Balanced Subdivisions with Boundary Condition of Two Sets of Points in the Plane

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Abstract

Let R and B be two disjoint sets of red points and blue points in the plane, respectively, such that no three points of $R \cup B$ are collinear, and let a, b and g be positive integers. We show that if $ag \leq |R| < (a+1)g$ and $bg \leq |B| < (b+1)g$, then we can subdivide the plane into g convex polygons so that every open convex polygon contains exactly a red points and b blue points and that the remaining points lie on the boundary of the subdivision. This is a generalization of equitable subdivision of ag red points and bg blue points in the plane.

1 Introduction

We consider two disjoint sets R and B of red points and blue points in the plane, respectively, such that no three points of $R \cup B$ are collinear. Throughout this paper, R and B always denote these sets. We consider a problem of subdividing the plane into some convex polygons so that each convex polygon contains prescribed numbers of red points and blue points, where some polygons might be infinite domains. We begin with some known results on this problem.

Theorem 1 (The Equitable Subdivision Theorem). *Let $a \geq 1$, $b \geq 1$ and $g \geq 2$ be integers. If $|R| = ag$ and $|B| = bg$, then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that every open X_i contains exactly a red points and b blue points.*

The above Theorem 1, which was conjectured in [5] and proved for $a = 1, 2$ in [5] and [6], was proved in full generality by Bespamyatnikh, Kirkpatrick and Snoeyink [2], Sakai [10] and by Ito, Uehara and Yokoyama [3], independently. Notice that this theorem with $g = 2$ is equivalent to the following Ham-sandwich Theorem with even numbers of red points and blue points.

Theorem 2 (The Ham-sandwich Theorem [4]). *For given R and B in the plane, there exists a line l, called a bisector, such that each of the open half-planes $left(l)$ and $right(l)$ contains exactly $\lfloor |R|/2 \rfloor$ red points and $\lfloor |B|/2 \rfloor$ blue points and the remaining points, if any, lie on the line l.*

When we want to apply the Equitable Subdivision Theorem to some problems, we sometimes meet configurations for which an integer g does not divide $|R|$ or $|B|$. In such cases, we can use the following theorem, which is the main theorem of this paper. It is easy to see that this theorem is a generalization of the Ham-Sandwich Theorem with $|R|$ or $|B|$ odd. Moreover, it seems to be impossible to derive our theorem from the Equitable Subdivision Theorem.

Theorem 3. *Let $a \geq 1,\ b \geq 1$ and $g \geq 2$ be integers. If $ag \leq |R| < (a+1)g$ and $bg \leq |B| < (b+1)g$, then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that every open X_i contains exactly a red points and b blue points and that the remaining points, if any, lie on the boundary (Figure 1).*

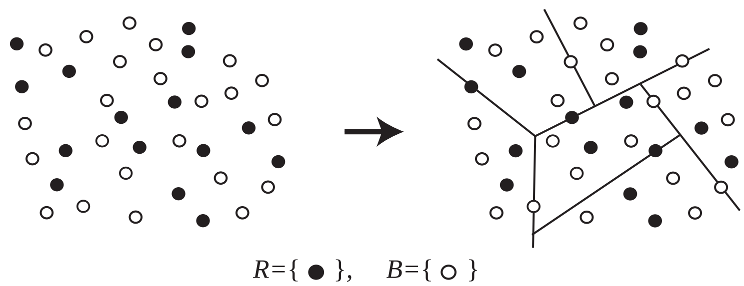

Figure 1: A subdivision with $a = 2,\ b = 3,\ g = 6$ given in Theorem 3

We conclude this section with two theorems related to our theorem, and remark that some other related results can be found in [1], [6] and [7].

Theorem 4 ([9]). *Let $a \geq 1,\ b \geq 1$ and $g \geq 2$ be integers. If $ag \leq |R| < (a+1)g$ and $bg \leq |B| < (b+1)g$, then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that each open X_i contains a or $a+1$ red points and b or $b+1$ blue points depending on X_i and that no point of $R \cup B$ lies on the boundary.*

Theorem 5 (Kaneko, Kano and Suzuki [8]). *Let $a \geq 1$, $g \geq 0$ and $h \geq 0$ be integers such that $g + h \geq 1$. If $|R| = ag + (a+1)h$ and $|B| = (a+1)g + ah$, then there exists a subdivision $X_1 \cup \cdots \cup X_g \cup Y_1 \cup \cdots \cup Y_h$ of the plane into $g + h$ disjoint convex polygons such that every open X_i contains exactly a red points and $a+1$ blue points and every open Y_j contains exactly $a+1$ red points and a blue points.*

2 Proof of Theorem 3

In this section, we shall prove Theorem 3. Consider open domains D_i's. We say that $D_1 \cup \cdots \cup D_k$ is a subdivision of the plane if any two distinct D_i's are disjoint and $\overline{D_1} \cup \cdots \cup \overline{D_k} = \mathcal{R}^2$, where $\overline{D_i}$ and $\mathcal{R}^2$ denote the closure of D_i and the plane (2-dimensional Euclidean space), respectively. The set $\mathcal{R}^2 - (D_1 \cup \cdots \cup D_k)$ is called the *boundary* of the subdivision $D_1 \cup \cdots \cup D_k$.

We say that the plane is subdivided into three convex wedges D_1, D_2, D_3 if the wedges D_i $(1 \leq i \leq 3)$ are the open domains defined by three rays emanating from the same point and every D_i is convex (see Figure 2). The following theorem plays an important role in our proof.

