1 2 3

Elementary graphs with respect to f-parity factors

Mikio Kano[1], Gyula Y. Katona[2], Jácint Szabó[3]

[1] Department of Computer and Information Sciences, Ibaraki University
Hitachi, Ibaraki 316-8511, Japan e-mail: kano@mx.ibaraki.ac.jp

[2] Department of Computer and Information Sciences, Technical University of Budapest
Budapest, P.O.Box 91, 1521 Hungary e-mail: kiskat@cs.bme.hu

[3] Egerváry Research Group (EGRES), Department of Operations Research, Eötvös University,
Budapest, Pázmány P. s. 1/C, Hungary H-1117

Abstract. This note concerns the f-parity subgraph problem, i.e., we are given an undirected graph G and a positive integer value function $f : V(G) \to \mathbb{N}$, and our goal is to find a spanning subgraph F of G with $\deg_F \le f$ and minimizing the number of vertices x with $\deg_F(x) \not\equiv f(x) \mod 2$. First we prove a Gallai–Edmonds type structure theorem and some other known results on the f-parity subgraph problem, using an easy reduction to the matching problem. Then we use this reduction to investigate barriers and elementary graphs with respect to f-parity factors, where an elementary graph is a graph such that the union of f-parity factors form a connected spanning subgraph.

Key words. f-parity factor, Parity factor, $(1, f)$-odd factor, Odd factor, Elementary graph, Structure theorem.

1. Introduction

In this paper we deal with a special case of the *degree prescribed subgraph problem*, introduced by Lovász [10]. This is as follows. Let G be an undirected graph and let $\emptyset \ne \mathcal{H}_v \subseteq \mathbb{N} \cup \{0\}$ be a degree prescription for each $v \in V(G)$. For a spanning subgraph F of G, define $\delta_{\mathcal{H}}^F(v) = \min\{|\deg_F(v) - i| : i \in \mathcal{H}_v\}$, and let

$$\delta_{\mathcal{H}}^F = \sum_{v \in V(G)} \delta_{\mathcal{H}}^F(v) \qquad \text{and} \qquad \delta_{\mathcal{H}}(G) = \min_F \delta_{\mathcal{H}}^F,$$

where the minimum is taken over all the spanning subgraphs F of G. A spanning subgraph F is called *$\mathcal{H}$-optimal* if $\delta_{\mathcal{H}}^F = \delta_{\mathcal{H}}(G)$, and it is an *$\mathcal{H}$-factor* if $\delta_{\mathcal{H}}^F = 0$, i.e., if $\deg_F(v) \in \mathcal{H}_v$ for all $v \in V(G)$. The *degree prescribed subgraph problem* is to determine the value of $\delta_{\mathcal{H}}(G)$.

An integer h is called a *gap* of $\mathcal{H} \subseteq \mathbb{N} \cup \{0\}$ if $h \notin \mathcal{H}$ but $\mathcal{H}$ contains an element less than h and an element greater than h. Lovász [12] gave a structural description on the degree

* Research is supported by Grant-in-Aid for Scientific Research of Japan

† Research is partially supported by the Hungarian National Research Fund and by the National Office for Research and Technology (Grant Number OTKA 67651 and 78439)

‡ Research is supported by OTKA grants T037547, K60802, TS 049788 and by European MCRTN Adonet, Contract Grant No. 504438.

prescribed subgraph problem in the case where $\mathcal{H}_v$ has no two consecutive gaps for all $v \in V(G)$. He showed that the problem is NP-complete without this restriction. The first polynomial time algorithm was given by Cornuéjols [2]. It is implicit in Cornuéjols [2] that this algorithm implies a Gallai–Edmonds type structure theorem for the degree prescribed subgraph problem (first stated in [14]), which is similar to – but in some respects much more compact than – that of Lovász'.

The case when an odd value function $f : V(G) \to \mathbb{N}$ is given and $\mathcal{H}_v = \{1, 3, 5, \ldots, f(v)\}$ for all $v \in V(G)$, is called the $(1, f)$*-odd subgraph problem*. We denote $\delta_{\mathcal{H}}(G) = \delta_f(G)$. This problem is much simpler than the general case due to the fact that only parity requirements are posed. The $(1, f)$-odd subgraph problem was first investigated by Amahashi [1], who gave a Tutte type characterization of graphs having a $[1, n]$-odd factor, where $n \geq 1$ is an odd integer. A Tutte type theorem for a general odd value function f was proved by Cui and Kano [3], and then a Berge type minimax formula on $\delta_f(G)$ by Kano and Katona [7]. A Gallai–Edmonds type theorem on the $(1, f)$-odd subgraph problem was given in [8] and [14].

