Bisections of Two Sets of Points in the Plane Lattice

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Abstract

Assume that $2m$ red points and $2n$ blue points are given on the lattice Z^2 in the plane R^2. We show that if they are in general position, that is, if at most one point lies on each vertical line and horizontal line, then there exists a rectangular cut that bisects both red points and blue points. Moreover, if they are not in general position, namely if some vertical and horizontal lines may contain more than one point, then there exists a semi-rectangular cut that bisects both red points and blue points. We also show that these results are best possible in some sense. Moreover, our proof gives $O(N \log N)$, $N = 2m + 2n$, time algorithm for finding the desired cut.

keywords red point, blue point, lattice, bisector, rectangular cut, semi-rectangular cut, two sets of points

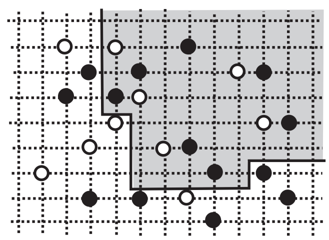
1 Introduction

It is the well-known Ham-sandwich Theorem that if $2m$ red points and $2n$ blue points are given in the plane R^2 in general position, then there exists a line that bisects both red points and blue points ([2]). Recently this theorem was generalized as follows.

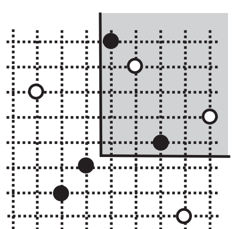
Theorem 1 ([1], [3], [6]) *Let $k \geq 2$, $m \geq 1$ and $n \geq 1$ be integers. If km red points and kn blue points are given in the plane in general position, then the plane can be subdivided into k convex polygons so that each polygon contains exactly m red points and n blue points.*

This theorem was conjectured and partially solved in [4], and this kind of theorem is called a balanced subdivision theorem. Some other results on balanced subdivisions can be found in [5]. We want to consider a similar problem in the plane lattice.

Assume that some red points and blue points are given on the lattice Z^2 in the plane R^2 (Fig. 1 (a)). Our aim in this paper is to find an orthogonal polygon (i.e., a rectilinear polygon) with small number of edges that bisects both red points and blue points. In the following, we call this orthogonal polygon an *orthogonal cut* since this polygon is usually an infinite region. Moreover, if the given points are in general position, we want to find a subdivision of the lattice into three convex orthogonal polygons so that each polygon contains the same number of red points and blue points.



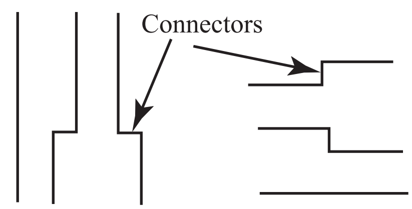
(a) R={●}, B={○}



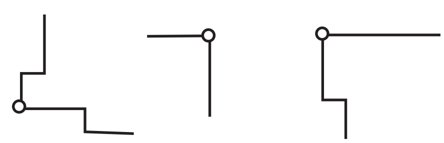
(b) R={●}, B={○}

Figure 1: (a): Red points and blue points on the lattice, and a semi-rectangular cut bisecting red points and blue points. (b): Red points and blue points on the lattice in general position, and a rectangular cut bisecting red points and blue points.

We now explain the situation more precisely and introduce some new notation. Assume that some red points and blue points are given on the lattice in the plane so that for each point of the lattice, at most one point lies on it. However it is allowed that two or more points lie on a vertical line or a horizontal line. Let R and B denote the set of red points and that of blue points, respectively. If for every vertical line and horizontal line, at most one point of $R \cup B$ lies on it, then we say that $R \cup B$ are *in general position* (see Fig. 1 (b)). A *semi-vertical line* in the plane consists of two vertical half-lines and one horizontal line segment with length one that joins the two end-points of the two vertical half-lines (Fig. 2 (a)). The horizontal line segment of length one is called the *connector*. A *semi-horizontal line* can be defined analogously (Fig. 2 (a)). A *semi-rectangular cut* consists of one semi-vertical line segment and one semi-horizontal line segment that have one common end-point, which is called the *corner*. (Fig. 2 (b)). A *rectangular cut* consists of one vertical line segment and one horizontal line segment that have one common end-point (Fig. 2 (b)). For convenience, a rectangular cut, a vertical line and a horizontal line are regarded as special semi-rectangular cuts.



