq\emptyset$ for some z, $w\in \mathrm{End}(F)$ with $z\neq w$. Let P and Q be the components of F that contain z and w, respectively (possibly P=Q). Let $a\in N_G(z)\cap N_G(w)-V(F)$. If $P\neq Q$, then since P and Q have hamiltonian paths which contain z and w as an end-vertex, respectively, $P+za+aw+Q$ contains a hamiltonian path. By replacing P and Q with this path, we obtain a new $(k+1)$-extended system F' with $V(F') = V(F) \cup \{a\}$. This contradicts the maximality of F. If $P = Q$, then we may assume $z = x_{P_0}$ and $w = y_{P_0}$ for some P_0. Then by replacing P with a cycle $P + az + zw$, we again obtain a $(k+1)$-extended system F' with $V(F') = V(F) \cup \{a\}$, a contradiction. Therefore, we have $N_G(z) \cap N_G(w) - V(F) = \emptyset$ for each distinct pair of vertices z and w in $\mathrm{End}(F)$. Hence

$$\sum_{z \in \mathrm{End}(F)} \bigl|N_G(z) \cap \bigl(V(G) - V(F)\bigr)\bigr| \le \bigl|V(G) - V(F)\bigr| = |G| - |F|.$$

Then by Claims 5 and 7, we obtain

$$\begin{aligned}
\sum_{z \in \mathrm{End}(F)} \deg_G(z) &= \sum_{C \in \mathcal{C}(F)} \sum_{z \in \mathrm{End}(F)} \bigl|N_G(z) \cap V(C)\bigr| \\
&\quad + \sum_{z \in \mathrm{End}(F)} \bigl|N_G(z) \cap \bigl(V(G) - V(F)\bigr)\bigr| \\
&\le \sum_{C \in \mathcal{C}(F)} \bigl(|C| - f(C)\bigr) + |G| - |F| \\
&= |F| - f(F) + |G| - |F| \\
&= |G| - k - 1.
\end{aligned}$$

This contradicts the condition $\sigma_{k+1}(G) \ge |G| - k$, and Theorem 8 follows. $\square$

3 Maximum Degree

A tree of maximum degree at most k is called a *k-tree*. Under the same assumption as that of Theorem 3, we can actually guarantee the existence of a 3-tree with at most k leaves.

Theorem 9 *Let $k \ge 2$ be an integer and let G be a connected claw-free graph. If $\sigma_{k+1}(G) \ge |G| - k$, then G has a spanning 3-tree with at most k leaves.*

In order to prove the above theorem, it suffices to prove the following lemma.

Lemma 10 *Let $k \geq 2$ be an integer. If a connected claw-free graph G has a spanning tree with at most k leaves, then G has a spanning 3-tree with at most k leaves.*

Proof. Let u be an arbitrary vertex in G, and consider every spanning tree as a rooted tree with root u. Choose a spanning tree T with at most k leaves so that $\sum_{x \in V(T)} \mathrm{dist}_T(u,x)$ is as large as possible, where $\mathrm{dist}_T(x,y)$ is the distence in T between two vertices x and y. Assume T has a vertex w of degree at least four. Then w has at least three children, and since G is claw-free, w has a pair of children v_1 and v_2 which are adjacent with each other in G. Let $T' = T - wv_1 + v_1v_2$. Then T' is a spanning tree of G, and $\deg_{T'}(w) = \deg_T(w) - 1$, $\deg_{T'}(v_2) = \deg_T(v_2) + 1$ and $\deg_{T'}(x) = \deg_T(x)$ for each $x \in V(G) - \{w, v_2\}$. Since $\deg_T(w) \geq 4$, T' does not have the larger number of leaves than T.

Let $x \in V(G)$. Then T has a unique ux-path P. If P still exists in T', we have $\mathrm{dist}_T(u,x) = \mathrm{dist}_{T'}(u,x)$. If P does not exist in T', then $wv_1 \in E(P)$ and $P' = P(u,w) + wv_2 + v_2v_1 + P(v_1,x)$ is a unique ux-path in T'. This implies $\mathrm{dist}_{T'}(u,x) = \mathrm{dist}_T(u,x) + 1$. Therefore, $\mathrm{dist}_{T'}(u,x) \geq \mathrm{dist}_T(u,x)$ for each $x \in V(G)$ and $\mathrm{dist}_{T'}(u,v) > \mathrm{dist}_T(u,v)$. These imply $\sum_{x \in V(G)} \mathrm{dist}_{T'}(u,x) > \sum_{x \in V(G)} \mathrm{dist}_T(u,x)$. This contradicts the choice of T, and hence we have $\Delta(T) \leq 3$ $\square$

4 Concluding Remarks

Matthews and Sumner [5] proved that a 2-connected claw-free graph of minimum degree at least $\frac{1}{3}\big(|G| - 2\big)$ has a hamiltonian cycle. This result was later extended by Zhang [11].

Theorem 11 (Zhang [11]) *A k-connected claw-free graph G with $\sigma_{k+1}(G) \geq |G| - k$ has a hamiltonian cycle.*

Interpreting a hamiltonian cycle as a "spanning tree with one leaf" and comparing Theorems 3 and 11, we may make the following conjecture.

Conjecture 12 *For integers k and m with $k \geq 2$ and $m \leq \min\{6, k-1\}$, every m-connected claw-free graph G with $\sigma_{k+1}(G) \geq |G|-k$ has a spanning tree with at most $k-m+1$ leaves.*

The assumption $m \leq 6$ in the above conjecture looks strange, but it comes from the following theorem by Ryjáček [7].

Theorem 13 (Ryjáček [7]) *Every 7-connected claw-free graph is hamiltonian.*

By the above theorem, a 7-connected claw-free graph has a spanning tree with two leaves without any degree sum condition.

References

[1] H. Broersma and H. Tuinstra, Independence trees and Hamilton cycles, J. Graph Theory **29** (1998), 227–237.

[2] Y. Egawa, H. Matsuda, T. Yamashita and K. Yoshimoto, On a spanning tree with specified leaves, Graphs Combin. **24** (2008), 13–18.

[3] H. Enomoto, Long paths and large cycles in finite graphs, J. Graph Theory **8** (1984), 287–301.

[4] H. A. Jung, On maximal circuits in finite graphs, Ann. Discrete Math. **3** (1978), 129–144.

[5] M. M. Matthews and D. P. Sumner, Longest paths and cycles in $K_{1,3}$-free graphs. J. Graph Theory **9** (1985), 269–277.

[6] C. Nara, On sufficient conditions for a graph to be hamiltonian, Natur. Sci. Rep. Ochanomizu Univ. **31** (1980), 75–80.

[7] Z. Ryjáček, On a closure concept in claw-free graphs, J. Combin. Theory Ser. B **70** (1997), 217–224.

[8] M. Tsugaki and T. Yamashita, Spanning trees with few leaves, Graphs Combin. **23** (2007), 585–598.

[9] D.B. West, Introduction to Graph Theory, Second Edition, Prentice-Hall, Inc. 1996.

[10] S. Win, On a conjecture of Las Vergnas concerning certain spanning trees in graphs, Resultane Math. **2** (1979), 215–224.

[11] C.Q. Zhang, Hamilton cycles in claw-free graphs, J. Graph Theory **12** (1988), 209–216.