In this note we show a new approach to the $(1, f)$-odd subgraph problem. Actually, it is worth allowing f to have also even values and defining $\mathcal{H}_v$ equal to $\{1, 3, \ldots, f(v)\}$ or $\{0, 2, \ldots, f(v)\}$, according to the parity of $f(v)$. We call this the f*-parity subgraph problem*. We show an easy reduction of the f-parity subgraph problem to the matching problem, and we show that this reduction easily yields the above mentioned Gallai–Edmonds and Berge type theorems on the f-parity subgraph problem. Then we investigate barriers with respect to the f-parity subgraph problem. As another application, we explore the graphs for which the edges belonging to some f-parity factor form a connected spanning subgraph. We call such a graph an f*-elementary graph*. We generalize some results on matching elementary graphs (proved by Lovász [11]) to f-elementary graphs. An attempt putting the f-parity subgraph problem into the general context of graph packing problems can be found in [15].

The f-parity subgraph problem can be reduced to the $(1, f)$-odd subgraph problem by the following construction: for every vertex $v \in V(G)$ with $f(v)$ even, connect a new vertex w_v to v in G, define $f(w_v) = 1$ and increase $f(v)$ by 1. Now $\delta_f(G)$ remains the same.

To avoid minor technical difficulties we assume that $f > 0$. Almost all results of the paper would hold without this restriction, too. Note that if G is a nontrivial f-elementary graph then $f > 0$ always holds.

The constant function $f \equiv 1$ is simply denoted by $\mathbf{1}$. For $X \subseteq V(G)$, let $G[X]$ be the subgraph of G induced by X, $\Gamma(X) = \{y \in V(G) - X : \exists x \in X, xy \in E(G)\}$, $f(X) = \sum\{f(x) : x \in X\}$ and χ_X denote the function with $\chi_X(x) = 1$ if $x \in X$ and $\chi_X(x) = 0$ otherwise. The number of components of a graph G is denoted by $\omega(G)$. The cardinality of a set X is written $|X|$, and $\mathbb{N}$ denotes the set of natural numbers. Two isomorphic graphs H and K are denote by $H \simeq K$. The graphs are finite and undirected, and have no loops, but may have multiple edges.

2. Reduction to matchings

In this section we show a reduction of the f-parity subgraph problem to matchings, which will then be used to prove the Gallai–Edmonds type structure theorem on the f-parity subgraph problem. The auxiliary graph we use is defined below.

Definition *For a graph G and a function $f : V(G) \to \mathbb{N}$, define G^f to be the following undirected graph. Replace every vertex $v \in V(G)$ by a new complete graph on $f(v)$ vertices, denoted by $K_{f(v)}$, and for each pair of vertices $u, v \in V(G)$ adjacent in G, add all possible $f(u)f(v)$ edges between $K_{f(u)}$ and $K_{f(v)}$. Let $V_{f(v)} = V(K_{f(v)})$.*

Observe that $G^{\mathbf{1}} = G$, $|V(G^f)| = f(V(G))$ and that $V_{f(v)} \neq \emptyset$ for every $v \in V(G)$. There is a strong connection between the maximum matchings of G^f and the optimal f-parity subgraphs of G. Note that the size of a maximum matching of G is just $|V(G)| - \delta_{\mathbf{1}}(G)$.

Lemma 1 *For every optimal f-parity subgraph F of G, there exists a matching M of G^f such that $|V(M)| = f(V(G)) - \delta_f^F$. Moreover, if $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$ for a vertex $w \in V(G)$ then M can be chosen to miss a prescribed vertex $x \in V_{f(w)}$. On the other hand, for every maximum matching M of G^f there exists a spanning subgraph F of G such that $\delta_f^F = f(V(G)) - |V(M)|$. Moreover, if M misses a vertex in $V_{f(w)}$ for some $w \in V(G)$ then F can be chosen such that $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$. In particular, $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$.*

