Spanning k-trees of n-connected Graphs

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Abstract

A tree is called a k-tree if the maximum degree is at most k. We prove the following theorem, by which a closure concept for spanning k-trees of n-connected graphs can be defined. Let $k \geq 2$ and $n \geq 1$ be integers, and let u and v be a pair of nonadjacent vertices of an n-connected graph G such that $\deg_G(u)+\deg_G(v) \geq |G|-1-(k-2)n$, where $|G|$ denotes the order of G. Then G has a spanning k-tree if and only if $G+uv$ has a spanning k-tree.

1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, i.e., $|G| = |V(G)|$. For a vertex v of G, let $N_G(v)$ denote the neighborhood of v in G, and denote the degree of v in G by $\deg_G(v)$, in particular, $\deg_G(v) = |N_G(v)|$. A set X of vertices of G is called an independent set if no two vertices of X are adjacent. For two vertices x and y of G, an edge joining them is denoted by xy or yx. For an integer $k \geq 2$, a tree is called a *k-tree* if the maximum degree is at most k. In particular, a Hamilton path of a graph is nothing but its spanning 2-tree.

We begin with some known results on spanning k-trees related to our theorem, and other results on spanning k-tree can be found in [3], [4], [5], and so on.

Theorem 1 (Ore [6]) *Let G be a connected graph. If every pair of nonadjacent vertices u and v of G satisfies $\deg_G(u) + \deg_G(v) \geq |G| - 1$, then G has a Hamilton path.*

The above theorem can be shown by the next theorem, which originally gives a similar result on Hamilton cycle and introduces a closure concept for Hamilton path.

Theorem 2 (Bondy and Chvátal [1]) *Let G be a connected graph, and u and v be a pair of nonadjacent vertices of G satisfying $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then G has a Hamilton path if and only if $G + uv$ has a Hamilton path.*

The next theorem is a generalization of Theorem 1.

Theorem 3 (Win [7]) *Let G be a connected graph and $k \geq 2$ be an integer. If $\sum_{x \in S} \deg_G(x) \geq |G| - 1$ for every independent set S of G with size k, then G has a spanning k-tree.*

The following theorem shows that if every pair of nonadjacent vertices of a graph G satisfies the condition of our main Theorem 5 with $n = 1$, then G has a special spanning tree or a Hamilton cycle.

Theorem 4 (Broersma and Tuinstra [2]) *Let $k \geq 2$ be an integer and G be a connected graph. If every pair of nonadjacent vertices u and v of G satisfies $\deg_G(u) + \deg_G(v) \geq |G| - k + 1$, then G has either a Hamilton cycle or an independence spanning tree with at most k end-vertices (leaves), where a spanning tree is called independence if the set of end-vertices is independent in G.*

The next theorem is our main result.

Theorem 5 *Let $k \geq 2$ and $n \geq 1$ be integers. Let G be an n-connected graph, and u and v be a pair of nonadjacent vertices of G such that*

$$\deg_G(u) + \deg_G(v) \geq |G| - 1 - (k - 2)n. \qquad (1)$$

Then G has a spanning k-tree if and only if $G + uv$ has a spanning k-tree.

Notice that the above theorem is a generalization of Theorem 2 since a Hamilton path is a spanning 2-tree, and that for the theorem with $k = 2$, the connectivity n of a graph does not contribute to the lower bound of the degree sum condition (1).

We conclude this section by showing that the lower bound of the degree sum condition (1) is sharp. Let G be the graph shown in Figure 1, which contains a complete graph K_t of order t and has $(k-1)n+1$ independent vertices including u that are adjacent to n vertices of K_t. Moreover t is sufficiently large, for example, $t \geq 2n+1$. It is clear that $|G| = t+(k-1)n+1$, G is n-connected, and $G+uv$ has a spanning k-tree. However G has no spanning k-tree since only n vertices of K_t are adjacent to $(k-1)n+1$ independent vertices of G. Furthermore, the two nonadjacent vertices u and v satisfy

$$\deg_G(u) + \deg_G(v) = |G| - 2 - (k-2)n.$$

Therefore the condition (1) in Theorem 5 is sharp.