Theorem 6 (The (2,3)-cutting Theorem, Sakai [10]). *Let $n \geq 2$ be an integer, and let μ_1 and μ_2 be measures in the plane $\mathcal{R}^2$ such that each μ_i is absolutely continuous with respect to*

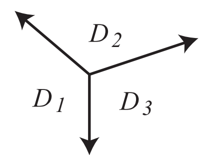

Figure 2: A subdivision of the plane into three convex open wedges D_1, D_2, D_3.

the Lebesgue measure. Suppose that there exists a bounded domain D in the plane such that $\mu_1(D) = \mu_2(D) = n$ and $\mu_1(\mathcal{R}^2 - D) = \mu_2(\mathcal{R}^2 - D) = 0$. Then at least one of the following two statements holds:

(i) There exist two positive integers n_1, n_2 with $n_1 + n_2 = n$ and a subdivision $D_1 \cup D_2$ of the plane into two open half-planes such that $\mu_1(D_1) = \mu_2(D_1) = n_1$ and $\mu_1(D_2) = \mu_2(D_2) = n_2$.

(ii) There exist three positive integers n_1, n_2, n_3 with $n_1 + n_2 + n_3 = n$ and a subdivision $D_1 \cup D_2 \cup D_3$ of the plane into three open convex wedges such that $\mu_1(D_1) = \mu_2(D_1) = n_1$, $\mu_1(D_2) = \mu_2(D_2) = n_2$ and $\mu_1(D_3) = \mu_2(D_3) = n_3$. Moreover, one of the three rays that determine these three convex wedges can be chosen as a vertical downward ray.

Proof of Theorem 3. We shall prove the theorem by induction on g. If $g = 2$, then $|R| \in \{2a, 2a+1\}$ and $|B| \in \{2b, 2b+1\}$, and so the theorem follows from the Ham-sandwich Theorem 2. Hence we may assume $g \geq 3$. We can express $|R|$ and $|B|$ as follows.

$$|R| = ag + k, \quad |B| = bg + h, \quad 0 \leq k < g \text{ and } 0 \leq h < g.$$

In order to apply the $(2,3)$-cutting Theorem, we define two measures μ_1 and μ_2 in the following way. We replace every point x in $R \cup B$ by a circle with center x and a constant radius, which will be defined later. A circle with center a red point is called a *red circle*, and a *blue circle* is defined similarly. The weights $w(red)$ and $w(blue)$ of a red circle and of a blue circle, respectively, are defined as follows:

$$w(red) = \frac{g}{|R|} \quad \text{and} \quad w(blue) = \frac{g}{|B|}.$$

More precisely, the weight of every circle is uniformly distributed in it. Finally the two measures μ_1 and μ_2 are defined as follows. For a domain D in the plane, $\mu_1(D)$ and $\mu_2(D)$ are the sum of weights of red circles and that of blue circles contained in D. Of course, if a part of a circle is covered by D, then its contribution to μ_1 or μ_2 is the weight of the part.

Let Y be the set of intersections of two lines each of which passes through two distinct points of $R \cup B$ (see Figure 3 (1)). By a suitable rotation of the plane, we may assume that every vertical line passing through a point of $R \cup B$ passes through no other points of $R \cup B \cup Y$. We determine the constant radius of all circles to be a sufficiently small positive real number so that the following three conditions hold.

(a) Every vertical line intersecting a circle intersects no other circles (Figure 3 (2));

(b) Every line intersecting two distinct circles intersects no other circles (Figure 3 (3);

(c) For four distinct point x, y, y_1 and y_2 of $R \cup B$, consider a vertical line l_x intersecting a circle with center x, and a line l' intersecting two circles with center

y_1 and y_2. Then any line l_y that intersects a circle with center y and passes through the intersection of l_x and l' intersects no other circles (Figure 3 (4)).

Notice that it is clear that if the radius becomes zero, then the conditions (a), (b), (c) are satisfied since no three points of $R \cup B$ are collinear and every vertical line passing through a point of $R \cup B$ passes through no other points of $R \cup B \cup Y$. Therefore we can find a positive radius that satisfies the above three conditions.

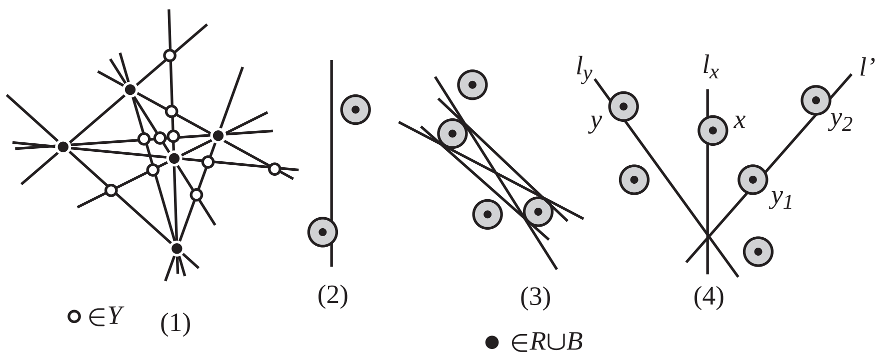

Figure 3: Some points of Y; the conditions (a), (b), and (c).