Proof. Let F be an optimal f-parity subgraph of G. Assume $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$. If $\deg_F(u) > f(u)$ for some $u \in V(G)$ then clearly $\delta_f^{F'} \le \delta_f^F$ for the spanning subgraph F' that is obtained from F by deleting an edge e incident to u. As F is f-parity optimal, e is not adjacent to w, so $\deg_{F'}(w) = \deg_F(w)$. Hence we assume that $\deg_F(v) \le f(v)$ for every vertex v. A matching M of G^f is obtained from F as follows. For every edge xy of F, M contains exactly one edge joining $K_{f(x)}$ to $K_{f(y)}$. Then for every $K_{f(v)}$, if $\deg_F(v) \equiv f(v)$, then M contains edges that covers all the remaining $f(v) - \deg_F(v)$ vertices of $K_{f(v)}$, and otherwise M contains edges covering all the remaining vertices but one vertex. Therefore a matching M of G^f misses exactly δ_f^F vertices. For a vertex w with $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$, M can be chosen to miss a prescribed vertex in $V_{f(w)}$.

For the second part, let M be a maximum matching of G^f. If M contains two edges between $K_{f(x)}$ and $K_{f(y)}$ for some $x, y \in V(G)$, then replace them by two edges, one inside $K_{f(x)}$ and the other one inside $K_{f(y)}$. Thus we may assume that M contains at most one edge between $K_{f(u)}$ and $K_{f(v)}$ for all distinct $u, v \in V(G)$. By contracting each $K_{f(u)}$ to one vertex u, we get a spanning subgraph F of G with $\delta_f^F = f(V(G)) - |V(M)|$. Moreover, $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$ in the case that M misses a vertex in $K_{f(w)}$.

We define critical graphs with respect to the f-parity subgraph problem as in the matching case. If $f = \mathbf{1}$ the graphs defined below are called *factor-critical*.

Definition *Given a graph G and a function $f : V(G) \to \mathbb{N}$, G is called f-critical if for every $w \in V(G)$ there exists a spanning subgraph F of G such that $\deg_F(w) \in \{\dots, f(w)-3, f(w)-1\}$ and $\deg_F(v) \in \{\dots, f(v)-2, f(v)\}$ for all $v \in V(G) - \{w\}$.*

By Lemma 1, G is f-critical if and only if G^f is factor-critical. The Gallai–Edmonds structure theorem for the f-parity subgraph problem follows from the classical Gallai–Edmonds theorem easily. We cite this latter result below.

Theorem 2. (Gallai, Edmonds [4], [5], [6]) *Let D consist of those vertices of a graph G which are missed by some maximum matching of G, let $A = \Gamma(D)$ and $C = V(G) - (D \cup A)$. Then*

1. *every component of $G[D]$ is factor-critical,*

2. $|\{K : K$ *is a component of* $G[D]$ *adjacent to* $A'\}| \geq |A'| + 1$ *for all* $\emptyset \neq A' \subseteq A$,
3. $\delta_{\mathbf{1}}(G) = \omega(G[D]) - |A|$,
4. $G[C]$ *has a perfect matching.*

A direct generalization of the above result is the version for the f-parity subgraph problem.

Theorem 3. ([8], [14]) *Let* G *be a graph and* $f : V(G) \to \mathbb{N}$ *be a function. Let* $D_f \subseteq V(G)$ *consist of those vertices* v *for which there exists an optimal* f*-parity subgraph* F *of* G *with* $\deg_F(v) \in \{\ldots, f(v)-3, f(v)-1\}$. *Let* $A_f = \Gamma(D_f)$ *and* $C_f = V(G) - (D_f \cup A_f)$. *Then*

1. every component of $G[D_f]$ *is* f*-critical,*
2. $|\{K : K$ *is a component of* $G[D_f]$ *adjacent to* $A'\}| \geq f(A') + 1$ *for all* $\emptyset \neq A' \subseteq A_f$,
3. $\delta_f(G) = \omega(G[D_f]) - f(A_f)$,
4. $G[C_f]$ *has an* f*-parity factor.*

The above Theorem 3 follows immediately from the next Lemma 4 and Lemma 1.

Lemma 4 *Consider the Gallai–Edmonds decomposition* $D \cup A \cup C$ *of* G^f *and the decompositin* $D_f \cup A_f \cup C_f$ *of* G *given in Theorem 3. Then for* $X = D, A, C$, *it holds that* $X_f = \{v \in V(G) : V_{f(v)} \subseteq X\}$.

