Draft

General balanced subdivision of two sets of points in the plane

[1]M. Kano and [2]Miyuki Uno

[1]kano@mx.ibaraki.ac.jp [2]umyu@mug.biglobe.ne.jp
Department of Computer and Information Sciences,
Ibaraki University, Hitachi, Ibaraki 316-8511, Japan
http://gorogoro.cis.ibaraki.ac.jp/

Abstract

Let R and B be two disjoint sets of red points and blue points, respectively, in the plane such that no three points of $R \cup B$ are co-linear. Suppose $ag \leq |R| \leq (a+1)g$, $bg \leq |B| \leq (b+1)g$. Then without loss of generality, we can express $|R| = a(g_1+g_2) + (a+1)g_3$, $|B| = bg_1 + (b+1)(g_2+g_3)$, where $g = g_1+g_2+g_3$, $g_1 \geq 0$, $g_2 \geq 0$, $g_3 \geq 0$ and $g_1+g_2+g_3 \geq 1$. We show that the plane can be subdivided into g disjoint convex polygons $X_1 \cup \cdots \cup X_{g_1} \cup Y_1 \cup \cdots \cup Y_{g_2} \cup Z_1 \cup \cdots \cup Z_{g_3}$ such that every X_i contains a red points and b blue points, every Y_i contains a red points and $b+1$ blue points and every Z_i contains $a+1$ red points and $b+1$ blue points.

1 Introduction

We consider two disjoint sets R and B of red points and blue points in the plane, respectively. We always assume that $R \cup B$ is in general position, that is, no three points of $R \cup B$ lie on the same line. We want to subdivide the plane into some disjoint convex polygons so that each polygon contains prescribed numbers of red points and blue points. We begin with some known results on this problem.

The following Theorem 1, which was conjectured in [5] and proved for $a = 1, 2$ in [5] and [6], has been established in full generality by Bespamyatnikh, Kirkpatrick and Snoeyink [2], Sakai [9] and by Ito, Uehara and Yokoyama [3], independently. Note that this theorem with $g = 2$ is equivalent to the famous Ham-sandwich Theorem for the plane [4].

Theorem 1 (The Equitable Subdivision Theorem [2], [3], [9]) *Let $a \geq 1$, $b \geq 1$ and $g \geq 2$ be integers. If $|R| = ag$ and $|B| = bg$, then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that every X_i contains exactly a red points and b blue points.*

The next theorem shows that if $a = 1$ in the above Theorem 1, we can obtain more general subdivision.

Theorem 2 (Kaneko and Kano [6]) *Let $b \geq 1$ be an integer. Suppose that R is a disjoint union of R_1 and R_2. If $|R_1| = g_1$, $|R_2| = g_2$ and $|B| = (b-1)g_1 + bg_2$, then we can subdivide the plane into $g_1 + g_2$ disjoint convex polygons $X_1 \cup \cdots \cup X_{g_1} \cup Y_1 \cup \cdots \cup Y_{g_2}$ so that every X_i contains exactly one red point of R_1 and $b-1$ blue points, and every Y_j contains exactly one red point of R_2 and b blue points.*

The next theorem shows another result on balanced subdivisions.

Theorem 3 (Kaneko, Kano and Suzuki [8]) *Let $a \geq 1$, $g \geq 0$ and $h \geq 0$ be integers such that $g + h \geq 1$. If $|R| = ag + (a+1)h$ and $|B| = (a+1)g + ah$, then there exists a subdivision $X_1 \cup \cdots \cup X_g \cup Y_1 \cup \cdots \cup Y_h$ of the plane into $g+h$ disjoint convex polygons such that every X_i contains exactly a red points and $a+1$ blue points and every Y_j contains exactly $a+1$ red points and a blue points.*

In this paper, we consider balanced subdivision problem in the case that $ag \leq |R| \leq a(g+1)$ and $bg \leq |B| \leq b(g+1)$. As we shall show in Lemma 5, in this case we can express $|R| = a(g_1+g_2) + (a+1)g_3$ and $|B| = bg_1 + (b+1)(g_2+g_3)$, or $|R| = ag_1 + (a+1)(g_2+g_3)$ and $|B| = b(g_1+g_2) + (b+1)g_3$ for some integers $g_1, g_2, g_3 \geq 0$ with $g_1 + g_2 + g_3 \geq 1$. By symmetry, we may assume that

$$|R| = a(g_1 + g_2) + (a+1)g_3 \quad \textit{and} \quad |B| = bg_1 + (b+1)(g_2 + g_3) \tag{1}$$

holds. The following theorem is our main result.