Because the number of circles is finite, we can easily find a bounded domain D such that the following equations hold.

$$\mu_1(\mathcal{R}^2 - D) = \mu_2(\mathcal{R}^2 - D) = 0 \;\; and$$
$$\mu_1(D) = |R|\, w(red) = \mu_2(D) = |B|\, w(blue) = g.$$

Hence by the $(2,3)$-cutting Theorem, one of the following two statements holds:

(I) There exist two positive integers g_1, g_2 with $g_1 + g_2 = g$ and a subdivision $D_1 \cup D_2$ of the plane into two open half-planes such that $\mu_1(D_1) = \mu_2(D_1) = g_1$ and $\mu_1(D_2) = \mu_2(D_2) = g_2$.

(II) There exist three positive integers g_1, g_2, g_3 with $g_1+g_2+g_3 = g$ and a subdivision $D_1 \cup D_2 \cup D_3$ of the plane into three open convex wedges such that $\mu_1(D_1) = \mu_2(D_1) = g_1$, $\mu_1(D_2) = \mu_2(D_2) = g_2$ and $\mu_1(D_3) = \mu_2(D_3) = g_3$. Moreover, one of the three rays that determine these three convex wedges is a vertical downward ray.

We consider two cases.

Case 1. (I) occurs.

Let l denote the boundary line of D_1 and D_2. Then for every $i \in \{1, 2\}$, it follows that

$$\frac{\mu_1(D_i)}{w(red)} = \frac{g_i|R|}{g} = ag_i + \frac{kg_i}{g}, \quad 0 \le \frac{kg_i}{g} < g_i, \tag{1}$$
$$\frac{\mu_2(D_i)}{w(blue)} = \frac{g_i|B|}{g} = bg_i + \frac{hg_i}{g}, \quad 0 \le \frac{hg_i}{g} < g_i. \tag{2}$$

We first consider the case that l intersects no circles. In this case, all kg_i/g and hg_i/g are integers, and so by the induction hypotheses, each D_i can be subdivided into g_i convex polygons satisfying the condition. Hence the plane can be subdivided into $g = g_1 + g_2$ convex polygons satisfying the condition.

Next we consider the case that l intersects exactly one circle C. Without loss of generality, we may assume that C is a red circle. Then $kg_i/g > 0$ as $\mu_1(D_i)/w(red)$ is not an integer. We can slightly move l to a new line l' that passes through the center of C and intersects no other circles by the condition (b) (Figure 4 (1)). Let D'_1 and D'_2 be the open half-planes determined by l', and let $\#(D'_i)$ denote the number of red points in D'_i. Since the contribution of the red circle C to $\mu_1(D_1)/w(red)$ and $\mu_1(D_2)/w(red)$ is less than one each, we have

$$\#(D'_i) = ag_i + \left\lfloor \frac{kg_i}{g} \right\rfloor \quad \text{and} \quad 0 \leq \left\lfloor \frac{kg_i}{g} \right\rfloor < g_i \qquad \text{for } i \in \{1, 2\}.$$

It is clear that each D'_i contains $bg_i + (hg_i)/g$ blue points, where $(hg_i)/g$ is an integer less than g_i. Thus we can apply the inductive hypotheses to D'_1 and D'_2, and hence we can obtain the desired subdivision of the plane.

We consider the case that l passes through one red circle and one blue circle. We slightly move l to the line l' that passes through the centers of these two circles (Figure 4 (2)). Since the contribution of the red circle and that of the blue circle to $\mu_1(D_i)/w(red)$ or $\mu_2(D_i)/w(blue)$ is less than one each, by the same argument as above, we can show that the new open half-planes D'_1 and D'_2 determined by l' satisfy the condition of the theorem, and thus the theorem holds by induction.

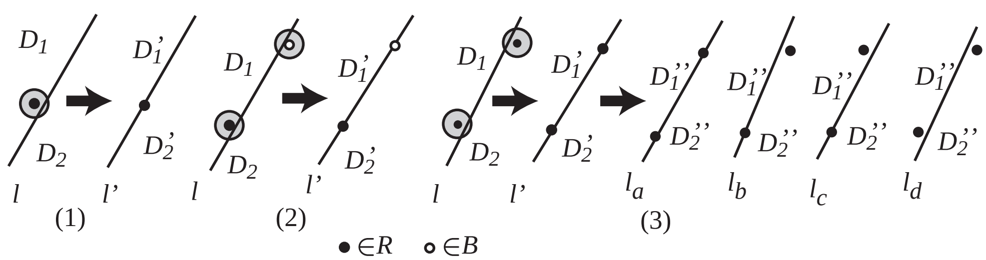

Figure 4: Line l, l'. Line l_a, l_b, l_c and l_d and open half planes D'_1 and D'_2.

We finally consider the case that l intersects two red circles or two blue ones. Without loss of generality, we may assume that l intersects two red circles. We first move l to the line l' that passes through the centers of these two red circles. Let D'_1 and D'_2 be the open half-planes determined by l'. Next we determine a new line l_a, l_b, l_c or l_d shown in Figure 4 (3), and show that the new open half-planes D''_1 and D''_2 defined by one of $\{l_a, l_b, l_c, l_d\}$ satisfy the condition of the theorem, which implies that the theorem holds by induction. Note that l_a, l_b, l_c and l_d may pass through some of the two red points on l', but for any point x of $R \cup B$ except the red points on l', $x \in D'_i$ implies $x \in D''_i$. Let $\#(D)$ denote the number of red points in an open domain D.

