Star-uniform graphs

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Abstract

A *star-factor* of a graph is a spanning subgraph each of whose components is a star. A graph G is called *star-uniform* if all star-factors of G have the same number of components. Motivated by the minimum cost spanning tree and the optimal assignment problems, Hartnell and Rall posed an open problem to characterize all the star-uniform graphs. In this paper, we show that a graph G is star-uniform if and only if G has equal domination and matching number. From this point of view, the star-uniform graphs were characterized by Randerath and Volkmann. Unfortunately, their characterization is incomplete. By deploying Gallai-Edmonds Matching Structure Theorem, we give a clear and complete characterization of star-unform graphs.

Key words: star-factor, star-uniform, Gallai-Edmonds decomposition, factor-criticality, domination number, covering number, matching number

1 Introduction

Throughout this paper, we consider simple finite graphs, which have neither loops nor multiple edges. Unless otherwise defined, we follow [6] for terminology and definitions.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The minimum degree of G is denoted by $\delta(G)$. We denote by $Iso(G)$ the set of isolated vertices of G, and by $End(G)$ the

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set of end-vertices (i.e., vertices of degree one) of G. An edge incident with an end-vertex is called *a pendant edge*. A vertex adjacent to an end-vertex is called a *stem*, and $Stem(G)$ denotes the set of stems of G. For a subset $S \subseteq V(G)$, let $N_G(S)$ denote the neighborhood of S in G, and $\langle S \rangle_G$ denote the subgraph of G induced by S. The subgraph $G - S$ of G is obtained from G by deleting the vertices in S and all the edges incident with them.

A graph with one vertex and no edge is called a *trivial graph*. The complete graph of order n is denoted by K_n, and the complete bipartite graph with bipartite sets of order n and m is denoted by $K(n, m)$. In particular, K_1 is a trivial graph, and $K(1, m)$ is called a *star*, where $m \geq 1$. When $m \geq 2$, the vertex of degree m in a star $K(1, m)$ is called the *center*, and only one vertex in $K(1, 1)$ can be called the *center*. The cycle and the path of order n are denoted by C_n and P_n, respectively.

The *corona* $H \circ K_1$ of a graph H is the graph obtained from H by adding a pendant edge to each vertex of H. A connected graph G of order at least three is called a *generalized corona* if $V(G) = End(G) \cup Stem(G)$. A *star-factor* of a graph is a spanning subgraph each of whose components is a star. It is not hard to see that every graph without trivial components admits a star-factor. Amahashi and Kano [1] and Las Vergnas [5], independently, obtained a criterion for the existence of star-factors with size at most n, i.e., $\{K_{1,1}, \cdots, K_{1,n}\}$-factor.

A graph G is called *star-uniform* if all the star-factors of G have the same number of components. Hartnell and Rall [3] posed an open problem to characterize all the star-uniform graphs. In the same paper, they characterized star-uniform graphs with girth at least five. It is obvious that all star-factors of G have the same number of components is equivalent to the property that every star-factor of G has the same size. For example, every generalized corona is a star-uniform graph. In this paper, we give a clear and complete characterization of all the star-uniform graphs.

2 Preliminary Results

We say that a vertex subset S of a graph G *dominates* a vertex v of G if $v \in S \cup N_G(S)$, and that S is a *dominating set* of G if every vertex of G is dominated by S. The cardinality of a smallest dominating set is called the *domination number* of G and denoted by $\gamma(G)$. For extensive bibliographies regarding work on domination in graphs the reader is referred to [4]. A set $D \subseteq V(G)$ is a *covering* of G if every edge of G has at least one end in D. The *covering number* $\beta(G)$ is the cardinality of a smallest covering of G. A *matching* M of G is a set of independent edges of G. The number of edges in M is called the *size* of M and denoted by $|M|$. If G has no matching M' such that $|M'| > |M|$, then M is called a *maximum matching*, and the size of a maximum matching is called the *matching number* of G and denoted by $\nu(G)$. The order of G is denoted by $|G|$.

The following results are well-known.