(a)



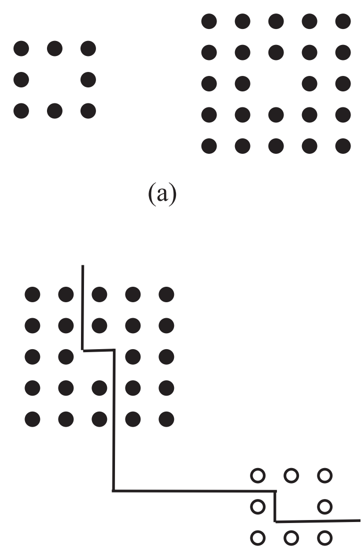
(b)

Figure 2: (a): Semi-vertical lines and semi-horizontal lines; (b) Three semi-rectangular cuts with their corners; the middle one is also a rectangular cut.

Our main result is the following theorem.

Theorem 2 *Let $m \geq 1$ and $n \geq 1$ be integers. If $2m$ red points and $2n$ blue points are given on the lattice in the plane, then there exists a semi-rectangular cut that bisects both red points and blue points (Fig. 1 (a)).*

We first show that the above theorem is best possible in the following sense. Namely, there exist configurations of red points and blue points that cannot be bisected by orthogonal cuts with five or less number of edges. It is easy to see that for each configuration of Fig. 3 (a), there exists neither a vertical line, a horizontal line nor a rectangular cut that bisects given points. Thus in order to bisect the points by an orthogonal cut, we need an orthogonal cut consisting of at least three edges. Therefore in order to bisect the red points and blue points of Fig. 3 (b) by an orthogonal cut, we need an orthogonal cut having at least six edges. Hence the result of Theorem 2 is best possible since a semi-rectangular cut consists of at most six edges.



(a)

(b)

Figure 3: (a): Two configurations that cannot be bisected by rectangular cuts; (b) A configuration of red points and blue points that cannot be bisected by an orthogonal cut with at most five edges.

If every vertical line and horizontal line contains at most one point of $R \cup B$, then we say that $R \cup B$ is *in general position*. If red points and blue points are given on the lattice in general position, then there exists a rectangular bisector as shown in the next our theorem.

Theorem 3 *Let $m \geq 1$ and $n \geq 1$ be integers. If $2m$ red points and $2n$ blue points are given on the lattice in the plane in general position, then there exists a rectangular cut that bisects both red points and blue points (see Fig. 1 (b)).*

We shall prove Theorem 3 in Sect. 2, and its proof gives us an $O(N \log N)$, $N = 2m + 2n$, time algorithm for finding the desired rectangular cut. By reducing Theorem 2 to Theorem 3, we shall prove Theorem 2 in Sect. 4. Finally the three partition problem will be considered in Sect. 5.

2 Proof of Theorem 3

For a rectangular cut $rcut$, let $Regi(rcut)$ denote one of the two regions determined by it, which will be explained precisely if necessary (Fig. 4). The next lemma plays an important role in the proof of Theorem 3.

Lemma 4 *Let $0 \le h_a \le h \le h_b$ and $1 \le k$ be integers. Suppose that some red points and blue points are given on the lattice in general position. If there are two rectangular cuts $rcut_a$ and $rcut_b$ such that $Regi(rcut_a)$ contains exactly k red points and h_a blue points, and that $Regi(rcut_b)$ contains exactly k red points and h_b blue points, then there exists a rectangular cut $rcut_c$ such that $Regi(rcut_c)$ contains exactly k red points and h blue points.*

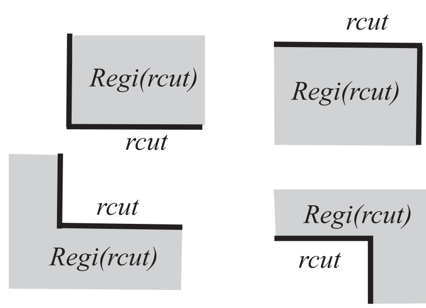


Figure 4: Some rectangular cuts $rcut$ and their regions $Regi(rcut)$

Proof. We shall show that we can move $rcut_a$ to $rcut_b$ as a sequence $(rcut_a = rcut_1, rcut_2, \ldots, rcut_{n-1}, rcut_n = rcut_b)$ in such a way that

> $(*)$ every $Regi(rcut_i)$ contains exactly k red points and the number of blue points in $Regi(rcut_i)$ is changed by at most ± 1 from $Regi(rcut_{i-1})$ for every $2 \le i \le n$.