If $\#(D'_1) \geq ag_1$ and $\#(D'_2) \geq ag_2$, then $D''_1 = D'_1$ and $D''_2 = D'_2$ determined by l_a satisfy $ag_i \leq \#(D''_i) < ag_i + g_i$ for $i \in \{1, 2\}$, and so these D''_i satisfy the condition of the theorem. If $\#(D'_1) \geq ag_1$ and $\#(D'_2) < ag_2$, then choose l_b, and let D''_1 and D''_2 be the open half-planes determined by l_b. Since $\#(D_2') > (\mu_1(D_2)/w(red)) - 2 \geq ag_2 - 2$, we have

$$ag_2 + 1 > \#(D_2'') = \#(D_2') + 1 > ag_2 - 2 + 1 = ag_2 - 1,$$

which implies $\#(D_2'') = ag_2$. Hence D_1'' and D_2'' satisfy the condition of the theorem. By symmetry, if $\#(D_1') < ag_1$ and $\#(D_2') \geq ag_2$, then D_1'' and D_2'' determined by l_c satisfy the condition of the theorem. If $\#(D_1') < ag_1$ and $\#(D_2') < ag_2$, then take open half-planes D_1'' and D_2'' determined by l_d. Since $\#(D_i') > ag_i - 2$ for every i, we have

$$ag_i + 1 > \#(D_i'') = \#(D_i') + 1 > ag_i - 2 + 1 = ag_i - 1,$$

which implies $\#(D_i'') = ag_i$. Hence D_1'' and D_2'' satisfy the condition of the theorem. Therefore the theorem follows by induction.

Case 2. (II) occurs.

We write D_1, D_2, D_3 for the convex wedges and r_1, r_2, r_3 for the rays in clockwise order, where r_1 is a vertical downward ray and D_1 lies to the left of r_1 (Figure 5 (1)). So the two boundary rays of D_i are r_i and r_{i+1}, where $r_4 = r_1$. For every $i \in \{1, 2, 3\}$, we have

$$\begin{aligned} \frac{\mu_1(D_i)}{w(red)} &= \frac{g_i|R|}{g} = ag_i + \frac{kg_i}{g}, \quad 0 \leq \frac{kg_i}{g} < g_i, \\ \frac{\mu_2(D_i)}{w(blue)} &= \frac{g_i|B|}{g} = bg_i + \frac{hg_i}{g}, \quad 0 \leq \frac{hg_i}{g} < g_i. \end{aligned}$$

If none of three rays intersects a circle, then we can apply the inductive hypotheses to each D_i, and thus the theorem follows. Hence we may assume that at least one of the three rays intersects a circle. We say that a 3-cutting consisting of three rays emanating from the same point is *convex* if the three wedges determined by these rays are convex.

Claim 1. *Let $1 \leq m \leq 4$ be an integer. If there exists a convex 3-cutting whose three rays r_1, r_2, r_3 intersect m circles $C_1, \cdots, C_m$, then there exists a convex 3-cutting composed by three rays r_1', r_2', r_3' that pass through the centers of $C_1, \cdots, C_m$ together with the centers of some other circles but at most four centers altogether, and intersect no other circles, where r_1' is a vertical downward ray. Moreover, if a point x of $R \cup B$ lies in D_i, then x lies in a new wedge D_i' or on the new three rays r_1', r_2', r_3'.*

Let x_i denote the center of the circle C_i for every $1 \leq i \leq m$. For every ray r_i or r_i', we denote by l_i or l_i' the line containing it. We first prove Claim 1 for $m = 4$. We consider the following two cases because of the condition (b).

Case 1. One ray intersects two circles and the other two rays intersect one circle each, and these four circles are distinct (i.e., $m = 4$).

Since r_1 intersects at most one circle, we may assume that r_1 and r_2 intersect circles C_1 and C_2 respectively, and that r_3 intersects two circles C_3 and C_4 (See Figure 5 (2)). Let p be the intersection of the vertical line passing through x_1 and the line passing through x_3 and x_4, and let r_1' and r_3' be the rays on these lines emanating from p (See Figure 5 (3)). Let r_2' be the ray emanating from p and passing through x_2. Then by the conditions (a), (b) and (c) on the radius of circles, it follows that every r_i' $(1 \leq i \leq 3)$ intersects no other circles mentioned above. We

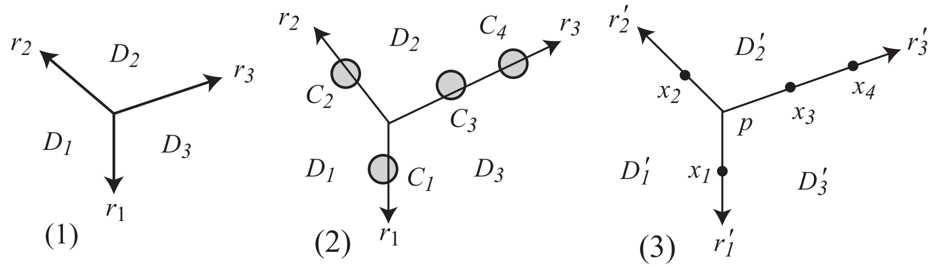

Figure 5: Rays r_i and r'_i, open convex wedges D_i and D'_i, and the centers x_j of the circles C_j.

now show that the new wedges D'_i are also convex. The line l'_1 containing r'_1 intersects none of C_2, C_3 and C_4 by (a), and so D'_1 and D'_3 are convex. Since D_2 is convex, C_2 lies above the line l_3 containing r_3, and so it also lies above l'_3 by (b), which implies that D'_2 is convex. Therefore Claim 1 follows.

Case 2. Two rays intersect two circles each, and these four circles are distinct (i.e., $m = 4$).