Theorem 1. (see [4]) *If a graph G has no isolated vertex, then $\gamma(G) \leq \nu(G) \leq \beta(G)$.*

Theorem 2. (Ore, 1962) *If a graph G has no isolated vertex, then $\gamma(G) \leq \lfloor |G|/2 \rfloor$.*

In 1998, Randerath and Volkmann [7] characterized all graphs with equal domination and covering number. To simplify matters we denote this class of extremal graphs by $\mathcal{G}_{\gamma=\beta}$. Another different structural characterization is due to Hartnell and Rall [2], but it is a bit complicated and some of the proofs are omitted. All graphs G with domination number $\lfloor |G|/2 \rfloor$ are determined in [7, 11]. Using the two known results, Randerath and Volkmann [8] gave a characterization of all the graphs with equal domination and matching number, for abbreviation $\mathcal{G}_{\gamma=\nu}$. Unfortunately, the characterizations of the family $\mathcal{G}_{\gamma=\nu}$ with minimum degree one in [7] and [8] are incomplete. For example, the graph G shown in Figure 1 has domination number 4, but covering number and matching number are 5. So it is not a member of $\mathcal{G}_{\gamma=\beta}$ or $\mathcal{G}_{\gamma=\nu}$. However, it is included in the characterizations of $\mathcal{G}_{\gamma=\beta}$ and $\mathcal{G}_{\gamma=\nu}$ in [7, 8].

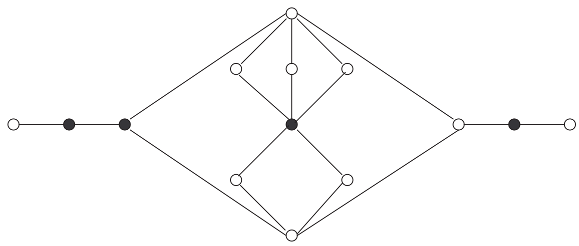

Figure 1: $\gamma(G) = 4$ and $\nu(G) = \beta(G) = 5$

In Section 3, we show that a graph G is star-uniform if and only if $G \in \mathcal{G}_{\gamma=\nu}$. In Section 4, we give a clear and complete characterization of all the star-unform graphs, i.e., all the graphs with equal domination and matching number. Since the tool in the proof of the characterization is the so-called "Gallai-Edmonds Structure Theorem", we need more definitions and notations.

A matching of G is called *perfect* if it covers all the vertices of G. A *near-perfect matching* of G is a matching that covers all but exactly one vertex of G. A graph G is said to be *factor-critical* if $G - v$ has a perfect matching for every $v \in V(G)$. Factor-criticality was first introduced by Gallai (see [6]) in 1963 and it is important in the study of matching theory. To be contrary to its apparent strong property, such graphs form a relatively rich family for study, which are the essential "building block" for Gallai-Edmonds Structure of the graphs with matchings.

Let M be a matching in G. An *M-alternating path* (or *M-alternating cycle*) in G is a path (or cycle) whose edges are alternately in M and $E(G) - M$. Let M_1 and M_2 be matchings in G and $M_1 \cup M_2$ denote the subgraph formed by the union of the two edge sets, so $V(M_1 \cup M_2) = V(M_1) \cup V(M_2)$ and $E(M_1 \cup M_2) = E(M_1) \cup E(M_2)$. The components of this subgraph are edges, alternating even cycles or alternating paths.

For a graph G, let $D(G)$ denote the set of vertices of G which are not covered by at least one maximum matching of G, $A(G)$ be the set of vertices in $V(G) - D(G)$ adjacent to at least one vertex in $D(G)$. Finally let $C(G) = V(G) - A(G) - D(G)$ (see Figure 2).

Gallai (1963) and Edmonds (1965), independently, obtained the following canonical decomposition theorem for maximum matchings in graphs. This result provides a complete structural characterization of maximum matchings in a graph.

Theorem 3 (Gallai-Edmonds Structure Theorem for Matchings)**.** *Let G be a graph and $D(G)$, $A(G)$ and $C(G)$ be the sets defined as above. Moreover, for convenience, let $D(G)$, $A(G)$ and $C(G)$ also denote the subgraphs of G induced by them. Then*
(*i*) *every component of $D(G)$ is factor-critical;*
(*ii*) *$C(G)$ has a perfect matching;*
(*iii*) *$|A(G)|$ is less than the number of components of $D(G)$; and*
(*iv*) *every maximum matching of G consists of a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and a matching which matches all the vertices of $A(G)$ with vertices in distinct components of $D(G)$.*

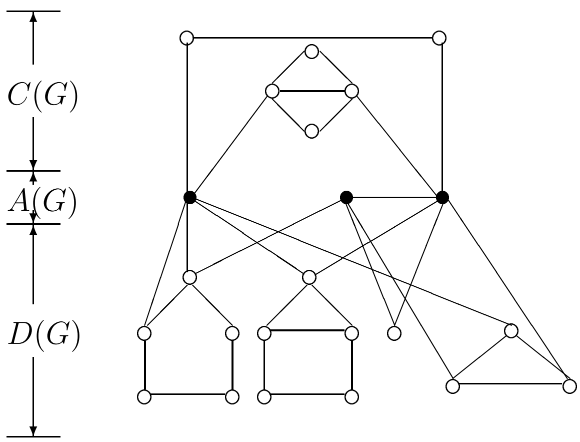

Figure 2: The Gallai-Edmonds decomposition of a graph G

3 Equivalent characterization of star-uniform graphs

Let $\mathcal{S}$ be a star-factor with the maximum number of components among all the star-factors of G. If we choose one edge from each component of $\mathcal{S}$, it yields a matching M. Conversely, suppose M is a maximum matching in G, then $G - V(M)$ is an independent set, and for each edge uv in M, u and v can not be adjacent to distinct vertices of $G - V(M)$ due to the maximality of M. For each isolated vertex x in $G - V(M)$, we add an edge $e \in E(G)$ joining