Theorem 4 *Let $a \geq 1$, $b \geq 1$, $g_1 \geq 0$, $g_2 \geq 0$ and $g_3 \geq 0$ be integers such that $g_1+g_2+g_3 \geq 1$. If $|R| = a(g_1 + g_2) + (a+1)g_3$ and $|B| = bg_1 + (b+1)(g_2 + g_3)$, then there exists a subdivision $X_1 \cup \cdots \cup X_{g_1} \cup Y_1 \cup \cdots \cup Y_{g_2} \cup Z_1 \cup \cdots \cup Z_{g_3}$ of the plane into $g_1+g_2+g_3$ disjoint convex polygons such that every X_i contains exactly a red points and b blue points, every Y_i contains exactly a red points and $b+1$ blue points, and every Z_i contains exactly $a+1$ red points and $b+1$ blue points, if $g_1 \geq 1$, $g_2 \geq 1$ and $g_3 \geq 1$, respectively (see Figure 1).*

We call the subdivision of the plane given in the above theorem a *general balanced subdivision*. Moreover, we notice that our proof of the above Theorem 4 gives $O(n^4)$ time algorithm for finding a balanced subdivision of the plane, where n is the total number of red and blue points.

Before giving proofs, we remark that it seems to be impossible to derive our Theorem 4 from Theorem 1. Namely, someone may consider in the following way. Add some new imaginary red points and blue points so that in the resulting plane, there are exactly $a(g+1)$ red points and $(b+1)g$ blue points. Then we apply Theorem 1 to obtain an equitable subdivision of the plane, and remove the imaginary points. However, it seems to be impossible to guarantee that we can add new points so that each polygon of an equitable subdivision contains at most one imaginary red point and at most one blue point. Namely, some polygon may contain more than one imaginary red point or more than one imaginary blue point. Therefore, it seems to be impossible to derive Theorem 4 from Theorem 1.

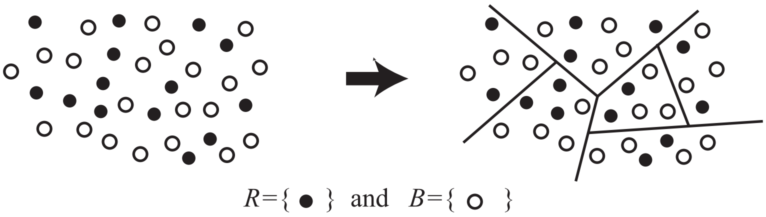

Figure 1: $g_1 = 3,\ g_2 = 1,\ g_3 = 2$, $|R| = 2(g_1 + g_2) + 3g_3$ and $|B| = 3g_1 + 4(g_2 + g_3)$; and a balanced subdivision of the plane.

2 Proof of Theorem 4

We begin with some definitions and notation. We deal only with *directed lines* in order to define the right side of a line and the left side of it. Thus a *line* means a directed line. A line l dissects the plane into three pieces: l and the two open half-planes $right(l)$ and $left(l)$ that are bounded to the left and to the right of l, respectively (see Figure 2). For a line l, we define l^* as the line lying on l and having the opposite direction of l (see Figure 2). Furthermore, we always assume that a line does not pass through any point in $R \cup B$.

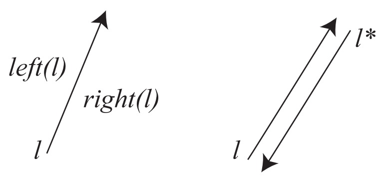

Figure 2: $right(l)$, $left(l)$ and the line l^*.

Lemma 5 *Let $a \geq 1$, $b \geq 1$ and $g \geq 1$ be integers. If $ag \leq |R| \leq (a+1)g$ and $bg \leq |B| \leq (b+1)g$, then we can uniquely express either $|R| = a(g_1 + g_2) + (a+1)g_3$ and $|B| = bg_1 + (b+1)(g_2 + g_3)$, or $|R| = ag_1 + (a+1)(g_2 + g_3)$ and $|B| = b(g_1 + g_2) + (b+1)g_3$, where $g_1 \geq 0$, $g_2 \geq 0$, $g_3 \geq 0$ and $g = g_1 + g_2 + g_3$.*

