

Pan-factorial Property in Regular Graphs*

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Abstract

Among other results, we show that if for any given edge e of an r -regular graph G of even order, G has a 1-factor containing e , then G has a k -factor containing e and another one avoiding e for all k , $1 \leq k \leq r - 1$.

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For a function $f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$, a spanning subgraph F of G with $\deg_F(x) = f(x)$ for all $x \in V(G)$ is called an f -factor of G , where $\deg_F(x)$ denotes the degree of x in F . If $f(x) = k$ for all vertices $x \in V(G)$, then an f -factor is also called a k -regular factor or a k -factor. An $[a, b]$ -factor is a spanning subgraph F of G such that $a \leq \deg_F(x) \leq b$ for all $x \in V(G)$.

A graph G is *pan-factorial* if G contains all k -factors for $1 \leq k \leq \delta(G)$. In this note, we investigate the pan-factor property in regular graphs. Moreover, we proved that the existence of 1-factor containing any given edge implies the existence of k -factors containing or avoiding any given edge.

The first of our main results is the following.

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Theorem 1 *Let G be a connected r -regular graph of even order. If for every edge e of G , G has a 1-factor containing e , then G has a k -factor containing e and another k -factor avoiding e for all integers k , $1 \leq k \leq r - 1$.*

The next theorem is also one of our main results.

Theorem 2 *Let G be a connected graph of even order, e be an edge of G , and a, b, c be odd integers such that $1 \leq a < c < b$. If G has both a a -factor and b -factor containing e , then G has a c -factor containing e . Similarly, if G has both a a -factor and b -factor avoiding e , then G has a c -factor avoiding e .*

The above theorem shows that there exists a kind of continuity relation among regular factors, which is an improvement of the following theorem obtained by Katerinis [2].

Theorem 3 (Katerinis [2]) *Let G be a connected graph of even order, and a, b and c be odd integers such that $1 \leq a < c < b$. If G has both a a -factor and b -factor, then G has a c -factor.*

We need a few known results as lemmas for the proof of our theorems. Firstly, we quote Petersen's classic decomposition theorem about regular graphs of even degree.

Lemma 1 (Petersen [3]) *Every $2r$ -regular graph can be decomposed into r disjoint 2-factors.*

For the introduction of Tutte's f -factors theorem, we require the following notation. For a graph G and $S, T \subseteq V(G)$ with $S \cap T = \emptyset$, define

$$\delta_G(S, T) = \sum_{x \in S} f(x) + \sum_{x \in T} (d_{G-S}(x) - f(x)) - h_G(S, T),$$

where $h_G(S, T)$ is the number of components C of $G - (S \cup T)$ such that $\sum_{x \in V(C)} f(x) + e_G(V(C), T) \equiv 1 \pmod{2}$ and such a component C is called an f -odd component of $G - (S \cup T)$.

Lemma 2 (Tutte's f -factor Theorem [4]) *Let G be a graph and $f : V(G) \rightarrow \{0, 1, 2, 3, \dots\}$ be a function. Then*

- (a) G has an f -factor if and only if $\delta_G(S, T) \geq 0$ for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$;
- (b) $\delta_G(S, T) \equiv \sum_{x \in V(G)} f(x) \pmod{2}$ for all $S, T \subseteq V(G)$ with $S \cap T = \emptyset$.

Lemma 3 *Let G be a connected graph. If for any edge e there exists a 1-factor containing e , then there exists another 1-factor avoiding e .*

Proof. For any edge $e \in E(G)$, we will show that there exists a 1-factor avoiding e . Choose an edge e' incident to the given edge e , then there exists a 1-factor F containing e' and thus F is the 1-factor avoiding e . \square

Now we are ready to show the main results. We start with the proof of Theorem 2 and then derive the proof of Theorem 1 from it.

Proof of Theorem 2. Let e be an edge of G . Assume that G has both a -factor and b -factor avoiding e . By applying Theorem 3 to $G - e$, we see that $G - e$ has a c -factor, which implies that G has a c -factor avoiding e .

We now prove that if G has both a -factor and b -factor containing e , then G has a c -factor containing e .

We define a new graph G^* by inserting a new vertex w on the edge e , and define an integer-value function $f_k : V(G^*) \rightarrow \{k, 2\}$ such that

$$f_k(x) = \begin{cases} k & \text{if } x \in V(G); \\ 2 & \text{if } x = w. \end{cases}$$

Then G has a k -factor containing e if and only if G^* has a f_k -factor. It is obvious that $\sum_{x \in V(G^*)} f_k(x) = k|V(G)| + 2 \equiv 0 \pmod{2}$ since G is of even order.