x to a vertex in $V(M)$ and obtain a star-factor with $|M|$ components. Hence we have the following proposition.

Proposition 4. *Let G be a connected graph. Then the maximum number of components of star-factors in G is equal to the number of edges of a maximum matching in G (i.e., the matching number).*

Proposition 4 shows the relationship between the maximum number of components of star-factors and the matching number. Similarly, we have a proposition to relate the minimum number of components of star-factors and the domination number.

Proposition 5. *Let G be a connected graph. Then the minimum number of components of star-factors in G is equal to the domination number $\gamma(G)$.*

Proof. Let $\mathcal{S}$ be a star-factor in G with the minimum number of components. Then all the centers in $\mathcal{S}$ form a dominating set, so $\gamma(G)$ is at most the minimum number of components of star-factors in G.

Conversely, suppose D is a dominating set of the minimum order. Then every vertex of $V(G) - D$ has at least one neighbor in D and every vertex of D has at least one neighbor in $V(G) - D$ since D is a minimum dominating set. Now we construct a bipartite graph $B = (V(G) - D, D)$ with edge set $E(B) = \{uv \mid u \in V(G) - D, v \in D \text{ and } uv \in E(G)\}$. Then B has a star-factor, which can be regarded as a star-factor of G. Since the number of components of a star-factor of B is at most $|D|$, it follows that $\gamma(G) = |D|$ is at least the minimum number of components of star-factors in G. Therefore the proposition is proved.□

Obviously, Theorem 2 is a corollary of Proposition 5. Combining Propositions 4 and 5, we are able to link star-uniform graphs with two well-studied graphic parameters - matching number and domination number.

Theorem 6. *A connected graph G is star-uniform if and only if $\nu(G) = \gamma(G)$. Moreover, every star-factor of a star-uniform graph G has $\gamma(G)$ components.*

4 Characterization of all star-uniform graphs

First, we present a lemma which shows the basic idea for dealing with star-uniform graphs.

Lemma 7. ([3, 9]) *Let G be a connected star-uniform graph, and H be a spanning subgraph of G without isolated vertices. Then H is also a star-uniform graph and $\gamma(H) = \gamma(G) = \nu(G) = \nu(H)$. In particular, each component of H is star-uniform.*

From Lemma 7, we can deduce the following lemma easily, which will be used several times in the proof of our main result.

Lemma 8. *Let G be a connected star-uniform graph, and H be a spanning subgraph of G without isolated vertices, and H_1 be a component of H and $X \subseteq V(H_1)$. Suppose that both $H_1 - X$ and $\langle (G-H_1) \cup X\rangle_G$ contain no isolated vertices, and $\gamma(\langle (G-H_1)\cup X\rangle_G) = \gamma(G-H_1)$, where $G - H_1$ denotes $G - V(H_1)$ or $V(G) - V(H_1)$. Then $H_1 - X$ is star-uniform and $\gamma(H_1 - X) = \gamma(H_1)$.*

Proof. By Lemma 7, H_1 is star-uniform. By Theorem 1 and the assumption of this lemma, we have

$$\begin{aligned}\gamma(G) &\leq \gamma(\langle (G-H_1)\cup X\rangle_G) + \gamma(H_1 - X)\\ &= \gamma(G-H_1) + \gamma(H_1 - X)\\ &\leq \nu(G-H_1) + \nu(H_1 - X)\\ &\leq \nu(G-H_1) + \nu(H_1).\end{aligned}$$

On the other hand, it follows

$$\gamma(G) = \nu(G) \geq \nu(G-H_1) + \nu(H_1).$$

Hence

$$\gamma(H_1 - X) = \nu(H_1 - X) = \nu(H_1) = \gamma(H_1)$$

as H_1 is star-uniform. Therefore $H_1 - X$ is star-uniform and the lemma holds. □

The following results are given by Randerath and Volkmann [8], which can also be proved by using Gallai-Edmonds Structure Theorem.