(*p*roof) We can uniquely express $|R| = ax + (a+1)(g-x)$ and $|B| = by + (b+1)(g-y)$ for some integers $0 \leq x \leq g$ and $0 \leq y \leq g$. If $x \geq y$, then by letting $g_1 = y$, $g_2 = x - y$ and $g_3 = g - x$, we can express $|R| = a(g_1 + g_2) + (a+1)g_3$ and $|B| = bg_1 + (b+1)(g_2 + g_3)$. Otherwise, by letting $g_1 = x$, $g_2 = y - x$ and $g_3 = g - y$, we have $|R| = ag_1 + (a+1)(g_2 + g_3)$ and $|B| = b(g_1 + g_2) + (b+1)g_3$. The uniqueness of the expression can be easily proved. □

The following theorem, called the 3-cutting Theorem, plays an important role. This theorem was proved by Bespamyatnikh, Kirkpatrick and Snoeyink [2] under the assumption that

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} = \frac{m_3}{n_3}.$$

However this condition can be removed without changing the arguments in the proof given in [2]. This relaxation is necessary to prove our Theorem 4. Note that similar results, which seem to be essentially equivalent to the original 3-cutting Theorem, were obtained in [3] and [9], respectively.

Theorem 6 (The 3-cutting Theorem [2]) *Let $m_1, m_2, m_3, n_1, n_2, n_3$ be positive integers such that $|R| = m_1 + m_2 + m_3$ and $|B| = n_1 + n_2 + n_3$. Suppose that one of the following statements (i) or (ii) is true:*

(i) For every integer $i \in \{1, 2, 3\}$ and for every line l such that $|right(l) \cap R| = m_i$, we have $|right(l) \cap B| < n_i$ (Figure 3 (a)).

(ii) For every integer $i \in \{1, 2, 3\}$ and for every line l such that $|right(l) \cap R| = m_i$, we have $|right(l) \cap B| > n_i$.

Then there exists three rays emanating from a certain same point such that the three open polygons W_i $(1 \leq i \leq 3)$ defined by these three rays are convex, and each W_i $(1 \leq i \leq 3)$ contains exactly m_i red points and h_i blue points (Figure 3 (b)). Moreover, one of the three rays can be chosen to be a vertically downward ray.

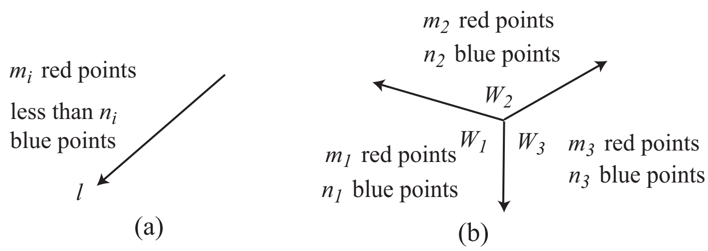

Figure 3: A subdivision $W_1 \cup W_2 \cup W_3$ of the 3-cutting Theorem.

Two different proofs of the next lemma can be found in [5] and [2].

Lemma 7 *If there exist two lines l_1 and l_2 such that $|right(l_1) \cap R| = |right(l_2) \cap R|$ and $|right(l_1) \cap B| < |right(l_2) \cap B|$, then for every integer n, $|right(l_1) \cap B| \leq n \leq |right(l_2) \cap B|$, there exists a line l_3 such that $|right(l_3) \cap R| = |right(l_1) \cap R|$ and $|right(l_3) \cap B| = n$.*

Proof of Theorem 4. Let $g = g_1 + g_2 + g_3$. We shall prove the theorem by induction on g. It is trivial that the theorem holds for $g = 1$, and so we may assume $g \geq 2$. Moreover, by Theorem 1, if $g_1 = g_2 = 0$, $g_2 = g_3 = 0$ or $g_1 = g_3 = 0$ then the theorem is true. So we may assume that

$$\text{at least two of } g_1,\ g_2 \text{ and } g_3 \text{ are greater than or equal to 1.} \tag{2}$$

Assume that there exist three integers $r \geq 0$, $s \geq 0$, $t \geq 0$ and two lines l_1 and l_2 such that $1 \leq r+s+t \leq g-1$, $|right(l_1) \cap R| = |right(l_2) \cap R| = a(r+s)+(a+1)t$, $|right(l_1) \cap B| \leq br+(b+1)(s+t)$ and $|right(l_2) \cap B| \geq br+(b+1)(s+t)$. Then by Lemma 7, there exists a line l_3 that satisfies

$$|right(l_3) \cap R| = a(r+s)+(a+1)t \quad \text{and} \quad |right(l_3) \cap B| = br+(b+1)(s+t). \tag{3}$$

By applying the inductive hypotheses to $right(l_3)$ and $left(l_3)$ respectively, we can obtain the desired balanced subdivision of the plane. Therefore we may assume that the next claim holds.