Proof. Assume $V_{f(v)}$ meets D. Then there exists a maximum matching M of G^f that misses exactly one vertex of $V_{f(v)}$. By Lemma 1, for any vertex z of $V_{f(v)}$, there exsists a maximum matching of G^f that misses z, and hence $V_{f(v)} \subseteq D$. Thus $D_f = \{v \in V(G) : V_{f(v)} \subseteq D\}$ by Lemma 1. By the construction of G^f, if $V_{f(v)}$ meets A, then $V_{f(v)} \subseteq A$, and so $A_f = \{v \in V(G) : V_{f(v)} \subseteq A\}$. Therefore the lemma also holds for $X = C$.

From Theorem 3 the Berge type minimax formula on the f-parity subgraph problem follows easily.

Definition *A component* K *of* G *is called* f*-odd or* f*-even when* $f(V(K))$ *is odd or even. Let* f*-odd*(G) *denote the number of* f*-odd components of* G. *Let*

$$\mathrm{def}_f(Y) = f\text{-}odd(G - Y) - f(Y) \qquad \text{for} \qquad Y \subseteq V(G).$$

Theorem 5. [7] *For a graph* G *and a function* $f : V(G) \to \mathbb{N}$, *it follows that*

$$\delta_f(G) = \max\{\mathrm{def}_f(Y) : Y \subseteq V(G)\}.$$

Proof. Let $Y \subseteq V(G)$. Since an f-odd component K of $G - Y$ has no f-parity factor, it follows that $\delta_f^F \geq f\text{-}odd(G - Y) - f(Y) = \mathrm{def}_f(Y)$ for every spanning subgraph F of G, and thus $\delta_f(G) \geq \mathrm{def}_f(Y)$.

By virtue of Theorem 3 and by the fact that every f-critical component of $G - A_f$ is f-odd, we have

$$\delta_f(G) = \omega(G[D_f]) - f(A_f) = f\text{-}odd(G - A_f) - f(A_f) = \mathrm{def}_f(A_f).$$

Hence the theorem holds.

Now we show how to use this approach to analyze barriers.

Definition *A set $Y \subseteq V(G)$ is called an f-*barrier *if* $\mathrm{def}_f(Y) = \delta_f(G)$.

As f-critical graphs are f-odd, the canonical Gallai–Edmonds set A_f is an f-barrier. A **1**-barrier is just an ordinary barrier in matching theory. One can observe that if $Y \subseteq V(G^f)$ satisfies $V_{f(v)} \cap Y \neq \emptyset$ and $V_{f(v)} \setminus Y \neq \emptyset$, then $V_{f(v)} \cap Y$ is adjacent to only one component of $G^f - Y$. Moreover, if Y is a barrier in G^f then each $X \subseteq Y$ is adjacent to at least $|X|$ odd components of $G^f - Y$ since otherwise

$$\begin{aligned}\mathrm{def}_{\mathbf{1}}(Y - X) &= f\text{-}odd(G - (Y - X)) - |Y - X| \\ &> f\text{-}odd(G - Y) - |X| - (|Y| - |X|) = \mathrm{def}_{\mathbf{1}}(Y),\end{aligned}$$

which is impossible. Hence if Y is a barrier in G^f then $|Y \cap V_{f(v)}| \in \{0, 1, f(v)\}$ for all $v \in V(G)$. It also follows that if $|Y \cap V_{f(v)}| = 1$ and $V_{f(v)} \setminus Y \neq \emptyset$ then $Y \setminus V_{f(v)}$ is a barrier of G^f since the unique vertex in $Y \cap V_{f(v)}$ is adjacent to exactly one odd component of $G^f - Y$ containing $V_{f(v)} \setminus Y$. Thus if Y is a barrier of G^f then $Y' = \{v \in V(G) : V_{f(v)} \subseteq Y\}$ is an f-barrier of G. On the other hand, if Y' is an f-barrier of G then $\bigcup\{V_{f(v)} : v \in Y'\}$ is clearly a barrier of G^f. The canonical Gallai–Edmonds barrier $A(G^f)$ of G^f has this form.

Definition *An f-barrier Y of G is called strong if the f-odd components of $G - Y$ are f-critical.*

It is obvious that A_f is a strong f-barrier. Since a graph K is f-critical if and only if K^f is factor-critical, we have the following.