Theorem 9. ([8]) *Let $G = (X, Y)$ be a connected* bipartite *graph with $|X| \leq |Y|$ and $\delta(G) \geq 2$. Then G is star-uniform if and only if G possesses the following properties:*
(i) $\nu(G) = \gamma(G) = |X|$;
(ii) for any two distinct vertices x_1, x_2 of X that are adjacent to a common vertex of Y, there exist two distinct vertices y_1 and y_2 in Y such that $N_G(y_i) = \{x_1, x_2\}$ for $i = 1, 2$.

Based on Randerath and Volkmann's original characterization of non-bipartite star-uniform graphs with $\delta(G) \geq 2$, it is not hard to determine the family of all possible star-uniform graphs.

Theorem 10. ([8]) *Let G be a connected* non-bipartite *graph with $\delta(G) \geq 2$. Then G is star-uniform if and only if G is one of the nine graphs shown in Figure* 3.

Based on Theorems 9 and 10, we characterize all the star-uniform graphs with *minimum degree one.*

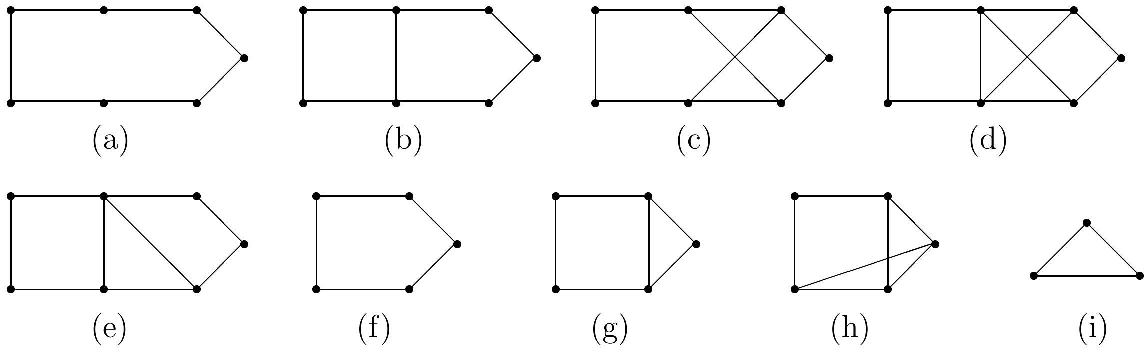

Figure 3. Non-bipartite star-uniform graphs with minimum degree at least two

Theorem 11. *Let G be a connected star-uniform graph with minimum degree one. Then G is K_2 or a generalized corona, or has the following structure: every component H of $G - (End(G) \cup Stem(G))$ is one of the following graphs.*

(i) *a trivial graph;*
(ii) *a star-uniform graph with minimum degree at least two;*
(iii) *a bipartite graph with minimum degree one, and $H - End(H)$ is a trivial graph or a connected bipartite star-uniform graph with minimum degree at least two such that $\nu(H - End(H)) = \nu(H)$.*

Moreover, if H satisfies (ii) *or* (iii)*, then*

$$\gamma(H - X) = \gamma(H) \qquad \text{for all} \quad X \subseteq V(H) \cap N_G(Stem(G)). \tag{1}$$

Conversely, if a connected graph G is K_2 or a generalized corona, or possesses the above structure including (1)*, then G is a star-uniform graph with minimum degree one.*

For a graph G, let $\omega(G)$ denote the number of components of G. To prove that a graph G is not star-uniform, we usually show that G contains two star-factors with different number of components. We now prove our main theorem.

Proof of Theorem 11. We first show the second part of Theorem 11. Since K_2 and every generalized corona are star-uniform, we may assume that G is not such a graph but possesses the given structure. So G has order at least three, and $End(G) \cap Stem(G) = \emptyset$. It follows that every maximum matching of G covers all vertices in $Stem(G)$. Moreover, there exists a maximum matching M such that every edge of M incident with a vertex of $Stem(G)$ is a pendant edge of G. So

$$\nu(G) = |Stem(G)| + \sum_H \nu(H), \tag{2}$$

where H runs over all non-trivial components of $G - (End(G) \cup Stem(G))$.

We next estimate the domination number of G.

Claim 1. Every non-trivial component H of $G - (End(G) \cup Stem(G))$ satisfies $\nu(H) = \gamma(H)$.

Let H be a non-trivial component of $G - (End(G) \cup Stem(G))$. If H is star-uniform, then $\nu(H) = \gamma(H)$ by Theorem 6. So we may assume that H is a bipartite graph with minimum degree one given in (iii). If $H - End(H)$ is a trivial graph, then H is a star, and so $\nu(H) = \gamma(H) = 1$. If $H - End(H)$ is a connected bipartite star-uniform graph with minimum degree at least two, then $\nu(H) = \nu(H - End(H)) = \gamma(H - End(H))$ by (iii) and Theorem 6. Since $End(H) \subseteq N_G(Stem(G))$, we have $\gamma(H - End(H)) = \gamma(H)$ by condition (1). Hence $\nu(H) = \gamma(H)$, and the claim is proved.