Since r_1 intersects at most one circle, we may assume that r_2 intersects two circles C_1 and C_2, r_3 intersects C_3 and C_4, and r_1 intersects no cycle. Let p be the intersection of the line passing through x_1 and x_2 and the line passing through x_3 and x_4, and let r'_2 and r'_3 be the rays on these lines emanating from p. Let r'_1 be the downward ray starting at p. Then by (b) and (c), every r'_i $(1 \leq i \leq 3)$ intersects no other circles mentioned above. Since D_1 is convex, the slope of l_2 containing r_2 is negative. So if the slope of l'_2 containing r'_2 is positive, there exists a vertical line that intersects two cycles C_1 and C_2, which contradicts (a). Hence the slope of l'_2 is negative, and by the same argument, the slope of l'_3 is positive. Hence D'_1, D'_2 and D'_3 are convex. Therefore Claim 1 holds in this case.

We now consider the case of $1 \leq m \leq 3$. Notice that in this case, it may appear that the common point of the three rays lies in a circle. We first consider the case that this situation does not appear. Since in each case the proof is almost the same, here we consider only the case that $m = 3$, r_1 intersects one circle C_1, r_2 intersects two circles C_2 and C_3, and r_3 intersects no circle. Notice that the remaining cases are considered in Appendix A. Let q be the point from which the three rays r_1, r_2 and r_3 emanate, and let p be the intersection of the vertical line passing through x_1 and the line passing through x_2 and x_3. Let r'_1 be the vertical downward ray emanating from p, r'_2 be the ray emanating from p and passing through x_2 and x_3, and r'_3 be the ray emanating from p and parallel to r_3. We continuously move the three rays and their common point from r_1, r_2, r_3 and q to r'_1, r'_2, r'_3 and p, respectively, in such a way that a ray corresponding to r_3 is always parallel to r_3 and a ray corresponding to r_1 is vertical downward. If a ray corresponding to r_3 touches a new cycle, say C_4, then stop moving after a little more move, and denote the ray that slightly intersects C_4 by r^*_3, and denote two other rays by r^*_1 and r^*_2.

If we can move r_1, r_2, r_3, q to r'_1, r'_2, r'_3, p, respectively, without touching any other circles, then r'_1, r'_2 and r'_3 are the desired rays as follows. Since the slope of r'_3 is positive and that of r'_2 is negative by (a), which implies that D'_1, D'_2 and D'_3 are convex. If a ray corresponding to r_3 touches a new cycle C_4, then the three rays r^*_1, r^*_2 and r^*_3 intersect four circles, and so we can apply the same argument in the case of $m = 4$, and obtain the desired three rays.

Claim 2. *Let $1 \le m \le 4$ be an integer. Let $C_1, C_2, \dots, C_m$ be m distinct circles, and x_i be the center of C_i for every $1 \le i \le m$. Suppose that the three rays r'_1, r'_2, r'_3, given in Claim 1, pass through the centers of these m circles altogether and intersect no other circles. Then we can slightly move r'_i to r''_i for every $1 \le i \le 3$ so that the following condition (iv) and (v) are satisfied and that for any choice of one of (i), (ii), (iii) for each $1 \le k \le m$, these chosen conditions are also satisfied (see Figure 6).*

For every $1 \le k \le m$, assume that x_k lies on a ray r'_j, and two wedges D'_s and D'_t have r'_j as their common boundary. (i) x_k lies on a new ray r''_j; (ii) x_k is contained in D''_s; or (iii) x_k is contained in D''_t. (iv) The three new open wedges D''_i's defined by r''_i's are convex; and (v) for every point $y \in R \cup B - \{x_1, x_2, \dots, x_m\}$, $y \in D'_i$ implies $y \in D''_i$.

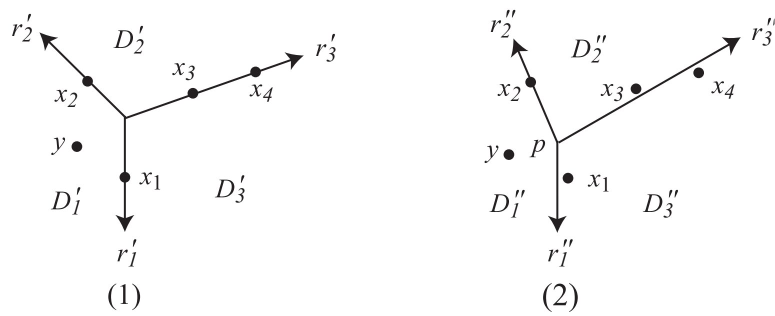

Figure 6: Consider conditions that $x_1 \in D''_3$, x_2 lies on r''_2, $x_3 \in D''_2$, and $x_4 \in D''_3$. Then there exist three rays r''_1, r''_2, r''_3 that satisfy these conditions and (iv) and (v).

Since no three points of $R \cup B$ lie on the same line and no two points of $R \cup B$ lie on the same vertical line, we can slightly move r'_i to r''_i for every $1 \le i \le 3$ so that the desired conditions are satisfied. Hence Claim 2 holds.