Observation *A set $Y \subseteq V(G)$ is a strong f-barrier in G if and only if $\bigcup\{V_{f(v)} : v \in Y\}$ is a strong* **1***-barrier in G^f.*

Király proved that the intersection of strong **1**-barriers is also a strong **1**-barrier [9]. This result holds for the f-parity subgraph problem as well.

Theorem 6. *The intersection of strong f-barriers is a strong f-barrier.*

Proof. Let Y_1, Y_2 be strong f-barriers of G. Then $Y_i' = \bigcup\{V_{f(v)} : v \in Y_i\}$ are strong **1**-barriers of G^f, hence their intersection, which is just $\bigcup\{V_{f(v)} : v \in Y_1 \cap Y_2\}$, is also a strong **1**-barrier by [9]. Thus $Y_1 \cap Y_2$ is a strong f-barrier of G.

By Tutte's theorem, maximal barriers for matching are strong. This remains true for f-barriers, too. Indeed, let Y be a maximal f-barrier of G and K be an f-odd component of $G - Y$. Then K has no f-parity factor, and so $C_f(K) \neq V(K)$ in its canonical Gallai–Edmonds decomposition. Hence either $D_f(K) = V(K)$ or $A_f(K) \neq \emptyset$. In the first case K is f-critical by Theorem 3, and in the second case $Y \cup A_f(K)$ would be a larger f-barrier than Y, which is impossible. Thus all f-odd components of $G - Y$ are f-critical, implying that Y is strong.

In the matching case, it holds that the canonical Gallai–Edmonds barrier A is the intersection of all maximal barriers. This fails for the general case: take a triangle together with a pendant vertex w of degree 1, and define $f \equiv \deg$. Then this graph is of order four and has an f-parity factor, which is a whole graph, and $A_f = \emptyset$. But it has exactly one nonempty barrier $\{w\}$.

However, the fact that in matchings the canonical Gallai–Edmonds barrier A is the intersection of all strong barriers remains true for f-parity subgraphs by the above observation and the fact that A_f itself is strong.

3. f-elementary graphs

In this section we generalize some results on elementary graphs, obtained in Lovász [11], to the f-parity case.

Definition *Let G be a connected graph and $f : V(G) \to \mathbb{N}$. An edge $e \in E(G)$ is said to be f-*allowed *if G has an f-parity factor containing e. Otherwise e is f-forbidden. The graph G is said to be f-elementary if the f-allowed edges induce a connected spanning subgraph of G. The graph G is weakly f-elementary if G_2 is f-elementary, where G_2 is the graph obtained from G by replacing every edge $e \in E(G)$ by two parallel edges.*

A $\mathbf{1}$-elementary graph is briefly called *elementary.* An f-elementary graph is weakly f-elementary, but not vice versa: $G = K_2$ with $f \equiv 2$ is weakly f-elementary but not f-elementary. These classes coincide if $f = \mathbf{1}$. Lemma 7 justifies why we introduced the weak version of f-elementary graphs.

Lemma 7 *G^f is elementary if and only if G is weakly f-elementary.*

Proof. Let M be a perfect matching of G^f. If M contains at least three edges between $K_{f(u)}$ and $K_{f(v)}$ for some $u, v \in V(G)$, then replace two of them by another two edges, one inside $K_{f(u)}$ and the other one inside $K_{f(v)}$. So the number of edges of M between $K_{f(u)}$ and $K_{f(v)}$ decreased by 2. This construction shows that if G^f is elementary then G is weakly f-elementary.

On the other hand, if G is weakly f-elementary then G^f is clearly elementary.

The $f = \mathbf{1}$ special cases of the following two theorems can be found in Lovász and Plummer [13] (Theorems 5.1.3 and 5.1.6). Using our reduction, these special cases together with Lemmas 4 and 7 imply both Theorem 8 and 9.