It is clear that there exists a dominating set L of order $\gamma(G)$ in G that includes $Stem(G)$. For every non-trivial component H of $G - (End(G) \cup Stem(G))$, let X_H denote the set of vertices of H dominated by $Stem(G)$. Then $H - X_H$ is dominated by $V(H) \cap L$, and thus we have

$$\begin{aligned}
\gamma(G) &= |L| = |Stem(G)| + \sum_H |L \cap V(H)| \\
&\geq |Stem(G)| + \sum_H \gamma(\langle (H - X_H) \cup (L \cap X_H) \rangle_G) \\
&= |Stem(G)| + \sum_H \gamma(H) \qquad \text{(by (1))} \\
&= |Stem(G)| + \sum_H \nu(H) = \nu(G), \qquad \text{(by Claim 1 and (2))}
\end{aligned}$$

where H runs over all non-trivial components of $G - (End(G) \cup Stem(G))$. Therefore $\gamma(G) = \nu(G)$ by Theorem 1. Consequently G is star-uniform by Theorem 6.

We now prove that a connected star-uniform graph G with $\delta(G) = 1$ is K_2 or a generalized corona, or has the structure given in Theorem 11. We may assume that G is neither K_2 nor a generalized corona, and hence G has order at least three. Let $\mathcal{S}$ denote a star-factor of G that consists of a star-factor of the induced subgraph

$$\left\langle\, End(G) \ \cup \ Stem(G) \ \cup \ Iso(G - (End(G) \cup Stem(G))) \,\right\rangle_G$$

and a star-factor of each non-trivial component of $G - (End(G) \cup Stem(G))$. We may assume that every vertex of $Stem(G)$ is the center of a component in $\mathcal{S}$.

Let H denote any non-trivial component of $G - (End(G) \cup Stem(G))$. Then H is a star-uniform graph by Lemma 7, and $\mathcal{S}_H = \mathcal{S} \cap H$ is a star-factor of H. It follows from Theorem 6 that $\omega(\mathcal{S}_H) = \gamma(H)$. Let

$$U = V(H) \cap N_G(Stem(G)).$$

Then U is non-empty as G is connected, and U is a proper subset of $V(H)$ since otherwise removing $\mathcal{S}_H$ from $\mathcal{S}$ and for each vertex u of H, by adding an edge joining u to a vertex in $Stem(G)$ to $\mathcal{S}$, we get another star-factor of G with $\omega(\mathcal{S}) - \omega(\mathcal{S}_H)$ components.

If $\delta(H) \geq 2$, then H is given in (ii). Hence we may assume $\delta(H) = 1$, which implies H has an end-vertex. Notice that every end-vertex of H is adjacent to some vertices of $Stem(G)$ since it is not contained in $End(G)$.

Claim 2. H has no perfect matchings.

Assume that H has a perfect matching. Since H is star-uniform, $\mathcal{S}_H$ is a perfect matching of H. If $|V(H)| = 2$, then $H = K_2$ and $U = V(H)$, which contradicts the fact that U is a proper subset of $V(H)$. Hence $|V(H)| \geq 4$, and H has a path $P_4 = (x_1, x_2, x_3, x_4)$ such that x_1x_2 and x_3x_4 are edges of $\mathcal{S}_H$ and a vertex y of $Stem(G)$ adjacent to x_1 in G. Then by removing x_1x_2 from $\mathcal{S}$ and by adding two edges x_2x_3 and x_1y to it, we obtain another star-factor of G with $\omega(\mathcal{S}) - 1$ components, a contradiction. Hence H has no perfect matchings.

Let M be a maximum matching of H, and let $A(H)$, $C(H)$ and $D(H)$ be the vertex sets defined in Gallai-Edmonds Structure Theorem. Also $A(H)$, $C(H)$ and $D(H)$ denote the subgraphs of H induced by them. We may assume that the star-factor $\mathcal{S}_H$ satisfies the following: (i) $\mathcal{S}_H$ includes M; (ii) $C(H) \cap \mathcal{S}_H$ is a perfect matching of $C(H)$; (iii) each vertex of $A(H)$ is the center of a star of $\mathcal{S}_H$, whose edges join $A(H)$ to $D(H)$; and (iv) $\mathcal{S}_H \cap D(H)$ is a collection of a near-perfect matching of each component of $D(H)$.

Claim 3. $C(H) = \emptyset$.