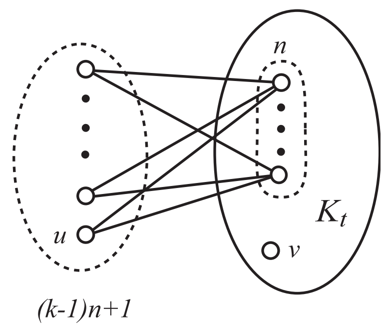

Figure 1: $G+uv$ has a spanning k-tree but not G. K_t denotes the complete graph of order t with $t \geq 2n+1$.

2 Proof of Theorem 5

Let a and b be two nonadjacent vertices of G that satisfy

$$\deg_G(a) + \deg_G(b) \geq |G| - 1 - (k-2)n. \tag{2}$$

We prove that if $G+ab$ has a spanning k-tree, then G also has a spanning k-tree.

Suppose that $G+ab$ has a spanning k-tree T. If $ab \notin E(T)$, then T is the desired spanning k-tree of G, and so we may assume $ab \in E(T)$. Let T_a and T_b be the components of $T - ab$ that contain a and b, respectively. We regard T_a and T_b as rooted trees with root a and b, respectively (see Figure 2). Hence for every vertex $x \in V(T_a) - \{a\}$, a child x^+ of x in T_a is defined, that is, x^+ is a vertex of T_a adjacent to x such that x lies on the path in T_a connecting x^+ and a. So x has $\deg_T(x) - 1$ children, and the set of children of x is denoted by $N_T(x)^+$. Similarly, for every vertex $y \in V(T_b) - \{b\}$, a child y^+ and the set $N_T(y)^+$ of children of y in T_b with root b are defined. Moreover, we briefly denote $V(T_a)$ and $V(T_b)$ by V_a and V_b, respectively.

Hereafter we assume that G has no spanning k-tree, and derive a contradiction. Let

$$N_G(a) \cap V_b = \{b_1, \cdots, b_r\},$$
$$N_G(b) \cap V_a = \{a_1, \cdots, a_s\} \quad \text{(see Figure 2).}$$

It might be occurred that $N_G(a) \cap V_b = \emptyset$ and $N_G(b) \cap V_a = \emptyset$. Note that $b \notin N_G(a) \cap V_b$ and $a \notin N_G(b) \cap V_a$. If $N_G(a) \cap V_b \neq \emptyset$ and $\deg_T(b_i) < k$ for some i, then $T - ab + ab_i$ is a spanning k-tree of G, a contradiction. Hence the following holds.

$$\begin{aligned} \deg_T(b_i) = k \quad &\text{for every} \quad 1 \le i \le r, \quad \text{and} \\ \deg_T(a_j) = k \quad &\text{for every} \quad 1 \le j \le s. \end{aligned} \tag{3}$$

If a and a certain a_j^+ are adjacent in G, then G has a spanning k-tree $T - ab - a_j a_j^+ + ba_j + aa_j^+$, a contradiction. Hence the following holds.

$$\begin{aligned} aa_j^+ \notin E(G) \quad &\text{for every} \quad a_j^+ \in N_T(a_j)^+, \quad 1 \le j \le s, \quad \text{and} \\ bb_i^+ \notin E(G) \quad &\text{for every} \quad b_i^+ \in N_T(b_i)^+, \quad 1 \le i \le r. \end{aligned} \tag{4}$$

We consider the following two cases.

Case 1. $r + s < n$.

Since G is n-connected, G has n internally disjoint a-b paths. Hence if $r+s < n$, then $n-(r+s)$ paths of the n internally disjoint a-b paths contain $n-(r+s)$ independent edges joining V_a to V_b. These independent edges are represented by

$$x_1y_1, x_2y_2, \cdots, x_ty_t \in E(G), \qquad t = n - (r+s) \ge 1,$$
$$x_i \in V_a - \{a\}, \quad y_i \in V_b - \{b\} \qquad \text{for all} \quad 1 \le i \le t. \quad \text{(see Figure 2)}$$

Then the following three claims hold.

Claim 1. For every $1 \le i \le t$, $\deg_T(x_i) = k$ or $\deg_T(y_i) = k$.