It is obvious that if the above statement holds, then the existence of the desired rectangular cut $rcut_c$ is guaranteed. We shall prove the above statement by showing the following claims.

Claim 1. If both rectangular cuts $rcut_a$ and $rcut_b$ are vertical lines and their regions are both to the right or to the left of their lines, then the desired sequence of vertical lines exists. Similar statement also holds for horizontal lines.

Since $R \cup B$ are in general position, Claim 1 follows immediately.

Claim 2. If $rcut_a$ consists of a vertical line segment $vline_a$ and a horizontal line segment $hline_a$ and $Regi(rcut_a)$ is the region to the right of $vline_a$ and below $hline_a$, then we can move $rcut_a$ to a vertical line $vline$ with region $Reg(vline)$ to the right of it in such a way that $(*)$ holds for $rcut_a$ and $vline$ (Fig. 5 (1)).

Move first $vline_a$ to right until it touches a red point (see $vline_5$ of Fig. 5 (1)). Since every vertical line contains at most one point, if the vertical line segment passes through a blue point, the number of blue point in the region decreases by one (see $vline_2$, $vline_3$, $vline_4$ of Fig. 5 (1)). Next we move the horizontal line segment of $hline_a$ to the right of $vline_5$ upward until it touches a red point (see $hline_2$, $hline_3$ of Fig. 5 (1)). If the horizontal line segment passes through a blue point, the number of blue point in the region increases by one. Then we simultaneously move the vertical line segment $vline_5$ and the horizontal line segment $hline_3$ touching red points so that the resulting region contains the same number of red points since it includes one new red point and excludes one old red point (see region determined by $vline_6$ and $hline_4$ of Fig. 5 (1)). We repeat the same procedure until there is no point above the horizontal line segment of the resulting region. In this case the vertical line segment of the region can be extended to a vertical line without changing the red points and blue points. Hence Claim 2 is proved.

Claim 3. We can move a vertical line $vline$ with region to the right of it to a horizontal line with region above it in such a way that $(*)$ holds ((2), (3), (4) of Fig. 5). Similarly, a horizontal line with region above it can be moved to a vertical line with region to the left of it in such a way that $(*)$ holds ((4), (5) of Fig. 5).

Assume that $Regi(vline)$ is to the right of $vline$ as shown in (2) in Fig. 5. We first add a horizontal line segment to make a new rectangular cut whose region contains the same red points and blue points as $Regi(vline)$. Then we can move this rectangular cut to a horizontal line as shown in the proof of Claim 2 (see (3) of Fig. 4). Thus the claim holds.

It is clear that if a rectangular cut $rcut_s$ is moved to a rectangular cut $rcut_t$ in such a way that $(*)$ holds, then conversely $rcut_t$ can be moved to $rcut_s$ in the same way. Hence by Claim 2, each of $rcut_a$ and $rcut_b$ can be moved to a vertical line or a horizontal line, and by Claims 3 and 1, they are moved each other in such a way that $(*)$ holds. Therefore Lemma 4 is proved. □

Proof of Theorem 3. There exists a horizontal line $hline$ that bisects red points, namely, each of two regions above and below it contains exactly m

red points. By applying Lemma 4 to *hline* with region above it and *hline* with region below it, we can find the desired rectangular cut. □

Let $N = 2m + 2n$. It takes $O(N \log N)$ time to sort all points in $R \cup B$ by x-coordinate and also by y-coordinate. The number of rectangular cuts in a sequence $(rcut_a = rcut_1, rcut_2, \ldots, rcut_{n-1}, rcut_n = rcut_b)$ given in the proof of Lemma 4 is $O(N)$, and each step it takes a constant time. Also we can find a bisector of red points in $O(N)$ time. Therefore we can find the desired rectangular cut that bisects both red points and blue points in $O(N \log N)$ time.

3 Proof of Theorem 2

We shall prove Theorem 2 by reducing it to Theorem 3.

Proof of Theorem 2. Choose any horizontal line *hline* that contains at least two points of $R \cup B$. Let $p_1, p_2, \ldots, p_k$ be the points of $R \cup B$ lying on *hline* totally ordered from left to right. Let $hline'$ be the next horizontal line of the lattice above *hline*. Then we add new $k-1$ distinct horizontal lines between *hline* and $hline'$. In the following, each point p_i, $i \geq 2$, will be moved on this $(i-1)$-th new horizontal line, while p_1 will be left on *hline*. We repeat the same procedure for all horizontal lines containing at least two points of $R \cup B$ (Fig. 6).