Claim 1. *Let (r,s,t) be a triple of integers such that $0 \leq r \leq g_1$, $0 \leq s \leq g_2$, $0 \leq t \leq g_3$ and $1 \leq r+s+t \leq g-1$. Then for every line l with $|right(l) \cap R| = a(r+s)+(a+1)t$, we always have either*

$$|right(l) \cap B| < br+(b+1)(s+t) \quad or \quad |right(l) \cap B| > br+(b+1)(s+t),$$

in particular, $|right(l) \cap B| \neq br+(b+1)(s+t)$.

By Claim 1, we can define the sign of every triple (i,j,k) with $0 \leq i \leq g_1$, $0 \leq j \leq g_2$, $0 \leq k \leq g_3$ and $1 \leq i+j+k \leq g-1$ as follows: For every line l with $|right(l) \cap R| = a(i+j)+(a+1)k$,

$$\begin{aligned}
&\text{if } |right(l) \cap B| > bi+(b+1)(j+k), \text{ then } sign(i,j,k) = +; \text{ and} \\
&\text{if } |right(l) \cap B| < bi+(b+1)(j+k), \text{ then } sign(i,j,k) = -.
\end{aligned}$$

The next claim gives an easy but useful property of $sign(i,j,k)$.

Claim 2. *Let i, j and k be integers such that $0 \leq i \leq g_1$, $0 \leq j \leq g_2$, $0 \leq k \leq g_3$ and $1 \leq i+j+k \leq g-1$. Then*

$$sign(g_1-i, g_2-j, g_3-k) = -sign(i,j,k). \tag{4}$$

We may assume that $sign(i,j,k) = +$ since otherwise we can similarly prove the claim. Let l be a line such that $|right(l) \cap R| = a(i+j)+(a+1)k$. Then $|right(l) \cap B| > bi+(b+1)(j+k)$ by $sign(i,j,k) = +$. This implies that

$$\begin{aligned}
|right(l^*) \cap R| &= a(g_1-i+g_2-j)+(a+1)(g_3-k), \quad \text{and} \\
|right(l^*) \cap B| &= |left(l) \cap B| = |B| - |right(l) \cap B| \\
&< b(g_1-i)+(b+1)(g_2-j+g_3-k).
\end{aligned}$$

Hence $sign(g_1-i, g_2-j, g_3-k) = - = -sign(i,j,k)$.

Claim 3. *We may assume that the following four statements hold. (i) If $g_1 \geq 1$ $g_2 \geq 1$ and $g_3 \geq 1$, then $sign(1,0,0) = sign(0,1,0) = sign(0,0,1)$. (ii) If $g_1 = 0$, then $g_2 \geq 1$, $g_3 \geq 1$ and $sign(0,1,0) = sign(0,0,1)$. (iii) If $g_2 = 0$, then $g_1 \geq 1$, $g_3 \geq 1$ and $sign(1,0,0) = sign(0,0,1)$. (iv) If $g_3 = 0$, then $g_1 \geq 1$, $g_2 \geq 1$ and $sign(1,0,0) = sign(0,1,0)$.*

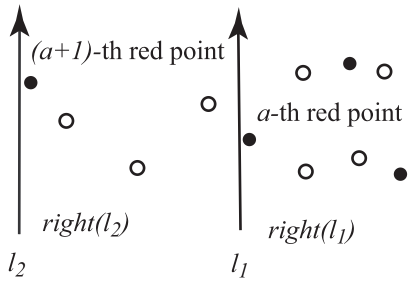

Figure 4: Lines l_1 and l_2.

(proof) We first consider the case where $g_1 \geq 1$, $g_2 \geq 1$, $g_3 \geq 1$ and $sign(1,0,0) = -$. Let l_1 and l_2 be two parallel lines such that $|right(l_1) \cap R| = a$, $|right(l_2) \cap R| = a+1$ and that l_1 and l_2 pass very close to the a-th red point and $(a+1)$-th red point, respectively (see Figure 4). Then $|right(l_1) \cap B| < b$ by $sign(1,0,0) = -$.