Assume that G^* has no f_c -factor. Then, by Tutte's f -factor Theorem, there exist two disjoint subsets $S, T \subseteq V(G^*)$ such that

$$\delta(S, T; f_c) = \sum_{x \in S} f_c(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_c(x)) - h(S, T; f_c) \leq -2. \quad (1)$$

On the other hand, since G^* has both f_a -factor and f_b -factor, we have

$$\delta(S, T; f_a) = \sum_{x \in S} f_a(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_a(x)) - h(S, T; f_a) \geq 0, \quad (2)$$

$$\delta(S, T; f_b) = \sum_{x \in S} f_b(x) + \sum_{x \in T} (\deg_{G^*-S}(x) - f_b(x)) - h(S, T; f_b) \geq 0. \quad (3)$$

Now depending on the location of w , we consider three cases:

Case 1. $w \notin S \cup T$.

(1), (2) and (3) can be rewritten as

$$c|S| + \sum_{x \in T} \deg_{G^*-S}(x) - c|T| - h(S, T; f_c) \leq -2, \quad (4)$$

$$a|S| + \sum_{x \in T} \deg_{G^*-S}(x) - a|T| - h(S, T; f_a) \geq 0, \quad (5)$$

$$b|S| + \sum_{x \in T} \deg_{G^*-S}(x) - b|T| - h(S, T; f_b) \geq 0. \quad (6)$$

Subtracting (5) from (4), we have

$$(c - a)(|S| - |T|) + h(S, T; f_a) - h(S, T; f_c) \leq -2. \quad (7)$$

Similarly, from (6) and (4), we have

$$(c - b)(|S| - |T|) + h(S, T; f_b) - h(S, T; f_c) \leq -2. \quad (8)$$

Recall that $h(S, T; f_k)$ is the number of f_k -odd components C of $G^* - (S \cup T)$, which satisfies $\sum_{x \in V(C)} f_k(x) + e_{G^*}(C, T) \equiv 1 \pmod{2}$. Since all a , b and c are odd integers, it follows that if $w \notin V(C)$, then

$$\begin{aligned} \sum_{x \in V(C)} f_a(x) + e_{G^*}(C, T) &= a|C| + e_{G^*}(C, T) \\ &\equiv b|C| + e_{G^*}(C, T) = \sum_{x \in V(C)} f_b(x) + e_{G^*}(C, T) \pmod{2} \\ &\equiv c|C| + e_{G^*}(C, T) = \sum_{x \in V(C)} f_c(x) + e_{G^*}(C, T) \pmod{2}. \end{aligned}$$

Therefore we obtain

$$h(S, T; f_c) - h(S, T; f_a) \leq 1 \quad \text{and} \quad h(S, T; f_c) - h(S, T; f_b) \leq 1.$$

If $|S| \geq |T|$, then (7) implies

$$-1 \leq (c - a)(|S| - |T|) + h(S, T; f_a) - h(S, T; f_c) \leq -2,$$

a contradiction. If $|S| < |T|$, then (8) implies

$$-1 \leq (c - b)(|S| - |T|) + h(S, T; f_b) - h(S, T; f_c) \leq -2,$$

a contradiction again.

Case 2. $w \in S$.

In this case, (1), (2) and (3) become

$$\begin{aligned} 2 + c(|S| - 1) + \sum_{x \in T} \deg_{G^* - S}(x) - c|T| - h(S, T; f_c) &\leq -2 \\ 2 + a(|S| - 1) + \sum_{x \in T} \deg_{G^* - S}(x) - a|T| - h(S, T; f_a) &\geq 0 \\ 2 + b(|S| - 1) + \sum_{x \in T} \deg_{G^* - S}(x) - b|T| - h(S, T; f_b) &\geq 0. \end{aligned}$$

It is clear that $h(S, T; f_c) = h(S, T; f_a) = h(S, T; f_b)$. If $|S| \geq |T| + 1$, we have $0 \leq (c - a)(|S| - 1 - |T|) \leq -2$, a contradiction; if $|S| < |T| + 1$, then $0 \leq (c - b)(|S| - 1 - |T|) \leq -2$, a contradiction as well.