As horizontal lines, we apply the similar procedure to every vertical line containing at least two points of $R \cup B$ so that in the resulting lattice, every vertical line contains at most one point of $R \cup B$. Therefore, it is easy to see that in the new lattice, $R \cup B$ is in general position (Fig. 6).

Since $R \cup B$ is in general position in the new lattice, by Theorem 3 there exists a rectangular cut that bisects both red points and blue points. As shown in Fig. 7, every rectangular cut in the new lattice is corresponding to a semi-rectangular cut in the original given lattice. Hence we can find the desired semi-rectangular that bisects both red points and blue points in the original lattice. □

4 Balanced Partition into three regions

An orthogonal polygon P, whose edges are vertical or horizontal, is called *convex* if for every vertical or horizontal line ℓ, the intersection $P \cap \ell$ consists of at most one line segment. Examples of a convex polygon and of a non-convex polygon are shown in Fig. 8.

In this section we prove the following theorem.

Theorem 5 *Let $m \geq 1$ and $n \geq 1$ be integers. If $3m$ red points and $3n$ blue points are given on the lattice in the plane in general position, then the plane can be subdivided into three convex orthogonal polygons so that each polygon contains exactly m red points and n blue points (Fig. 9).*

Note that if we consider a configuration which consists of three red points and three blue points lying on the same horizontal line, then it is impossible to subdivide the plane into three convex orthogonal polygons so that each polygon contains exactly one red point and one blue point. Hence the condition that $R \cup B$ lies in general position is necessary.

Proof of Theorem 5. There exists a horizontal line $hline$ such that the region above $hline$ contains exactly m red points. The remaining region, which is below $hline$, can be divided into two regions by a vertical half-line so that each region contains exactly m red points (Fig. 10 (1)). Then by Lemma 4, there exists a rectangular cut $rcut_1$ with region $Regi(rcut)$ containing exactly m red points and n blue points. If $Regi(rcut)$ is a infinite region with interior angle $3\pi/2$, then by Theorem 3 the red points and blue points in the remaining region can be bisected by a rectangular cut, and so we can obtain the desired subdivision of the plane (Fig. 10 (2)). Hence we may assume that $Regi(rcut)$ is a region with interior angle $\pi/2$ ((Fig. 11 (1)).

It is easy to see that there exists either a horizontal line without intersecting $Regi(rcut)$ such that the region above it contains exactly m red points or a vertical line without intersecting $Regi(rcut)$ such that the region to the right of it contains exactly m red points. By symmetry, we may assume that there exists such a horizontal line $hline$ (Fig. 11 (1)). Then the plane is subdivided into three regions $Regi(rcut)$, $Regi(hline)$ and the remaining region, which is called an *L-shaped region*.

We now show that the L-shaped region can be moved to $Regi(hline)$ as a sequence (L-*shaped region* $= rcut_1, rcut_2, \ldots, rcut_{n-1}, rcut_n = hline$) in such a way that

> ($*$) every $Regi(rcut_i)$ contains exactly m red points and the number of blue points in $Regi(rcut_i)$ is changed by at most ± 1 from $Regi(rcut_{i-1})$ for every $2 \leq i \leq n$.

This part can be proved in the same way as in the proof of Theorem 3, and so we only give some figures (Fig. 11 (2), (3)). Therefore we can find a rectangular cut such that the region determined by it contains exactly m red points and n blue points since one of the regions of the L-shaped region and $Regi(hline)$ contains at least n blue points and the other one contains at most n blue points. Consequently Theorem 5 is proved.

