Draft

A neighborhood condition for graphs to have $[a, b]$-factors III

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Abstract

Let a, b, k, and m be positive integers such that $1 \leq a < b$ and $2 \leq k \leq (b+1-m)/a$. Let $G = (V(G), E(G))$ be a graph of order $|G|$. Suppose that $|G| > (a+b)(k(a+b-1)-1)/b$ and $|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \geq a|G|/(a+b)$ for every independent set $\{x_1, x_2, \ldots, x_k\} \subseteq V(G)$. Then for any subgraph H of G with m edges and $\delta(G - E(H)) \geq a$, G has an $[a, b]$-factor F such that $E(H) \cap E(F) = \emptyset$. This result is best possible in some sense and it is an extension of the result of H. Matsuda (Discrete Mathematics **224** (2000) 289–292).

1 Introduction

We consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G. For a vertex v of G, let $\deg_G(v)$ and $N_G(v)$ denote the degree of v in G and the neighborhood of v in G, respectively. Furthermore, $\delta(G)$ denotes the minimum degree of G, and $N_G(S) = \bigcup_{x \in S} N_G(x)$ for $S \subset V(G)$. We write $N_G[v]$ for $N_G(v) \cup \{v\}$. For two disjoint vertex subsets A and B of G, the number of edges of G joining A to B is denoted by $e_G(A, B)$. For a subset $S \subset V(G)$, let $G - S$ denote the subgraph of G induced by $V(G) - S$.

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Let a and b be integers such that $1 \le a \le b$. An $[a, b]$-factor of G is a spanning subgraph F of G such that

$$a \le \deg_F(x) \le b \quad \text{for all} \quad x \in V(G).$$

Note that if $a = b$, then an $[a, b]$-factor is a regular a-factor.

2 Background and Results

The following results on a k-factor are known.

Theorem 1 (Iida and Nishimura [1]) *Let $k \ge 2$ be an integer and let G be a connected graph of order $|G|$ such that $|G| \ge 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, $k|G|$ is even, and $\delta(G) \ge k$. If G satisfies $|N_G(x) \cup N_G(y)| \ge (|G| + k - 2)/2$ for all non-adjacent vertices x and y of G, then G has a k-factor.*

Theorem 2 (Niessen [4]) *Let G be a connected graph of order $|G|$ and $\delta(G) \ge k \ge 2$, where k is an integer with $k|G|$ is even and $|G| \ge 8k-7$. If $|N_G(x) \cup N_G(y)| \ge |G|/2$ for all non-adjacent vertices x and y of G, then G has a k-factor or G belongs to some exceptional families.*

One of the authors showed a neighborhood condition for the existence of an $[a, b]$-factor.

Theorem 3 (Matsuda [5]) *Let a and b be integers such that $1 \le a < b$ and let G be a graph of order $|G|$ with $|G| \ge 2(a+b)(a+b-1)/b$ and $\delta(G) \ge a$. If*

$$|N_G(x) \cup N_G(y)| \ge \frac{a|G|}{a+b}$$

for any two non-adjacent vertices x and y of G, then G has an $[a, b]$-factor.

The following theorem gurantees the existence of an $[a, b]$-factor which includes some specified edges.

Theorem 4 (Matsuda [6]) *Let a, b, m, and t be integers such that $1 \le a < b$ and $2 \le t \le \lceil (b - m + 1)/a \rceil$. Suppose that G is a graph of order $|G| > ((a+b)(t(a+b-1)-1)+2m)/b$ and $\delta(G) \ge a$. Let H be any subgraph of G with $|E(H)| = m$. If*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_t)| \ge \frac{a|G| + 2m}{a+b}$$

for every independent set $\{x_1, x_2, \ldots, x_t\} \subseteq V(G)$, then G has an $[a, b]$-factor including H.

In this paper, we prove the following two theorems for the existence of an $[a, b]$-factor which excludes some specified edges.

Theorem 5 *Let a, b, m, and k be positive integers such that $1 \le a < b$ and $2 \le k < (a+b+1-m)/a$. Let G be a graph with $|G| > (a+b)((k+m)(a+b-1)-1)/b$. If*

$$|N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| \ge \frac{a|G|}{a+b} \tag{1}$$

for every independent set $\{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, then for any subgraph H of G with m edges and $\delta(G - E(H)) \ge a$, G has an $[a,b]$-factor F excluding H (i.e. $E(H) \cap E(F) = \emptyset$).