Theorem 8. *A graph G is weakly f-elementary if and only if $\delta_f(G) = 0$ and $C_{f-\chi_w}(G) = \emptyset$ for all $w \in V(G)$.*

Proof. A graph G is weakly f-elementary if and only if G^f is elementary by Lemma 7, and G^f is elementary if and only if $\delta_{\mathbf{1}}(G^f) = 0$ and $C(G^f - x) = \emptyset$ for all $x \in V(G^f)$ ([13], Theorem 5.1.3). Since $\delta_f(G) = \delta_{\mathbf{1}}(G^f)$, it is enough to prove that under the assumption $\delta_f(G) = \delta_{\mathbf{1}}(G^f) = 0$, for every $w \in V(G)$ it follows that

$$C(G^f - x) = \emptyset \text{ for every } x \in V_{f(w)} \quad \Longleftrightarrow \quad C_{f-\chi_w}(G) = \emptyset. \tag{1}$$

If $f(w) \geq 2$, then $G^f - x \simeq G^{f-\chi_w}$ and so (1) follows from Lemma 4. Thus assume that $f(w) = 1$. As $G^f - x \simeq (G - w)^f$, Lemma 4 implies that $C(G^f - x) = \emptyset \Leftrightarrow C_f(G - w) = \emptyset$. Hence it suffices to show that

$$C_f(G - w) = \emptyset \quad \Longleftrightarrow \quad C_{f-\chi_w}(G) = \emptyset. \tag{2}$$

Since $\delta_f(G) = 0$ and $f(w) = 1$, it is easy to see that an optimal $(f - \chi_w)$-parity subgraph of G is either an f-parity factor of G or an optimal f-parity subgraph of $G - w$ enlarged by w as an isolated vertex, and vice versa. Since $(f - \chi_w)(w) = 0$, we have $w \notin D_{f-\chi_w}(G)$ by the definition of $D_{f-\chi_w}(G)$. Thus $D_{f-\chi_w}(G) = D_f(G - w)$. For the edge $e = wu$ of an f-parity factor F of G, $F - e$ is an optimal $(f - \chi_w)$-parity subgraph of G, and hence $u \in D_{f-\chi_w}(G)$ and $w \in A_{f-\chi_w}(G)$. It is immediate that $A_{f-\chi_w}(G) - \{w\} = A_f(G - w)$. Therefore (2) holds.

Theorem 9. *A graph G is weakly f-elementary if and only if f-odd$(G-Y) \le f(Y)$ for all $Y \subseteq V(G)$, and if equality holds for some $Y \neq \emptyset$ then $G-Y$ has no f-even components.*

Proof. Call $Y \subseteq V(G)$ *f-bad* if either f-*odd*$(G-Y) > f(Y)$ or equality holds here and $G-Y$ has an f-even component. It follows from Lemma 7 that the graph G is weakly f-elementary if and only if G^f is elementary, which is equivalent to that G^f has no $\mathbf{1}$-bad set ([13], Theorem 5.1.6). So we only have to prove that G has an f-bad set Y if and only if G^f has a $\mathbf{1}$-bad set Y'. If $Y \subseteq V(G)$ is f-bad then $Y' = \bigcup\{V_{f(v)} : v \in Y\}$ is $\mathbf{1}$-bad in G^f. On the other hand, assume that $Y' \subseteq V(G^f)$ is $\mathbf{1}$-bad in G^f. If $V_{f(v)} \cap Y' \neq \emptyset$ and $V_{f(v)} \setminus Y' \neq \emptyset$ for some $v \in V(G)$ then let $x \in V_v \cap Y'$. Now x is adjacent to only one component of $G^f - Y'$ hence $Y' - x$ is also $\mathbf{1}$-bad. So we can assume that Y' is a union of some $V_{f(v)}$. Now $Y = \{v \in V(G) : V_{f(v)} \subseteq Y'\}$ is f-bad in G.

In the matching case the existence of a certain canonical partition of the vertex set was revealed by Lovász [11] (Lovász, Plummer [13], Theorem 5.2.2). We cite this result.

Definition *A set $X \subseteq V(G)$ is called nearly f-extreme if $\delta_{f-\chi_X}(G) = \delta_f(G) + |X|$. Besides, X is f-extreme if $\delta_f(G-X) = \delta_f(G) + f(X)$.*

It is clear that $\delta_{f-\chi_X}(G) \le \delta_f(G) + |X|$ and $\delta_f(G-X) \le \delta_f(G) + f(X)$ for every $X \subseteq V(G)$. Nearly $\mathbf{1}$-extreme and $\mathbf{1}$-extreme sets coincide.