5 Acknowledgment

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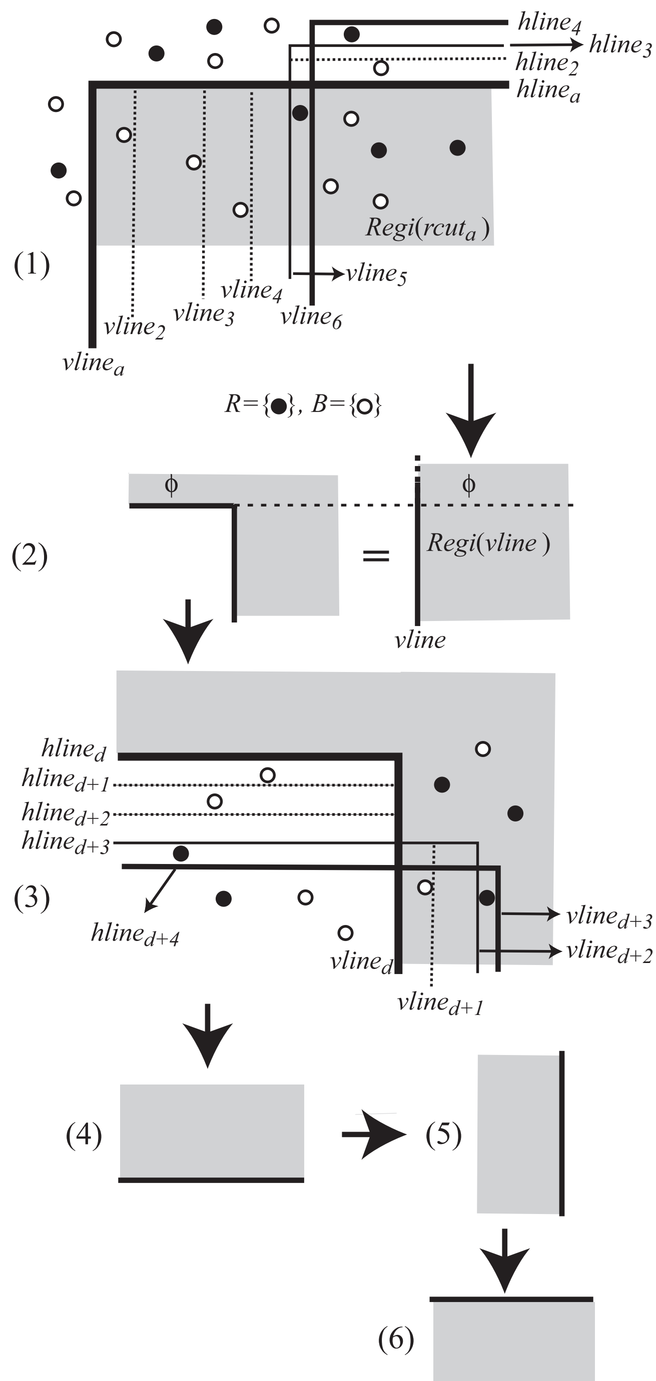

Figure 5: Some rectangular cuts $rcut_i$ and their regions $Regi(rcut_i)$

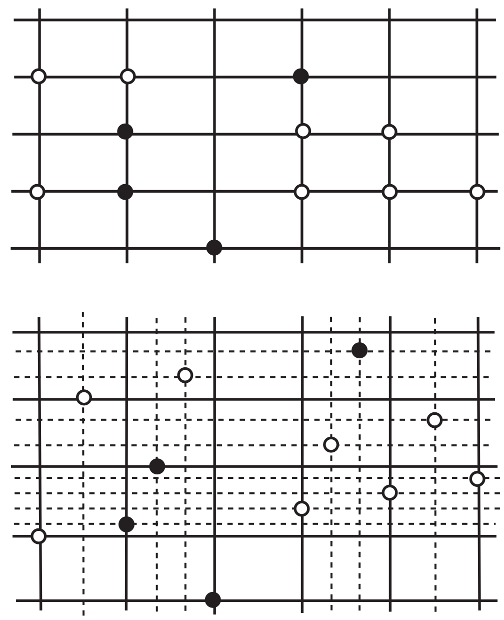

Figure 6: (1) Red points and blue points on the lattice; (2) Red points and blue points on the new lattice in general position.

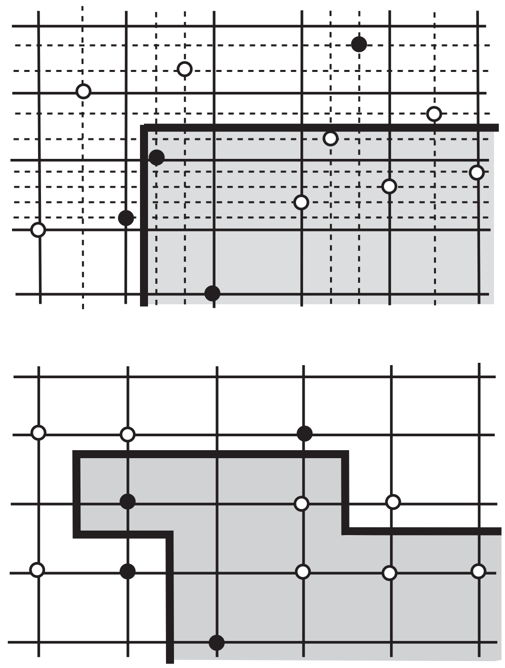

Figure 7: (1) A rectangular cut in the new lattice; (2) The corresponding semi-rectangular cut in the original lattice.