The condition (1) is best possible in the sense that we cannot replace $a|G|/(a+b)$ by $a|G|/(a+b) - 1$, which is shown in the following example: Let $t \ge 2m$ be a sufficiently large integer. Consider the join of two graphs $G = A + B$, where A consists of $at - 2m$ isolated vertices and m independent edges, and B consists of $bt + 1$ isolated vertices. Then it follows that $|G| = |A| + |B| = (a+b)t + 1$ and

$$\frac{a|G|}{a+b} > |N_G(x_1) \cup N_G(x_2) \cup \cdots \cup N_G(x_k)| = at > \frac{a|G|}{a+b} - 1$$

for a subset $\{x_1, x_2, \ldots, x_k\} \subseteq B$ with $2 \le k < (a+b+1-m)/a$. However, G has no $[a,b]$-factor excluding the m edges in A because $b|A| < a|B|$.

The next theorem corresponds to the case $k = 1$ of Theorem 5.

Theorem 6 *Let a, b, and m be integers such that $1 \le a < b$ and $m \ge 1$. Suppose that G is a graph with $\delta(G) \ge a|G|/(a+b)$ and $|G| > (a+b)((m+1)(a+b+1)-5)/b$. Then for any subgraph H of G with m edges, G has an $[a,b]$-factor excluding H.*

3 Proofs of Theorem 5 and 6

For a vertex v and a vertex subset T of G, for convenience, we write $N_T(v)$ and $N_T[v]$ for $N_G(v) \cap T$ and $N_G[v] \cap T$, respectively. Our proofs of the theorems depend on the following criterion.

Theorem 7 (Lam, Liu, Li and Shiu [2]) *Let $1 \le a < b$ be integers, and let G be a graph and H a subgraph of G. Then G has an $[a,b]$-factor F such that $E(H) \cap E(F) = \emptyset$ if and only if*

$$b|S| + \sum_{x \in T} \deg_{G-S}(x) - a|T| \ge \sum_{x \in T} \deg_H(x) - e_H(S,T)$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 5. Suppose that G satisfies the assumption of the theorem, but has no desired $[a,b]$-factor for some subgraph H with m edges and $\delta(G-H) \ge a$. Then by Theorem 7, there exist two disjoint subsets S and T of $V(G)$ such that

$$b|S| + \sum_{x \in T} \big(\deg_{G-S}(x) - \deg_H(x) + e_H(x,S) - a \big) \le -1. \tag{2}$$

We choose such subsets S and T so that $|T|$ is minimum.

Claim 1 $|S| \geq 1$.

If $S = \emptyset$, then by (2) we obtain

$$-1 \geq \sum_{x \in T} \big(\deg_G(x) - \deg_H(x) - a \big) \geq \sum_{x \in T} (\delta(G - E(H)) - a) \geq 0,$$

which is a contradiction. ∎

Claim 2 $|T| \geq b + 1$.

Suppose that $|T| \leq b$. Since $|S| + \deg_{G-S}(x) - \deg_H(x) \geq \deg_{G-H}(x) \geq \delta(G - E(H)) \geq a$ for all $x \in T$, it follows from (2) that

$$\begin{aligned} -1 &\geq b|S| + \sum_{x \in T} \big(\deg_{G-S}(x) - \deg_H(x) + e_H(x, S) - a \big) \\ &\geq \sum_{x \in T} \big(|S| + \deg_{G-S}(x) - \deg_H(x) + e_H(x, S) - a \big) \geq 0. \end{aligned}$$

This is a contradiction. ∎

Claim 3 $\deg_{G-S}(x) - \deg_H(x) + e_H(x, S) \leq a - 1$ *for all* $x \in T$.

Suppose that there exists a vertex $u \in T$ such that $\deg_{G-S}(u) - \deg_H(u) + e_H(u, S) \geq a$. Then the subsets S and $T - \{u\}$ satisfy (2), which contradicts the choice of T. Hence the claim holds. ∎

By Claim 3, we obtain

$$|N_T[x]| \leq \deg_{G-S}(x) + 1 \leq \deg_H(x) - e_H(x, S) + a \qquad \text{for all } x \in T.$$

Now we obtain a set $\{x_1, x_2, \ldots, x_k\}$ of independent vertices of G as follows: First define

$$h_1 = \min\{\deg_{G-S}(x) - \deg_H(x) + e_H(x, S) \mid x \in T\},$$

and choose $x_1 \in T$ such that $\deg_{G-S}(x_1) - \deg_H(x_1) + e_H(x_1, S) = h_1$ and $\deg_H(x_1) - e_H(x_1, S)$ is minimum. Next, for $i = 2, \cdots, k$, where $k < (a + b + 1 - m)/a$, we define

$$h_i = \min\{\deg_{G-S}(x) - \deg_H(x) + e_H(x, S) \mid x \in T - \bigcup_{j=1}^{i-1} N_T[x_j] \},$$

and choose $x_i \in T - \bigcup_{j=1}^{i-1} N_T[x_j]$ such that $\deg_{G-S}(x_i) - \deg_H(x_i) + e_H(x_i, S) = h_i$ and $\deg_H(x_i) + e_H(x_i, S)$ is minimum. Then we have $h_1 \leq h_2 \leq \cdots \leq h_k \leq a - 1$ by