We now come to the last stage of the proof. By (II), there exists a 3-cutting with three convex wedges D_1, D_2 and D_3 that satisfy

$$\frac{\mu_1(D_i)}{w(red)} = \frac{g_i|R|}{g} = ag_i + \frac{kg_i}{g}, \quad 0 \le \frac{kg_i}{g} < g_i, \quad \text{and} \tag{3}$$

$$\frac{\mu_2(D_i)}{w(blue)} = \frac{g_i|B|}{g} = bg_i + \frac{hg_i}{g}, \quad 0 \le \frac{hg_i}{g} < g_i \quad \text{for all } 1 \le i \le 3. \tag{4}$$

Let D'_1, D'_2 and D'_3 be three convex open wedges obtained in Claim 1. Since every D'_i contains no centers of the circles given in Claim 1, it follows that

$$\mu_j(D'_i) \le \mu_j(D_i) \qquad \text{for all } j \in \{1, 2\} \text{ and } 1 \le i \le 3.$$

We first consider red points. For an open wedge D and a ray r, let $\#(D)$ and $\#(r)$ denote the numbers of red points in D and r, respectively. If

$$\#(D'_i) \ge ag_i \qquad \text{for all } 1 \le i \le 3,$$

then we require that every red center x_i lies on a new ray r''_j in Claim 2. Since

$$ag_i \le \#(D''_i) = \#(D'_i) \le \frac{\mu_1(D_i)}{w(red)} < ag_i + g_i,$$

D''_1, D''_2 and D''_3 satisfy the condition of the theorem about red points. Next assume that

$$\#(D'_1) < ag_1 \quad \text{and} \quad \#(D'_i) \ge ag_i \qquad \text{for } i \in \{2,3\}.$$

Then there are at least $ag_1 - \#(D'_1)$ red points lying on $r'_1 \cup r'_2$ as $\mu_1(D_1)/w(red) \ge ag_1$. So we make a request that $ag_1 - \#(D'_1)$ red points on $r'_1 \cup r'_2$ belong to D'_1. Then

$$\#(D''_1) = \#(D'_1) + ag_1 - \#(D'_1) = ag_1.$$

For all the other red points on $r'_1 \cup r'_2 \cup r'_3$, we require that they are on $r''_1 \cup r''_2 \cup r''_3$. So for each $i \in \{2,3\}$, it follows that

$$ag_i \le \#(D''_i) = \#(D'_i) \le \frac{\mu_1(D_i)}{w(red)} < ag_i + g_i.$$

Therefore D''_1, D''_2 and D''_3 satisfy the condition of the theorem about red points.

Since the case that $\#(D'_1) < ag_1$, $\#(D'_2) < ag_2$ and $\#(D'_3) \ge ag_3$ can be dealt with in the same way as in the next case, we skip this case here. Notice that all the remaining cases including the above case will be considered in Appendix B.

We finally consider the case that

$$\#(D'_i) < ag_i \qquad \text{for all } 1 \le i \le 3.$$

Let $m_i = ag_i - \#(D'_i)$ for all $1 \le i \le 3$. Then all m_i are positive integers and satisfy

$$\begin{aligned} &m_1 + m_2 + m_3 \le \#(r'_1) + \#(r'_2) + \#(r'_3) \le 4, \quad \#(r'_1) \le 1, \\ &m_j \le \#(r'_j) + \#(r'_{j+1}) \;\text{ and }\; \#(r'_i) \le 2 \quad \text{for } j \in \{1,2,3\} \text{ and } i \in \{2,3\}\ . \end{aligned}$$

By symmetry, we may assume that $m_2 \ge m_3$. Then $m_3 = 1$.

First assume $\#(r'_3) = 0$. Then $m_1 = m_2 = m_3 = 1$, $\#(r'_1) = 1$ and $\#(r'_2) = 2$. Hence we require that the red point on r'_1 belongs to D''_3, two red points of r'_2 belong to D''_1 and D''_2 one each. We next consider the case of $\#(r_3) \ge 1$. We require that a red point on r'_1, if any, belongs to D''_1 and $m_1 - \#(r'_1)$ red points on r'_2 belongs to D''_1. This can be done since $m_1 \le \#(r'_1) + \#(r'_2)$. Next if m_2 is less than or equal to the number of remaining red points on r'_2, then we require that m_2 red points on r'_2 belongs to D''_2 and one red point of r'_3 belongs to D''_3 as $m_3 = 1$. So we may assume that m_2 is greater than the number of remaining red points on r'_2. Then we require that all the remaining red points on r'_2 belongs to D''_2 and the necessary number of red points of r'_3 belongs to D''_2. This can be done since $m_1 + m_2 \le \#(r'_1) + \#(r'_2) + \#'(r'_3)$. Finally we require that one red points of r'_3 belongs to D''_3. This can be done since r'_3 contains still $\#(r'_1) + \#(r'_2) + \#(r'_3) - m_1 - m_2 \ge m_3 = 1$ red points.

Since we can arbitrarily choose one of (i), (ii), (iii) in Claim 2 for any blue point x_k keeping the choices given above for red points, we can carry out the same procedure for blue points as above. Note that the total number of red points and blue points lying $r'_1 \cup r'_2 \cup r'_3$ is m in Claim 2, which is at most 4. Therefore by Claim 2, we can obtain a convex subdivision $D''_1 \cup D''_2 \cup D''_3$ of the plane such that every D''_i satisfies the condition of the theorem. Consequently, by applying the inductive hypotheses to each D''_i, we can obtain the desired subdivision of the theorem, and hence the theorem is proved.

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A Appendix : Proof of Claim 1

We shall complete the proof of Claim 1 in the proof of Theorem 3, that is, we prove Claim 1 in the remaining cases. In order to do so, we first prove the following two cases, which are generalizations of Case 1 and Case 2 in the proof of Claim 1.

We use the same notations. For example, x_i denotes the center of a circle C_i for every $1 \leq i \leq m$. For every ray r_i, we denote by l_i the line containing it.

Case A. One line l_i intersects two circles and the other two lines l_j's intersect one circle each, and these four circles are distinct.