Assume $C(H) \neq \emptyset$. Let C be a component of $C(H)$. Since H is connected, there exists an edge $uv \in \mathcal{S}_H \cap E(C)$ such that u is adjacent to a vertex x in $A(H)$, and v is adjacent to a vertex y in $(C - u) \cup A(H) \cup Stem(G)$. By deleting edge uv from $\mathcal{S}$ and by adding two edges ux and vy to $\mathcal{S}$, we obtain another star-factor of G with $\omega(\mathcal{S}) - 1$ components, a contradiction.

Claim 4. $A(H)$ is an independent set.

Suppose that there exists an edge uv in $A(H)$. Let T_u be a star in $\mathcal{S}_H$ with center u. For each end-vertex x in T_u, where x is a vertex of a component D of $D(H)$, we perform the following operation. If D is singleton, then x is adjacent to another vertex y in $A(H) \cup Stem(G)$. In this case, we remove the edge ux from $\mathcal{S}$ and add the edge xy to $\mathcal{S}$. If D is non-trivial, then $\delta(D) \geq 2$ as D is factor-critical. So x is adjacent to another vertex z in D, then we remove the edge ux from $\mathcal{S}$ and add the edge xz to it. Finally, by adding the edge uv to $\mathcal{S}$, we obtain another star-factor of G with $\omega(\mathcal{S}) - 1$ components since v is the center of a component of $\mathcal{S}$, a contradiction.

Claim 5. H is a connected bipartite graph with bipartition $A(H)$ and $D(H)$.

If $A(H) = \emptyset$, then $H = D(H)$ is factor-critical, which implies $\delta(H) \geq 2$. But this is contrary to $\delta(H) = 1$. Hence $A(H) \neq \emptyset$. Suppose a component D in $D(H)$ is non-trivial. Then there exists a star T_v in $\mathcal{S}_H$ with end-vertex x in D and center v in $A(H)$. If T_v has another end-vertex z, which is in another component of $D(H)$, we remove the edge vz from $\mathcal{S}$ and add another edge zt to $\mathcal{S}$, where $t \in A(H) \cup D(H) \cup Stem(G)$. Now we obtain another star-factor with the same number of components as $\mathcal{S}$. So, without loss of generality, we may assume that $V(T_v) = \{v, x\}$. Take $y \in N_H(x) \cap V(D)$ and let M_y be a perfect matchings of $D - y$. Then since $\mathcal{S}_H \cap E(D)$ is a perfect matching of $D - x$, it follows that $(\mathcal{S}_H - E(D)) \cup M_y \cup xy$ is a star-factor of H with $\omega(\mathcal{S}_H) - 1$ components, a contradiction. Therefore every component of $D(H)$ is a trivial graph. By Claims 3 and 4, H is a bipartite graph with bipartition $A(H)$ and $D(H)$.

Claim 6. $\gamma(H) = |A(H)|$.

By Gallai-Edmonds Structure Theorem, $1 \leq |A(H)| < |D(H)|$ and every maximum matching of H covers all vertices in $A(H)$. So $\gamma(H) = |A(H)|$ by Theorem 6.

Claim 7. If $|A(H)| = 1$, then H is a star $K(1, m)$ for $m \geq 2$ and the center of H is not adjacent to any vertex of $Stem(G)$.

If $|A(H)|$=1, then $H = K(1, m)$ for $m \geq 2$ as H is not K_2. If the center of H is adjacent to a vertex in $Stem(G)$, then we remove $\mathcal{S}_H$ from $\mathcal{S}$, and for each vertex in H, add an edge joining it to a vertex in $Stem(G)$. Since every vertex of $Stem(G)$ is a center of some star in $\mathcal{S}$, it yields a star-factor of G with $\omega(\mathcal{S}) - 1$ components appears, a contradiction.

Claim 8. If $|A(H)| \geq 2$, then $\nu(H - End(H)) = \nu(H)$ and $H - End(H)$ is a connected bipartite star-uniform graph with minimum degree at least two.

If a vertex v of $A(H)$ has degree one in H, then for a vertex u of $D(H)$ adjacent to v, $H - u$ has no matching covering v, which is a contradiction (see Theorem 3). Hence every vertex of $A(H)$ has degree at least two in H. Assume $|A(H)| \geq 2$. By the above property, $End(H) \subset D(H)$ and $H - End(H)$ is a non-trivial connected bipartite graph.