Claim 3 and we have $\sum_{i=1}^{k} \deg_H(x_i) \le m$ since $|E(H)| = m$ and $\{x_1, x_2, \dots, x_k\}$ is an independent set of G. Note that by Claim 3 and $k < (a+b+1-m)/a$, we have

$$\begin{aligned}\Big|\bigcup_{j=1}^{k-1} N_T[x_j]\Big| &\le \sum_{j=1}^{k-1}(\deg_{G-S}(x_j)+1) \le \sum_{j=1}^{k-1}\big(a+\deg_H(x_i)\big)\\ &\le a(k-1)+|E(H)| \le a(k-1)+m < b+1 \le |T|.\end{aligned}$$

Hence we can take an independent set $\{x_1, x_2, \dots, x_k\}$.

By the condition of Theorem 5, the following inequalities hold:

$$\begin{aligned}\frac{a|G|}{a+b} &\le |N_G(x_1)\cup N_G(x_2)\cup\cdots\cup N_G(x_k)|\\ &\le \sum_{i=1}^{k}\deg_{G-S}(x_i)+|S|\\ &\le \sum_{i=1}^{k}(h_i+\deg_H(x_i)-e_H(x_i,S))+|S|,\end{aligned}$$

which implies

$$|S| \ge \frac{a|G|}{a+b} - \sum_{i=1}^{k}(h_i+\deg_H(x_i)-e_H(x_i,S)). \tag{3}$$

Since $|G|-|S|-|T| \ge 0$ and $a-h_k \ge 1$, we obtain $(|G|-|S|-|T|)(a-h_k) \ge 0$. This inequality together with (2) gives us the following:

$$\begin{aligned}&(|G|-|S|-|T|)(a-h_k)\\ &\ge b|S| + \sum_{x\in T}(\deg_{G-S}(x)-\deg_H(x)+e_H(x,S)-a)+1\\ &\ge b|S| + \sum_{i=1}^{k-1} h_i|N_T[x_i]| + h_k(|T| - \sum_{i=1}^{k-1}|N_T[x_i]|) - a|T| + 1\\ &= b|S| + \sum_{i=1}^{k-1}(h_i-h_k)|N_T[x_i]| + (h_k-a)|T|+1\\ &\ge b|S| + \sum_{i=1}^{k-1}(h_i-h_k)(h_i+1+\deg_H(x_i)) + (h_k-a)|T|+1\\ &= b|S| + \sum_{i=1}^{k}(h_i-h_k)(h_i+1+\deg_H(x_i)) + (h_k-a)|T|+1,\end{aligned}$$

where $h_i - h_k \le 0$ and $h_i + 1 + \deg_H(x_i) \ge |N_T[x_i]|$. Then it follows from the above inequality that

$$0 \le (a-h_k)|G| - (a+b-h_k)|S| + \sum_{i=1}^{k-1}(h_k-h_i)(h_i+1+\deg_H(x_i)) - 1. \tag{4}$$

Substituting (3) into (4), we have

$$\begin{aligned}
0 &\le (a-h_k)|G| - (a+b-h_k)\left(\frac{a|G|}{a+b} - \sum_{i=1}^{k}(h_i + \deg_H(x_i) - e_H(x_i,S))\right) \\
&\quad + \sum_{i=1}^{k}(h_k - h_i)(h_i + 1 + \deg_H(x_i)) - 1 \\
&= -\frac{b|G|}{a+b}h_k - \sum_{i=1}^{k}(h_i^2 - (a+b-1-\deg_H(x_i))h_i - h_t - (a+b)\deg_H(x_i)) - 1.
\end{aligned}$$