The existence of l_1 implies $sign(0,1,0) = -$ by Claim 1. Assume $sign(0,0,1) = +$. Then $|right(l_2) \cap B| > b+1$, and thus we can find a line l_3 between l_1 and l_2 such that $right(l_3)$ contains exactly a red points and $b+1$ blue points. This contradicts Claim 1 with $(0,1,0)$. Hence $sign(0,0,1) = -$.

We next consider the case where $g_1 \geq 1$, $g_2 \geq 1$, $g_3 \geq 1$ and $sign(1,0,0) = +$. Let l_1 and l_2 be the two parallel lines given above. Then $|right(l_1) \cap B| > b$ by $sign(1,0,0) = +$. Since $|right(l_1) \cap B| \neq b+1$ by Claim 1 with $(0,1,0)$, we have $|right(l_1) \cap B| \geq b+2$. Hence $sign(0,1,0) = +$. Furthermore, since $|right(l_2) \cap B| \geq |right(l_1) \cap B| \geq b+2$, we have $sign(0,0,1) = +$.

By the above two results, we can say that for every $\alpha \in \{+,-\}$, $sign(0,1,0) = \alpha$ implies $sign(1,0,0) = \alpha$ and $sign(0,0,1) = \alpha$. Consequently, the statement (i) is proved.

Suppose $g_1 = 0$. Then $g_2 \geq 1$ and $g_3 \geq 1$ by (2). Let l_1 and l_2 be the same lines as given above. Assume $sign(0,1,0) = -$. Then $|right(l_1) \cap B| < b+1$ by $sign(0,1,0) = -$. If $sign(0,0,1) = +$, then $|right(l_2) \cap B| > b+1$, and so we can find a line l_4 between l_1 and l_2 such that $|right(l_4) \cap R| = a$ and $|right(l_4) \cap B| = b+1$, which contradicts Claim 1 with (0,1,0). Hence $sign(0,0,1) = -$. Next, assume $sign(0,1,0) = +$. Then $|right(l_1) \cap B| > b+1$, and so $|right(l_2) \cap B| > b+1$. Hence $sign(0,0,1) = +$. Therefore (ii) is proved.

Suppose that $g_2 = 0$. Then $g_1 \geq 1$ and $g_3 \geq 1$ by (2). Let l_1 and l_2 be the same lines as given above. Assume $sign(1,0,0) = -$. Then $|right(l_1) \cap B| < b$. If $sign(0,0,1) = +$, then $|right(l_2) \cap B| > b+1$, and so we can find a line l_5 between l_1 and l_2 such that $|right(l_5) \cap R| = a$ and $|right(l_5) \cap B| = b$, which contradicts Claim 1 with $(1,0,0)$. Hence $sign(0,0,1) = -$. Assume $sign(1,0,0) = +$. Then $|right(l_1) \cap B| > b$, and thus $|right(l_2) \cap B| > b+1$ by Claim 1 with (0,0,1). Hence $sign(0,0,1) = +$. Consequently (iii) is true.

We finally consider the case where $g_3 = 0$. Then $g_1 \geq 1$ and $g_2 \geq 1$. Let l_1 and l_2 be the same line as given above. If $sign(1,0,0) = -$, then $sign(0,1,0) = -$ as $|right(l_1) \cap B| < b$.

If $sign(1,0,0) = +$, then $|right(l_1) \cap B| > b$, and so $|right(l_1) \cap B| > b+1$ by Claim 1 with $(0,1,0)$, which implies $sign(0,1,0) = +$. Consequently, (iv) holds, and hence Claim 3 is proved.

Because of symmetry, hereafter we assume

$$sign(1,0,0) = sign(0,1,0) = sign(0,0,1) = -, \tag{5}$$

when we can consider these signs. We say that three triples (r_1, s_1, t_1), (r_2, s_2, t_2), (r_3, s_3, t_3) satisfy the condition of the 3-cutting Theorem if

$$\begin{aligned} &g_1 = r_1 + r_2 + r_3, \quad g_2 = s_1 + s_2 + s_3, \quad g_3 = t_1 + t_2 + t_3 \\ &sign(r_1, s_1, t_1) = sign(r_2, s_2, t_2) = sign(r_3, s_3, t_3), \quad \text{and} \\ &0 \le r_i, s_i, t_i \quad \text{and} \quad 1 \le r_i + s_i + t_i \quad \text{for every} \quad i \in \{1,2,3\}. \end{aligned}$$