Since l_1 intersects at most one circle by the condition (a) on the radius of circles, we may assume that l_1 and l_2 intersect circles C_1 and C_2, respectively, and that l_3 intersects two circles C_3 and C_4 (See Figure 7 (1)). Let l'_1 be the vertical line passing through x_1, l'_3 the line passing through x_3 and x_4, and p the intersection of l'_1 and l'_3 (See Figure 7 (2)). Let l'_2 be the line passing through p and x_2, and let r'_i be the ray on l'_i emanating from p. Then by the conditions

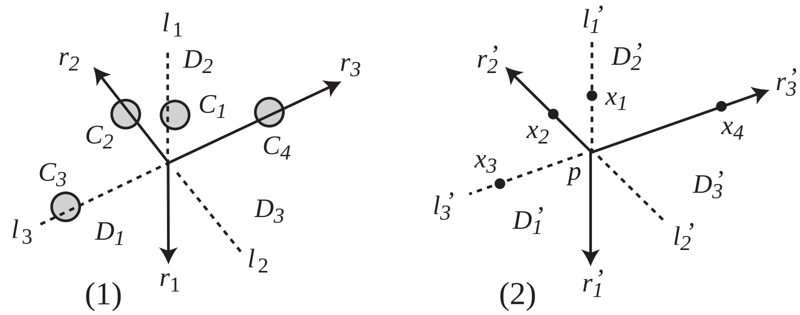

Figure 7: Lines l_i and l'_i and the centers x_j of C_j.

(a), (b) and (c), it follows that every l'_i $(1 \le i \le 3)$ intersects no other circles mentioned above. We now show that the new wedges D'_i determined by the rays r'_i are also convex. Since the slope of l_3 is positive, the slope of l'_3 is also positive since otherwise there exists a vertical line passing through two circles, which contradicts (a). If the slope of l'_2 is positive, then there exists a vertical line passing through two circle C_1 and C_2, which contradicts (a). Hence the slope of l'_2 is negative. Therefore D'_1, D'_2 and D'_3 are convex, and Claim 1 follows.

Case B. Two lines l_i's intersect two circles each, and these four circles are distinct.

Since l_1 intersects at most one circle, we may assume that l_2 intersects two circles C_1 and C_2, l_3 intersects C_3 and C_4, and l_1 intersects no cycle. Let l'_2 be the line passing through x_1 and x_2, and l'_3 the line passing through x_3 and x_4. Le p be the intersection of l'_2 and l'_3, and let l'_1 be the vertical line passing through p. Let r'_i be the ray on line l'_i emanating from p for every $1 \le i \le 3$. Then by (b) and (c), every r'_i $(1 \le i \le 3)$ intersects no other circles mentioned above. Since D_1 is convex, the slope of l_2 is negative, and so the slope of l'_2 is negative by (a). By the same argument, the slope of l'_3 is positive. Hence D'_1, D'_2 and D'_3 are convex. Therefore Claim 1 holds in this case.

The next Case 3 was proved before.

Case 3. $m = 3$, r_1 intersects on circle, r_2 intersects two circles and r_3 intersects no circle.

We now consider the remaining cases of $m = 3$.

Case 4. $m = 3$ and every r_i intersects exactly one circle each.

Since $m = 3$, the circles in the case are distinct. We may assume that r_1 intersects a circle C_1, r_2 intersects a cycle C_2, r_3 intersects a circle C_3 (Figure 8 (1)). Let l'_1 be the vertical line passing through x_1, and l'_2 be the line parallel to r_2 and passing through x_2. Let p be the intersection of l'_1 and l'_2, and let r'_1 and r'_2 be the rays on l'_1 and l'_2, respectively, emanating from p. Let r'_3 be the ray emanating from p and passing through x_3 (Figure 8 (2)). If some line l'_i containing r'_i intersects another circle C_4, then by the above Cases A or B, the claim holds. Hence we may assume that every line l'_i intersect no other circle except C_1, C_2 and C_3. If the slope of r'_3 is negative, then there exists a vertical line intersecting C_1 and C_3, a contradiction. Hence the slope of r'_3 is positive, and that of r'_2 is negative as r'_2 and r_2 are parallel. Hence D'_1, D'_2 and D'_3 are convex, and the claim holds.

Case 5. $m = 3$ and r_1 intersects no circle.

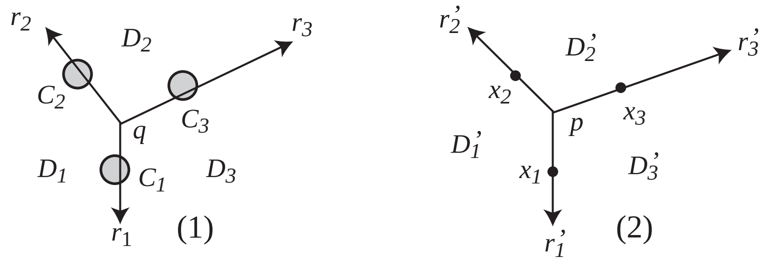

Figure 8: Rays r_i, open convex wedges D_i, and the circles C_j.

Without loss of generality, we may assume that r_2 intersects two circles C_1 and C_2, and r_3 intersects a circle C_3. If the vertical line l'_1 containing r_1 intersects another circle C_4, then by Case A the claim holds. Hence we may assume that l'_1 does not intersect any circle. Let l'_2 be the line passing through x_1 and x_2, and let p be the intersection of l'_2 and l'_1. Let r'_2 be the ray on l'_2 emanating from p, and r'_3 the ray emanating from p and passing through x_3. If r'_3 intersects another cycle, then the claim holds by Case B (or Case 2). So we may assume that r'_3 intersects no circle except C_3. Since l'_1 does not intersect any cycle, D'_1 and D'_3 are convex. Since C_3 lies above l_2 as D_2 is convex, C_3 lies above l'_2 by (b), and thus D'_2 is convex. Therefore the claim holds.