By the condition $2 < (a+b+1-m)/a$, we have $m < b-a+1$ and hence $a+b-1-\deg_H(x_i) \ge 2(a-1)$ for each $i = 1, 2, \ldots, k$. This together with the inequalities $h_1 \le h_2 \le \cdots \le h_k \le a-1$ of Claim 3 yields the fact $h_i^2 - (a+b-1-\deg_H(x_i))h_i$ attains its minimum at $h_i = h_k$. Suppose that $h_k \ge 1$. By $|G| > (a+b)((k+m)(a+b-1)-1)/b$, we obtain

$$\begin{aligned}
0 &\le -\frac{b|G|}{a+b}h_k - \sum_{i=1}^{k}(h_i^2 - (a+b-1-\deg_H(x_i))h_i - h_k - (a+b)\deg_H(x_i)) - 1 \\
&\le -\frac{b|G|}{a+b}h_k - \sum_{i=1}^{k}(h_k^2 - (a+b-1-\deg_H(x_i))h_k - h_k - (a+b)\deg_H(x_i)) - 1 \\
&= -\frac{b|G|}{a+b}h_k - kh_k^2 + k(a+b)h_k + (a+b-h_k)\sum_{i=1}^{k}\deg_H(x_i) - 1 \\
&\le -\frac{b|G|}{a+b}h_k - kh_k^2 + k(a+b)h_k + (a+b-h_k)m - 1 \\
&\le -kh_k^2 + \left(k(a+b) - \frac{b|G|}{a+b} - m\right)h_k + (a+b)m - 1 \\
&< -kh_k^2 + (k-(a+b)m+1)\,h_k + (a+b)m - 1 \\
&= -(h_k-1)\,(kh_k + (a+b)m - 1) \le 0.
\end{aligned}$$

This is a contradiction. Hence we consider the case $h_1 = h_2 = \cdots = h_k = 0$. By (3) and (4), $\sum_{i=1}^{k}(\deg_H(x_i) - e_H(x_i,S)) \ge 1$. By the choice of $\{x_1, x_2, \ldots, x_k\}$, one of (i) and (ii) holds for any $w \in T \setminus (\{x_1, x_2, \ldots, x_k\} \cup N_H(\{x_1, x_2, \ldots, x_k\})$: (i) $\deg_{G-S}(w) - \deg_H(w) + e_H(w,S) \ge 1$ or (ii) $\deg_{G-S}(w) - \deg_H(w) + e_H(w,S) = 0$ and $\deg_H(w) - e_H(w,S) \ge 1$. Since $\{x_1, x_2, \ldots, x_k\} \cap V(H) \ne \emptyset$ and any vertices $v \in T \setminus (\{x_1, x_2, \ldots, x_k\} \cup V(H))$ satisfy (i), we have

$$\sum_{x\in T}(\deg_{G-S}(x) - \deg_H(x) + e_H(x,S)) \ge |T| - k - 2m + 1.$$

By this inequality, (3), $\sum_{i=1}^{k} \deg_H(x_i) \leq m$, $2 \leq k < (a+b+1-m)/a$, and $|G| > (a+b)((k+m)(a+b-1)-1)/b$, we obtain

$$\begin{aligned}
-1 &\geq b|S| + |T| - k - 2m + 1 - a|T| = b|S| + (1-a)|T| - k - 2m + 1 \\
&\geq b|S| + (1-a)(|G| - |S|) - k - 2m + 1 \\
&= (a+b-1)|S| - (a-1)|G| - k - 2m + 1 \\
&\geq (a+b-1)\left(\frac{a|G|}{a+b} - m\right) - (a-1)|G| - k - 2m + 1 \\
&= \frac{b|G|}{a+b} - m(a+b+1) - k + 1 \\
&> (k+m)(a+b-1) - 1 - m(a+b+1) - k + 1 \\
&= k(a+b-2) - 2m \\
&\geq k(a+b-2) - 2(a+b-ak) \\
&= k(3a+b-2) - 2(a+b) \\
&\geq 2(3a+b-2) - 2(a+b) = 4(a-1) \geq 0.
\end{aligned}$$

Therefore Theorem 5 is proved. ∎

Proof of Theorem 6. Suppose that G satisfies the assumption of the theorem, but has no desired $[a,b]$-factor for some subgraph H with m edges. Note that $\delta(G-H) \geq a|G|/(a+b) - m \geq a$ hold by the conditions of Theorem 6. Then by Theorem 7, there exist two disjoint subsets S and T of $V(G)$ such that

$$b|S| + \sum_{x \in T} \big(\deg_{G-S}(x) - \deg_H(x) + e_H(x,S) - a\big) \leq -1. \tag{5}$$

We choose such subsets S and T so that $|T|$ is minimum.