If $\deg_T(x_i) < k$ and $\deg_T(y_i) < k$ for some $1 \le i \le t$, then G has a spanning k-tree $T - ab + x_iy_i$, a contradiction. Thus this claim holds.

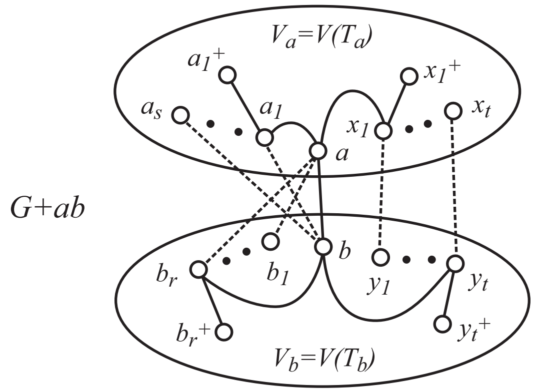

Figure 2: $G + ab$ has a spanning k-tree T containing ab.

Claim 2. If $\deg_T(y_i) < k$, then $ax_i^+ \notin E(G)$ for all $x_i^+ \in N_T(x_i)^+$. Similarly, if $\deg_T(x_j) < k$, then $by_j^+ \notin E(G)$ for all $y_j^+ \in N_T(y_j)^+$.

Assume that $\deg_T(y_i) < k$ and $ax_i^+ \in E(G)$ for some x_i^+. Then $T - ab - x_i x_i^+ + x_i y_i + ax_i^+$ is a spanning k-tree of G, a contradiction. Hence $ax_i^+ \notin E(G)$ for all $x_i^+ \in N_T(x_i)^+$. By the same argument, the latter also holds.

Claim 3. For every $1 \leq i \leq t$, either no child of x_i is adjacent to a in G or no child of y_i is adjacent to b in G.

Assume that a child x_i^+ of x_i is adjacent to a and a child y_i^+ of y_i is adjacent to b in G for some i. Then G has a spanning k-tree $T - ab - x_i x_i^+ - y_i y_i^+ + x_i y_i + ax_i^+ + by_i^+$, a contradiction. Hence the claim holds.

Let α, β and γ be the numbers of $i \in \{1, 2, \dots, t\}$ that satisfy the following (i), (ii) and (iii), respectively.

$$\begin{array}{llcl} \text{(i)} & \deg_T(x_i) = k & \text{and} & \deg_T(y_i) < k, \\ \text{(ii)} & \deg_T(x_i) < k & \text{and} & \deg_T(y_i) = k, \\ \text{(iii)} & \deg_T(x_i) = k & \text{and} & \deg_T(y_i) = k. \end{array}$$

By Claim 1, $t = \alpha + \beta + \gamma$. Then by Claims 2 and 3 and by (3), (4),

$t = \alpha + \beta + \gamma$ and $n = r + s + t$, we obtain

$$\begin{aligned}
\deg_G(a) + \deg_G(b) &= |N_G(a) \cap V_b| + |N_G(a) \cap V_a| \\
&\quad + |N_G(b) \cap V_a| + |N_G(b) \cap V_b| \\
&\le r + |V_a| - 1 - (k-1)s - (k-1)\alpha \\
&\quad + s + |V_b| - 1 - (k-1)r - (k-1)\beta - (k-1)\gamma \\
&= |V_a| + |V_b| - 2 - (k-2)(r+s) - (k-1)(\alpha + \beta + \gamma) \\
&= |G| - 2 - (k-2)(r+s+t) - t \\
&\le |G| - 2 - (k-2)n.
\end{aligned}$$

This contradicts the condition (2). Therefore the proof is complete in this case.

Case 2. $r + s \ge n$**.**

By (3) and (4), we obtain

$$\begin{aligned}
\deg_G(a) + \deg_G(b) &= |N_G(a) \cap V_b| + |N_G(a) \cap V_a| \\
&\quad + |N_G(b) \cap V_a| + |N_G(b) \cap V_b| \\
&\le r + |V_a| - 1 - (k-1)s \\
&\quad + s + |V_b| - 1 - (k-1)r \\
&\le |G| - 2 - (k-2)n.
\end{aligned}$$

Again, we derive a contradiction. Consequently the theorem is proved.

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