Every vertex of $Stem(G)$ is assumed to be the center of a star in $\mathcal{S}$ and every vertex of $End(H)$ is adjacent to a vertex in $Stem(G)$, so the set of all centers of stars in $\mathcal{S}_G$ that are not in H is a minimum dominating set of graphs $G - H$ and $\langle (G - H) \cup End(H) \rangle_G$, i.e. $\gamma(\langle (G - H) \cup End(H) \rangle_G) = \gamma(G - H)$. Then $H - End(H)$ is a star-uniform graph and $\gamma(H - End(H)) = \nu(H - End(H)) = \nu(H) = \gamma(H) = |A(H)|$ by Lemma 8.

Now it suffices to show that $\delta(H - End(H)) \geq 2$. Suppose $\delta(H - End(H)) = 1$. Let v be an end-vertex of $H - End(H)$, that is, all neighbors of v except one, say u, in H are end-vertices of H. Then $v \in A(H)$, and let T_v be a star with center v in $\mathcal{S}_H$. If $u \in T_v$, let ux be an edge in H and T_x be a star with center x in $\mathcal{S}_H$. Remove T_v and T_x from $\mathcal{S}$, and add edges uv and ux to $\mathcal{S}$. Moreover, for each vertex y in $T_v \cup T_x - \{v, x, u\}$, if $y \in End(H)$, add an edge joining y to a vertex in $Stem(G)$, otherwise joining y to a vertex in $A(H) - \{v, x\}$. Then we obtain another star-factor of G with $\omega(\mathcal{S}) - 1$ components, a contradiction. If $u \notin T_v$, then there exists a $z \in A(H)$ such that $u \in T_z$. By deleting two components T_v and T_z from $\mathcal{S}$ and adding a star $\{zu, uv\}$ and some edges to $\mathcal{S}$ as the above, we can similarly obtain a star-factor with $\omega(\mathcal{S}) - 1$ components. Thus we prove the claim.

In the following, for a component H of $G - (End(G) \cup Stem(G))$ satisfying the condition (ii) or (iii), we show that $\gamma(H - X) = \gamma(H)$ for all $X \subseteq U = V(H) \cap N_G(Stem(G))$.

Claim 9. If H is a non-bipartite star-uniform graph with minimum degree at least two, then $\gamma(H - X) = \gamma(H)$ for all $X \subseteq U$.

Suppose that H is a graph shown in Figure 3, which is factor-critical. Assume that there exists a subset $X \subseteq U$ such that $\gamma(H - X) \neq \gamma(H)$. Then $H - X$ contains isolated vertices by Lemma 8. Let Y be the set of isolated vertices of $H - X$, and B be a bipartite graph with vertex set $Y \cup N_H(Y)$ and edge set consisting of the edges between Y and $N_H(Y)$ in H. Then for each subset $Y' \subseteq Y$, we have $|N_B(Y')| > |Y'|$ since H is factor-critical. By Hall's Theorem, B has a matching that covers Y, and so B has a star-factor $\mathcal{B}$ with $|Y|$ components by Proposition 4. Since $H - (X \cup Y)$ has no isolated vertices, it has a star-factor with $\gamma(H - (X \cup Y))$ components by Proposition 5. Now removing $\mathcal{S}_H$ from $\mathcal{S}$, joining each vertex in $X - N_H(Y)$ to a vertex in $Stem(G)$ and adding the above two star-factors of $H - (X \cup Y)$ and B to $\mathcal{S}$, we get a star-factor of G which has $\omega(\mathcal{S}) - \gamma(H) + \gamma(H - (X \cup Y)) + |Y| = \omega(\mathcal{S}) - \gamma(H) + \gamma(H - X)$ components, a contradiction. Hence $\gamma(H - X) = \gamma(H)$.

Claim 10. If $H = (A(H), D(H))$ is a connected bipartite star-uniform graph with minimum degree at least two and $|A(H)| < |D(H)|$, then, for all $X \subseteq U$, $H - X$ is a connected graph with minimum degree at least two such that $\gamma(H - X) = \gamma(H)$.

Note that every vertex $u \in U$ is contained in $D(H)$; otherwise, $A(H) - u$ is a dominating set of $H - u$ and so G contains a star-factor having at most $\omega(\mathcal{S}) - 1$ components by Proposition 5 and $u \in N_G(Stem(G))$, a contradiction. Denote the vertices of X by $v_1, v_2, \cdots, v_{|X|}$. Then $H' = H - v_1$ is a connected star-uniform graph and $\gamma(H') = \gamma(H) = |A(H)|$ by Lemma 8 and Theorem 9. Suppose x_1 and x_2 are two neighbors of v_1 in H. Then by Theorem 9, there exists a vertex $y_1 \neq v_1$ in $A(H)$ such that $N_H(y_1) = \{x_1, x_2\}$. Moreover, there is another vertex y_2 distinct from v_1 and y_1 such that $N_H(y_2) = \{x_1, x_2\}$; otherwise, $A(H) \cup \{y_1\} - \{x_1, x_2\}$ is a dominating set for H' of order $|A(H)| - 1$, which is contradictory to $\gamma(H') = |A(H)|$. Hence $\delta(H') \geq 2$. We apply the previous procedure to H' generating $H'' = H' - v_2$, and repeating this procedure until we get the graph $H - X$, which is a connected bipartite star-uniform graph with minimum degree at least two and $\gamma(H - X) = \gamma(H) = |A(H)|$.