If a set of three triples $\{(r_i, s_i, t_i) \mid 1 \le i \le 3\}$ satisfies the above conditions, then

$$m_i = a(r_i + s_i) + (a+1)t_i \quad \text{and} \quad n_i = br_i + (b+1)(s_i + t_i), \quad (1 \le i \le 3)$$

satisfy the condition of the 3-cutting Theorem, and so we can subdivide the plane into three convex polygons, each of which contains exactly m_i red points and n_i blue points, and thus we can obtain the desired subdivision by applying the inductive hypotheses to each convex polygon.

Claim 4. *If $g_1 = 0$, then there exist three triples (r_i, s_i, t_i) $(1 \le i \le 3)$ that satisfy the condition of the 3-cutting Theorem, and thus the theorem 4 holds.*

Since $g_1 = 0$, we have $g_2 \ge 1$ and $g_3 \ge 1$ by (2). Then $sign(0, g_2 - 1, g_3) = -sign(0,1,0)$ by Claim 2. Choose the lowest element $(0,s,t)$ in lexicographical order such that $sign(0,s,t) \ne sign(0,1,0)$, that is, $sign(0,s',t') = sign(0,1,0)$ if $s' < s$ or $s' = s$ and $t' < t$. If $s = 0$, then $t \ge 2$ by (5), and so we obtain the following three triples that satisfy the conditions of the 3-cutting Theorem.

$$(0, g_2, g_3 - t), \quad (0,0,t-1), \quad (0,0,1),$$

where $sign(0, g_2, g_3 - t) = -sign(0,0,t) = sign(0,1,0) = sign(0,0,1) = -$ by Claims 2 and 3, and $sign(0,0,t-1) = sign(0,1,0) = -$ by the choice of $(0,s,t)$ and $s = 0$. If $s \ge 1$, then $s \le g_2 - 1$ by the above fact that $sign(0, g_2 - 1, g_3) = -sign(0,1,0)$, and so $1 \le g_2 - s$. Furthermore, if $s = 1$, then $1 \le t$ by $sign(0,s,t) \ne sign(0,1,0)$ and (5). Hence by claim 2, we can obtain the following desired three triples:

$$(0, g_2 - s, g_3 - t), \quad (0, s-1, t), \quad (0,1,0),$$

where $sign(0, g_2 - s, g_3 - t) = -sign(0,s,t) = sign(0,1,0) = -$. Therefore Claim 4 is proved.

Claim 5. *If $g_1 \ge 1$, then we may assume that $sign(0,j,k) = sign(1,0,0)$ for all $0 \le j \le g_2$ and $0 \le k \le g_3$ with $1 \le j + k$ since otherwise the theorem 4 holds.*

Assume that $sign(0,j,k) \neq sign(1,0,0)$ for some $0 \leq j \leq g_2$ and $0 \leq k \leq g_3$ with $1 \leq j+k$. Choose the lowest element $(0,s,t)$ in lexicographical order such that $sign(0,s,t) \neq sign(1,0,0)$. By (5), it follows that $2 \leq s+t$. By Claim 2, we have $sign(g_1, g_2-s, g_3-t) = -sign(0,s,t) = sign(1,0,0) = sign(0,1,0)$. If $s \geq 2$, then by (5) we obtain the following three triples that satisfy the conditions of the 3-cutting Theorem.

$$(g_1, g_2-s, g_3-t), \quad (0, s-1, t), \quad (0,1,0).$$

If $s=1$, then $t \geq 1$ and so we have the following three triples:

$$(g_1, g_2-1, g_3-t), \quad (0,1,t-1), \quad (0,0,1).$$

If $s=0$, then $t \geq 2$, and we obtain the desired three triples as follows:

$$(g_1, g_2, g_3-t), \quad (0,0,t-1), \quad (0,0,1).$$

In each case, the theorem is proved by induction, and hence we may assume that Claim 5 holds.