Theorem 10. (Lovász [11]) *If G is elementary then the maximal barriers of G form a partition $\mathcal{S}$ of $V(G)$. Moreover, it holds that*

1. *for $u, v \in V(G)$, the graph $G-u-v$ has a perfect matching if and only if u and v are contained in different classes of $\mathcal{S}$, (hence an edge xy of G is $\mathbf{1}$-allowed in G if and only if x and y are contained in different classes of $\mathcal{S}$),*
2. *$S \subseteq V(G)$ is a class of $\mathcal{S}$ if and only if $G-S$ has $|S|$ components, each factor-critical,*
3. *$X \subseteq V(G)$ is $\mathbf{1}$-extreme if and only if $X \subseteq S$ for some $S \in \mathcal{S}$.*

Lemma 7 implies the analogue of this result.

Theorem 11. *If G is weakly f-elementary then its maximal f-barriers form a subpartition $\mathcal{S}'$ of $V(G)$. Call the classes of $\mathcal{S}'$ proper, and add all elements $v \in V(G)$ not in a class of $\mathcal{S}'$ as a singleton class yielding the partition $\mathcal{S}$ of $V(G)$. Now it holds that*

1. *for $u, v \in V(G)$, the graph G has an $(f-\chi_{\{u,v\}})$-parity factor if and only if u and v are contained in different classes of $\mathcal{S}$ (hence an edge xy of G is f-allowed in G_2 if and only if x and y are contained in different classes of $\mathcal{S}$),*
2. *$S \subseteq V(G)$ is a class of $\mathcal{S}'$ if and only if $G-S$ has $f(S)$ components, each f-critical,*
3. *$X \subseteq V(G)$ is nearly f-extreme (f-extreme, resp.) if and only if $X \subseteq S$ for some $S \in \mathcal{S}$ ($S \in \mathcal{S}'$, resp.).*

Proof. Suppose that G is weakly f-elementary. Then G^f is elementary. As we already observed, every barrier Y of G^f satisfies $|Y \cap V_{f(v)}| \in \{0, 1, f(v)\}$ for all $v \in V(G)$. Since G^f is elementary, its maximal barriers form a partition $\mathcal{S}^f$ of $V(G^f)$ by Theorem 10. Thus if a maximal barrier of G^f contains exactly one vertex x of $V_{f(u)}$ and whole $V_{f(v)}$, then by symmetry, another maximal barrier contains one vertex $y \in V_{f(u)} - \{x\}$ and $V_{f(v)}$, which contradicts the above fact that the maximal barriers form a partition $V(G^f)$. Hence every maximal barrier of G^f is either a union of some $V_{f(v)}$ or a singleton. If Y' is an f-barrier of G then $\bigcup\{V_{f(v)} : v \in Y'\}$ is a barrier of G^f. On the other hand, if Y is a maximal barrier of G^f of the form $\bigcup V_{f(v)}$ then $Y' = \{v \in V(G) : V_{f(v)} \subseteq Y\}$ is clearly a maximal f-barrier of G. So these barriers Y' form the proper classes of $\mathcal{S}$, and for a singleton class $\{v\} \in \mathcal{S} - \mathcal{S}'$, it holds that each vertex $x \in V_{f(v)}$ is a maximal barrier of G^f. Now statement (i) is immediate from Theorem 10 (i), and (ii) also follows from Theorem 10 (ii) since if $S \in \mathcal{S}$ and $|S| \geq 2$ then $\bigcup\{V_v : v \in S\} \in \mathcal{S}^f$, and by the fact that a graph K is f-critical if and only if K^f is factor-critical.

Finally, (iii) follows from Theorem 10 and from the observation that $X \subseteq V(G)$ is f-extreme if and only if G^f has an extreme set X' consisting of one vertex from each V_v, $v \in X$.

Remark *It follows from Theorem 11 that $\mathcal{S}$ could be introduced as the partition $\{X \subseteq V(G) : X$ is a maximal nearly f-extreme set of $G\}$. Besides, if $X \subseteq V(G)$, $|X| \geq 2$ is maximal nearly f-extreme, then X is an f-barrier of G.*

Corollary 12 *If G is f-elementary then an edge e is f-allowed if and only if e joins two classes of $\mathcal{S}$.*

Proof. Suppose that e joins u to v and let $g = f - \chi_{\{u,v\}}$. By Theorem 11 (i), we only have to prove that G has a g-parity factor if and only if e is f-allowed. Assume that G has a g-parity factor but e is not f-allowed. (The other direction is trivial.) If $G-e$ had a g-parity factor F then $F+e$ would be an f-parity factor of G, which is impossible. Thus by Theorem 5 there exists a set $Y \subseteq V(G)$ such that $g\text{-}odd(G-e-Y) > g(Y)$. Since G has a g-parity factor, it follows from parity reasons that $g\text{-}odd(G-e-Y) = g(Y)+2$, and e joins two g-odd components Q_1 and Q_2 of $G-e-Y$. But then clearly no edge joining Y to $V(Q_1) \cup V(Q_2)$ is f-allowed in G. Since G is f-elementary, we have $V(G) = V(Q_1) \cup V(Q_2)$ and $Y = \emptyset$, but then e is an f-forbidden cut edge, which contradicts that G is f-elementary.