Case 6. $m = 2$.

Assume that r_1 intersects a circle C_1, r_2 intersects a circle C_2 and r_3 intersects no circle. Let l'_1 be the vertical line passing through x_1, and l'_2 the line parallel to r_2 and passing through x_2. Let p be the intersection of l'_1 and l'_2, and let r'_1 be the vertical ray on l'_1 emanating from p, and r'_2 the ray on l'_2 emanating from p. Let l'_3 be the line parallel to r_3 and passing through p, and r'_3 be the ray on l'_3 emanating from p. If r'_2 or r'_3 intersects another cycle, then then we can apply the case of $m = 3$ or $m = 4$, and hence the claim holds. So we may assume that r'_2 and r'_3 intersect no other circle. Since the slope of r'_2 is negative and that of r'_3 is positive, three domains D'_1, D'_2, D'_3 are convex, and hence the claim holds. We can similar prove other cases with $m = 2$.

Case 4. $m = 1$.

Assume that r_2 intersects a circle C_1, but none of r_1 and r_3 intersects a cycle. Let l'_2 the line parallel to r_2 and passing through x_1, and let p the intersection of l'_2 and the vertical line containing r_1. Let r'_2 be the ray on l'_2 emanating from p, r'_3 the ray parallel to r_3 and emanating from p, and r'_1 the vertical ray emanating from p. If none of r'_1, r'_2 and r'_3 intersects another circle, then r'_1, r_2, r'_3 are the desired rays. If r'_1, r'_2 and/or r'_3 intersect another circles, then we can apply the case of $m = 2,\ 3$ or 4, and hence the claim holds.

Case 5. At least two rays intersect a common circle.

Assume first the common point of r_1, r_2 and r_3 lies in a cycle C_1, that is, every ray intersects the same circle (Figure 9). Let l_1 be the vertical line containing r_1. Let p be a point on l_1 slightly above C_1, and r'_2 and r'_3 be rays emanating from p and parallel to r_2 and r_3, respectively. If none of r'_2 and r'_3 intersects another circle, then by Case 4 we can obtain the desired rays. If r'_2

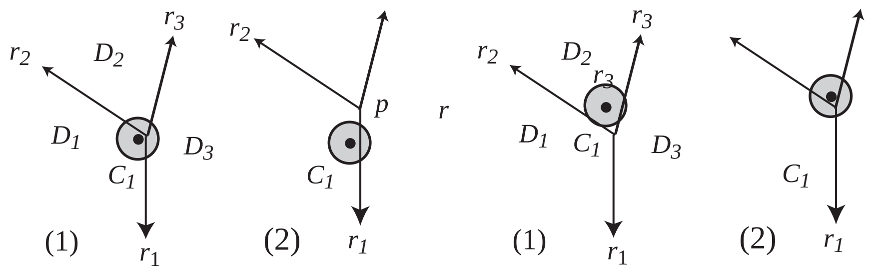

Figure 9: Rays r_i, open convex wedges D_i, and the circles C_j.

and/or r_3' intersects another circles, then we can apply the case of $m = 2, 3$ or 4 since these three rays r_1, r_2' and r_3' intersect at most four circles altogether by (c), and hence the claim holds.

Assume next that two of the three rays r_1, r_2 and r_3 intersect the same circle. In this case, we slightly move three rays so that it emanating from the new common point in the circle (Figure 9). Then we can apply the above argument. Consequently the proof of Claim 1 is complete.

B Appendix : The last stage of the proof of Theorem 3

We consider the following two cases, and complete the last stage of the proof of Theorem 3. Let $m_i = ag_i - \#(D_i')$, and $r_4' = r_1'$.

Case 1. Exactly one of $\#(D_i')$'s is less than ag_i, and the others two $\#(D_i')$'s are greater than or equal to ag_i.

Assume that $\#(D_s') < ag_s$ and $\#(D_j') \geq ag_j$ for $j \in \{1, 2, 3\} - \{s\}$. Since the number of red points on $r_s' \cup r_{s+1}'$ is greater than or equal to m_s, we can arrange m_s red points on $r_s' \cup r_{s+1}'$ to D_s''. Then the conditions of Claim 2 are satisfied about red points.

Case 2. Exactly two of $\#(D_i')$'s is less than ag_i, and the other $\#(D_i')$ is greater than or equal to ag_i.

Suppose $\#(D_2') < ag_2$, $\#(D_3') < ag_3$ and $\#(D_1') \geq ag_1$. We consider the rays in the order r_2', r_3', r_1' since we don't need to take care of D_1'. We require that m_2 red points on $r_2' \cup r_3'$ belongs to D_2'', where we assign the desired red points first from r_2' and next from r_3'. Then we require that m_3 red points of the remaining red points of r_3' and those of r_1' belongs to D_3''. This arrangement can be done since $m_2 \leq \#(r_2') + \#(r_3')$, $m_3 \leq \#(r_1') + \#(r_3')$ and $m_2 + m_3 \leq \#(r_1') + \#(r_2') + \#(r_3')$.

In the other case, we can apply the above argument since in each case, there is one wedge D_i which we don't need to take care.

Consequently the last stage of the proof of Theorem 3 is complete.