Claim 6. *If $g_1 \geq 1$, then there exist three triples (r_i, s_i, t_i) $(1 \leq i \leq 3)$ that satisfy the condition of the 3-cutting Theorem, and hence the theorem 4 holds.*

By (2), we have $g_2 \geq 1$ or $g_3 \geq 1$. Here we assume $g_2 \geq 1$ since we can similarly prove the claim in the case of $g_3 \geq 1$. Since $sign(g_1, g_2-1, g_3) = -sign(0,1,0) = -sign(1,0,0)$, there exists (i,j,k) such that $sign(i,j,k) \neq sign(1,0,0)$, $0 \leq i \leq g_1$, $0 \leq j \leq g_2$, $0 \leq k \leq g_3$ and $1 \leq i+j+k \leq g-1$. Choose the lowest element (r,s,t) in lexicographical order such that $sign(r,s,t) \neq sign(1,0,0)$, in particular, $sign(r',s',t') = sign(1,0,0)$ if $r' < r$. By Claim 5, we have $r \geq 1$, and if $r=1$, then $s+t \geq 1$ by $sign(1,s,t) \neq sign(1,0,0)$. Hence we obtain the following desired three triples that satisfy the conditions of the 3-cutting Theorem.

$$(g_1-r, g_2-s, g_3-t), \quad (r-1, s, t), \quad (1,0,0),$$

where $sign(g_1-r, g_2-s, g_3-t) = -sign(r,s,t) = sign(1,0,0)$ and $sign(r-1,s,t) = sign(1,0,0)$.

By Claims 4 and 6, the proof is complete. □

We now analyze the time complexity of an algorithm for finding a balanced subdivision of the plane, which is directly obtained from our proof. We first consider an algorithm for finding a line l_3 that satisfies (3). Since there are $O(n^2)$ lines passing through two points of $R \cup B$, we can find a line l_3 satisfying (3), if any, in $O(n^3)$ time. Notice that for a line l passing through two points x and y of $R \cup B$, we consider the four cases; (i) $x, y \in right(l)$, (ii) $x \in right(l)$ and $y \notin right(l)$, (iii) $x \notin right(l)$ and $y \in right(l)$; and (iv) $x, y \notin right(l)$. If there exists no such line l, then we can define $sign(i,j,k)$, and there exist three rays given in the 3-cutting Theorem. We can find such three rays in

$O(n^{\frac{4}{3}}(\log n)^2)$ by [2]. Therefore in any case, if we denote by $f(n)$ the time complexity of finding a balanced subdivision of the plane with $n = |R| + |B|$ points, then we have

$$f(n) \leq O(n^3) + f(n_1) + f(n_2) + f(n_3),$$

where $n_1 + n_2 + n_3 = n$, $n_1 \geq a + b$, $n_2 \geq a + b$, $n_3 \geq 0$, and $n_3 = 0$ if and only if there exists a line l_3. Consequently, we obtain $f(n) \leq O(n^4)$.

References

[1] Bárány, I., Matoušek, J.: Simultaneous partitions of measures by k-fans. Discrete Comput. Geom. **25** (2001)317–334.

[2] Bespamyatnikh, S., Kirkpatrick, D., Snoeyink, J.: Generalizing ham sandwich cuts to equitable subdivisions. Discrete Comput. Geom. **24** (2000) 605–622.

[3] Ito, H., Uehara, H., Yokoyama, M.: 2-dimensional ham-sandwich theorem for partitioning into three convex pieces. Discrete Comput. Geom. LNCS **1763** (2000) 129-157.

[4] *Handbook of Discrete and Computational Geometry*, edited by J. Goodman and J. O'Rourke, CRC Press, (2004) Chapter 14 Topological Methods written by Rade T. Živaljević , 305–329.

[5] Kaneko, A., Kano, M.: Balanced partitions of two sets of points in the plane. Computational Geometry: Theory and Applications, **13** (1999), 253–261.

[6] Kaneko, A., Kano, M.: Semi-balanced partitions of two sets of points in the plane and embeddings of rooted forests. Internat. J. Comput. Geom. Appl. in print.

[7] Kaneko, A., Kano, M.: Discrete Geometry on Red and Blue Points in the Plane – A Survey –. Discrete and Computational Geometry, Algorithms Combin., **25**, Springer (2003) 551-570.

[8] Kaneko, A., Kano, M., Suzuki, H.: Path Coverings of Two Sets of Points in the Plane. Towards a theory of geometric graphs, ed by J. Pach Contemporary Mathematics series of AMS, **342** (2004) 99-111.

[9] Sakai, T.: Balanced Convex Partitions of Measures in R^2. Graphs and Combinatorics, **18** (2002), 169–192.