Case 3. $w \in T$.

In this case, (1), (2) and (3) become

$$\begin{aligned} c|S| + \sum_{x \in T} \deg_{G^* - S}(x) - 2 - c(|T| - 1) - h(S, T; f_c) &\leq -2 \\ a|S| + \sum_{x \in T} \deg_{G^* - S}(x) - 2 - a(|T| - 1) - h(S, T; f_a) &\geq 0 \\ b|S| + \sum_{x \in T} \deg_{G^* - S}(x) - 2 - b(|T| - 1) - h(S, T; f_b) &\geq 0. \end{aligned}$$

Discussing similarly as in Case 2, we yield contradictions. Consequently the theorem is proved. \square

With help of Theorem 2 and Petersen's Theorem (Lemma 1), we can provide a clean proof for Theorem 1.

Proof of Theorem 1. For any edge e of G , let F_1 be a 1-factor containing e . From Lemma 3, there exists another 1-factor F_2 avoiding e . According to the parity of r we consider two cases.

Case 1. r is odd.

Since $G - F_1$ is an even regular graph, by Lemma 1, $G - F_1$ can be decomposed into 2-factors T_1, T_2, \dots, T_m , where $m = (r - 1)/2$. For an integer k ($1 \leq k \leq m - 1$), $F_1 \cup T_1 \cup \dots \cup T_k$ is a $(2k + 1)$ -factor containing e . In the mean time, $T_1 \cup \dots \cup T_k$ is a $2k$ -factor avoiding e . Moreover, $G - F_1$ is a $2m$ -factor avoiding e .

Similarly, $G - F_2$ has disjoint 2-factors T_1, T_2, \dots, T_m . Without loss of generality, we may assume $e \in T_1$. Then $F_2 \cup T_2 \cup \dots \cup T_{k+1}$ is a $(2k + 1)$ -factor avoiding e , and $T_1 \cup \dots \cup T_k$ is a $2k$ -factor containing e . Furthermore, $G - F_2$ is a $2m$ -factor containing e . Therefore the theorem holds in this case.

Case 2. r is even.

For even k , similar to Case 1, G can be decomposed into 2-factors T_1, T_2, \dots, T_m , where $m = r/2$. Without loss of generality, assume $e \in T_1$. Then $T_1, T_1 \cup T_2, \dots, T_1 \cup \dots \cup T_m$ are 2-factor, 4-factor, \dots , r -factor containing e , respectively. Moreover, $T_2, T_2 \cup T_3, \dots, T_2 \cup T_3 \cup \dots \cup T_m$ are 2-factor, 4-factor, \dots , $(r - 2)$ -factor avoiding e , respectively.

For odd k , it is clear that $G - F_2$ is a $(r - 1)$ -factor containing e and $G - F_1$ is an $(r - 1)$ -factor avoiding e . By Theorem 2, the odd-factors F_1 and $G - F_2$ containing e , respectively, imply the existence of k -factors containing e , $1 \leq k \leq r - 1$. Similarly, we obtain k -factors avoiding e , $1 \leq k \leq r - 1$.

So the desired statement holds and consequently the theorem is proved. \square

Next we consider the existence of factors containing or avoiding a given edge in a regular graph of *odd* order and prove a similar but slightly weaker result than Theorem 1.

Theorem 4 *Let G be a connected $2r$ -regular graph of odd order. For any given edge e and any vertex $v \in V(G) - V(e)$, if $G - v$ has a 1-factor containing e , then $G - v$ has a $[k, k + 1]$ -factor containing or avoiding e for $1 \leq k \leq 2r - 2$.*

Proof. For any edge e of G and any vertex $u \in V(G) - V(e)$, let the neighbor vertices of u be x_1, x_2, \dots, x_{2r} . We construct a new graph G^* by using two copies of $G - u$ and joining two sets of vertices $\{x_1, x_2, \dots, x_{2r}\}$ by a matching M . Then the resulting graph G^* is a $2m$ -regular graph with $2(|V(G)| - 1)$ vertices. Since $G - u$ has a 1-factor containing e , so does G^* . By Theorem 1, G^* has a k -factor containing e and another k -factor avoiding e for all k , $1 \leq k \leq 2r - 1$. Deleting the matching M from G^* , we obtain a $[k, k + 1]$ -factor

containing or avoiding e for $1 \leq k \leq 2r - 2$. □

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