Our last subject is generalizing bicritical graphs.

Definition *Let G be a graph and $f : V(G) \to \mathbb{N}$ be a function. Then G is said to be f-bicritical if G has an $(f - \chi_{\{u,v\}})$-parity factor for all pairs $u, v \in V(G)$.*

Theorem 13. *If G is weakly f-elementary then the following statements are equivalent.*

1. *G is f-bicritical.*
2. *All classes of $\mathcal{S}$ are singletons.*
3. *If $Y \subseteq V(G)$ and $|Y| \geq 2$ then $f\text{-}odd(G-Y) \leq f(Y) - 2$.*

Proof. (i) $\Rightarrow$ (ii): Each edge in G_2 is allowed, and thus Theorem 11 (i) implies the equivalence.

(ii) $\Rightarrow$ (iii) : Assume otherwise. By parity reasons, we have a set $Y \subseteq V(G)$ with $|Y| \geq 2$ such that $f\text{-}odd(G-Y) = f(Y)$. So Y is an f-barrier, and is contained in a set $S \in \mathcal{S}$ with $|S| \geq 2$ by Theorem 11, which contradicts (ii).

(iii) $\Rightarrow$ (i): Assume that G is not f-bicritical. Let $g = f - \chi_{\{u,v\}}$. Then G has no g-parity factor for some $u, v \in V(G)$. Thus there exists a set $Y \subseteq V(G)$ such that $g\text{-}odd(G-Y) > g(Y)$. Recall that G has an f-parity factor. If u or v belongs to a g-odd component Q of $G-Y$ then Y is an f-barrier of G and Q is an f-even component of $G-Y$, contradicting to Theorem 9. Hence both u and v belong to Y, thus $|Y| \geq 2$ and $f\text{-}odd(G-Y) = f(Y)$, a contradiction.

References

1. A. Amahashi, On factors with all degrees odd, Graphs and Combin. **1,** (1985) 111–114.
2. G. Cornuéjols, General factors of graphs, J. Combin. Theory Ser. B, **45,** (1988) 185–198.
3. Y. Cui and M. Kano, Some results on odd factors of graphs, J. of Graph Theory **12,** (1988) 327–333.
4. J. Edmonds, Paths, trees, and flowers, Canadian J. of Math. **17,** (1965) 449–467
5. T. Gallai, *Kritische Graphen II*, A Magyar Tud. Akad. – Mat. Kut. Int. Közl, **8,** (1963) 135–139.
6. T. Gallai, Maximale Systeme unabhängiger Kanten, A Magyar Tud. Akad. – Mat. Kut. Int. Közl. **9,** (1964) 401–413.
7. M. Kano and G. Y. Katona, Odd subgraphs and matchings, Discrete Math. **250,** (2002) 265–272.
8. M. Kano and G. Y. Katona, Structure theorem and algorithm on $(1, f)$-odd subgraphs, Discrete Math. **307,** (2007) 1404-1417.
9. Z. Király, The calculus of barriers, to appear.
10. L. Lovász, The factorization of graphs, Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969) (1970) 243–246.
11. L. Lovász, On the structure of factorizable graphs, Acta Math. Acad. Sci. Hungar **23** (1972) 179–195.
12. L. Lovász, The factorization of graphs. II, Acta Math. Acad. Sci. Hungar. **23** (1972) 223–246.
13. L. Lovász and M. D. Plummer, *Matching Theory*, North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1986.
14. J. Szabó, A note on the degree prescribed factor problem, EGRES Technical Reports 2004-19. `www.cs.elte.hu/egres`
15. J. Szabó, Graph packings and the degree prescribed subgraph problem. *PhD thesis, Eötvös University*, Budapest, 2006.

Received: xx , 2009
Final version received: xx , 2009