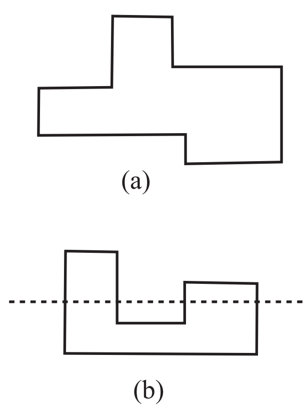

Figure 8: (a) A convex orthogonal polygon; (b) Non-convex orthogonal polygon.

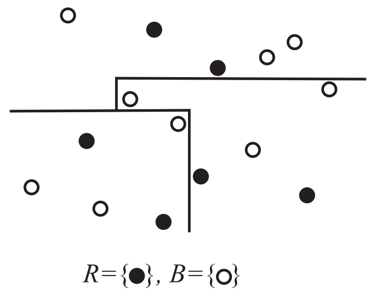

Figure 9: The plane is subdivided into three convex orthogonal polygons so that each polygon contains the same numbers of red points and blue points.

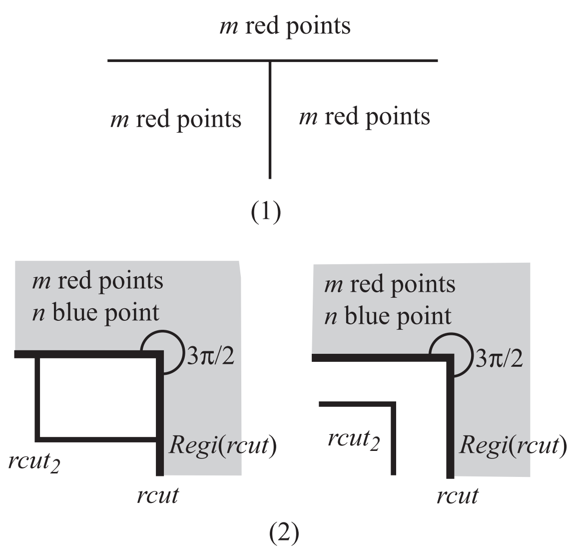

Figure 10: The plane is subdivided into three rectangular regions each of which contains exactly m red points.

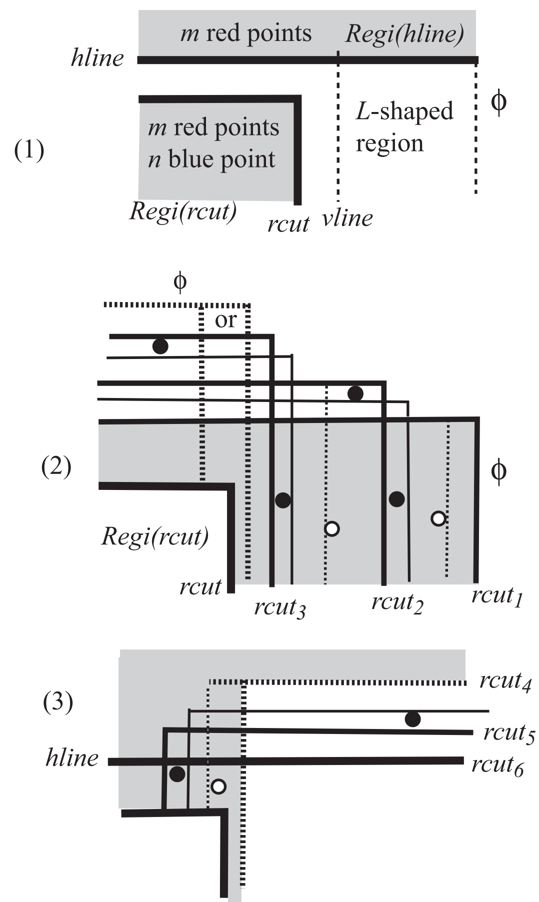

Figure 11: Red points and blue points on the lattice; (2) Red points and blue points on the new lattice in general position.