By the argument of Claims 1, 2, and 3 in the proof of Theorem 5, we obtain $|S| \geq 1$, $|T| \geq b+1$, and $\deg_{G-S}(x) - \deg_H(x) + e_H(x,S) \leq a-1$ for all $x \in T$. We now define

$$u_1 = \min\{\deg_{G-S}(x) - \deg_H(x) + e_H(x,S) \mid x \in T\},$$

and choose $x_1 \in T$ such that $\deg_{G-S}(x_1) - \deg_H(x_1) + e_H(x_1,S) = u_1$ and $\deg_H(x_1) - e_H(x_1,S)$ is minimum. For $i = 2, \cdots, |T|$, we define

$$u_i = \min\{\deg_{G-S}(x) - \deg_H(x) + e_H(x,S) \mid x \in T \setminus \{u_1, \ldots, u_{i-1}\}\ \},$$

and choose $x_i \in T \setminus \{x_1, \ldots, x_{i-1}\}$ such that $\deg_{G-S}(x_i) - \deg_H(x_i) + e_H(x_i,S) = u_i$ and $\deg_H(x_i) + e_H(x_i,S)$ is minimum. Then we have $u_1 \leq u_2 \leq \cdots \leq u_{|T|} \leq a-1$.

By the condition of Theorem 6, the following inequalities hold:

$$\frac{a|G|}{a+b} \leq \delta(G) \leq \deg_G(x_1) \leq \deg_{G-S}(x_1) + |S| \leq u_1 + \deg_H(x_1) - e_H(x_1,S) + |S|,$$

which implies

$$|S| \geq \frac{a|G|}{a+b} - (u_1 + \deg_H(x_1) - e_H(x_1, S)). \tag{6}$$

On the other hand, by (5) and $u_1 \leq u_2 \leq \cdots \leq u_{|T|}$, we have

$$\begin{aligned} 0 &\geq b|S| + \sum_{i=1}^{|T|} u_i - a|T| \geq b|S| + (u_1 - a)|T| + 1 \\ &\geq b|S| + (u_1 - a)(|G| - |S|) + 1 = (a+b-u_1)|S| - (a-u_1)|G| + 1, \end{aligned}$$

which implies

$$0 \geq (a+b-u_1)|S| - (a-u_1)|G| + 1. \tag{7}$$

By (6), (7), $u_1 \leq u_2 \leq \cdots \leq u_{|T|} \leq a-1$, and $|G| > (a+b)((m+1)(a+b+1)-5)/b$,

$$\begin{aligned} 0 &\geq (a+b-u_1)\left(\frac{a|G|}{a+b} - (u_1 + \deg_H(x_1) - e_H(x_1, S))\right) - (a-u_1)|G| + 1 \\ &= \frac{bu_1}{a+b}|G| - (a+b-u_1)(u_1 + \deg_H(x_1) - e_H(x_1, S)) + 1 \\ &\geq \frac{bu_1}{a+b}|G| - (a+b-u_1)(u_1+m) + 1 \\ &> u_1((m+1)(a+b+1)-5) - (a+b-u_1)(u_1+m) + 1 \\ &= u_1^2 + (m(a+b+2)-4)u_1 - m(a+b) + 1 \\ &= (u_1-1)^2 + m(a+b)(u_1-1) + 2(m-1)u_1. \end{aligned}$$

If $u_1 \geq 1$, then the above inequalities imply $0 > 0$, a contradiction. Hence we must consider the case $u_1 = 0$. By (6) and (7), $\deg_H(x_1) - e_H(x_1, S) \geq 1$. By the definition of $x_1, x_2, \ldots, x_{|T|}$, one of (i) and (ii) holds for any $w \in \{x_2, \ldots, x_{|T|}\}$: (i) $\deg_{G-S}(w) - \deg_H(w) + e_H(w, S) \geq 1$ or (ii) $\deg_{G-S}(w) - \deg_H(w) + e_H(w, S) = 0$ and $\deg_H(w) - e_H(w, S) \geq 1$. Therefore we have

$$\sum_{x \in T} (\deg_{G-S}(x) - \deg_H(x) + e_H(x, S)) \geq |T| - 2m.$$

By this inequality, (5), and $|G| > (a+b)((m+1)(a+b+1)-5)/b$, we obtain

$$\begin{aligned} -1 &\geq b|S| + |T| - 2m - a|T| = b|S| + (1-a)|T| - 2m \\ &\geq b|S| + (1-a)(|G| - |S|) - 2m \\ &= (a+b-1)|S| - (a-1)|G| - 2m \\ &\geq (a+b-1)\left(\frac{a|G|}{a+b} - m\right) - (a-1)|G| - 2m \\ &= \frac{b|G|}{a+b} - m(a+b+1) > 0 \\ &> (m+1)(a+b+1) - 5 - m(a+b+1) \\ &= a+b-4 \geq -1. \end{aligned}$$

Finally the proof of Theorem 6 is complete. ∎

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