Claim 11. If $H = (A(H), D(H))$ is a connected bipartite star-uniform graph with minimum degree one and $|A(H)| < |D(H)|$, then $\gamma(H - X) = \gamma(H)$ for all $X \subseteq U$.

If $H - End(H)$ is a trivial graph, then by Claim 7, the claim holds. So we may assume $H - End(H)$ is a connected bipartite star-uniform graph with minimum degree at least two by Claim 8. We can similarly show, as in Claim 10, that $U \subseteq D(H)$. By Claims 6 and 10, $H' = H - (End(H) \cup X)$ is a connected bipartite star-uniform graph with minimum degree at least two, and $A(H)$ is a minimum dominating set for both H and H'. $H - X = \langle V(H') \cup (End(H) - X) \rangle_H$ is obtained by adding some pendant edges incident with $A(H)$ to H', thus $A(H)$ is also a minimum dominating set for $H - X$, i.e., $\gamma(H - X) = \gamma(H') = \gamma(H)$.

Consequently, the proof is complete. □

From the proof of Theorem 11 and combining Theorem 9, we are able to give a much clearer characterization for all the star-uniform graphs with minimum degree one.

Theorem 12. *Let G be a connected graph with $\delta(G) = 1$. Then G is star-uniform if and only if G is K_2 or a generalized corona, or every component H of $G - (End(G) \cup Stem(G))$ is one of the following:*

(i) H is a trivial graph;
(ii) H is a connected bipartite graph with bipartition X and Y, where $1 \leq |X| < |Y|$. Let $U = V(H) \cap N_G(Stem(G))$. Then $\emptyset \neq U \subseteq Y$ and for any two distinct vertices x_1, x_2 of X that are adjacent to a common vertex of Y, there exist two distinct vertices y_1 and y_2 in $Y - U$ such that $N_H(y_i) = \{x_1, x_2\}$, for $i = 1, 2$;
(iii) H is a graph isomorphic to (f), (g), (h) or (i) shown in Figure 3, and $\gamma(H - X) = \gamma(H)$ for all $\emptyset \neq X \subseteq U \subset V(H)$, where $U = V(H) \cap N_G(Stem(G))$.

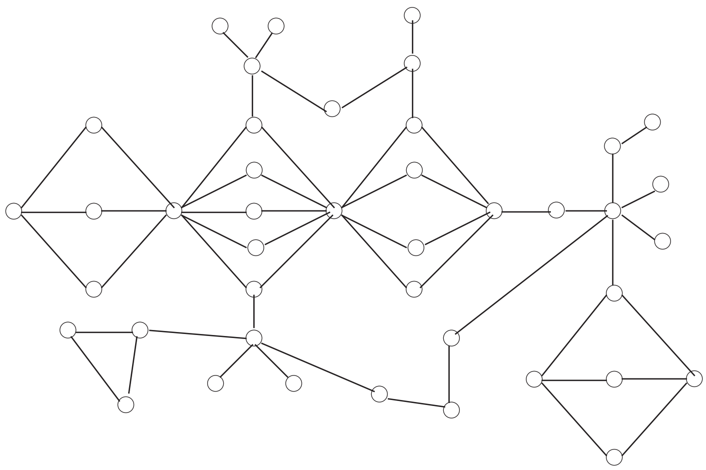

Figure 4: A star-uniform graph with minimum degree one.

As an example, we present a star-uniform graph with minimum degree one in Figure 4.

5 Conclusion

An *edge-weighting* of a graph G is a function $w : E(G) \longrightarrow \mathbb{N}^+$, where $\mathbb{N}^+$ is the set of positive integers. The *weight* of a star-factor $\mathcal{S}$ of G under w is the sum of all weight on edges of $\mathcal{S}$, i.e., $w(\mathcal{S}) = \Sigma_{e \in E(\mathcal{S})} w(e)$. So characterization of all the star-uniform graphs is a special case of the following question, which was proposed by Hartnell and Rall [3].

Question 13. *Which graph G has a non-constant edge-weighting w of G such that every star-factor of G has the same weight?*

Wu and Yu [10] characterized this family of graphs when girth is at least five. It seems that the structures of the graphs with girth less than five are much more complicated. New ideas and tools are required to identify all such graphs.

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