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A Balanced Interval of Two Sets of Points on a Line

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Abstract. Let n, m, k, h be positive integers such that $1 \leq n \leq m$, $1 \leq k \leq n$ and $1 \leq h \leq m$. Then we give a necessary and sufficient condition for a configuration with n red points and m blue points on a line to have an interval containing precisely k red points and h blue points.

1 Introduction

In this paper we shall prove the following theorem:

Theorem 1. *Let n, m, k, h be integers such that $1 \leq n \leq m$, $1 \leq k \leq n$ and $1 \leq h \leq m$. Then for any n red points and m blue points on a line in general position (i.e., no two points lie on the same position.), there exists an interval that contains precisely k red points and h blue points if and only if*

$$\left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1\right)(h-1) < m < \left(\left\lfloor \frac{n-1}{k-1} \right\rfloor\right)(h+1), \tag{1}$$

where the rightmost term is an infinite number when $k = 1$.

Before giving proofs, let us give an example and explain results related to our theorem. Consider a configuration consisting of 10 red points and 20 blue points on a line in general position. Then by the above theorem, we can easily show that if $k \in \{1, 2, 3, 5, 10\}$, then such a configuration has an interval containing exactly k red points and $2k$ blue points; otherwise (i.e., $k \in \{4, 6, 7, 8, 9\}$) there exist a configuration that has no such an interval (Fig. 1). We call an interval that contains given number of red points and blue points *a balanced interval*.

Let n and $m = \lambda n$ be positive integers, where $\lambda = \frac{m}{n}$ is a rational number. Suppose that there are n red points and λn blue points in the plane in general position. Then for given point P in the plane and for any integer $1 \leq k \leq n$, there exist two rays r_1 and r_2 emanating from P such that one of the open regions determined by r_1 and r_2 contains precisely k red points and $\lfloor \lambda k \rfloor$ blue points [1] (Fig. 2 (a)) . If we choose a point P infinitely far away, then two rays

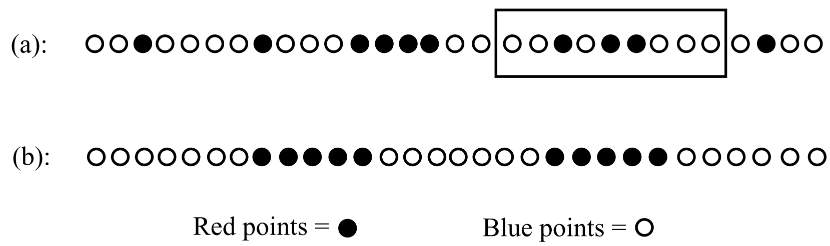

Fig. 1. (a): An interval containing 3 red points and 6 blue points; (b): A configuration that has no interval containing exactly 4 red points and 8 blue points.

become parallel lines, and two open regions become one connected convex region and one disconnected region consisting of two half planes ((Fig. 2 (b)). We want to know when we can always find a connected convex region determined by two parallel lines with given direction that contains precisely k red points and $\lfloor \lambda k \rfloor$ blue points. By considering the orthogonal projections of the red and blue points in the plane onto the line orthogonal to the direction, then our theorem gives an answer to this question. Other results on red and blue points in the plane can be found in [3], and definitions and notation not defined here are given in [2].

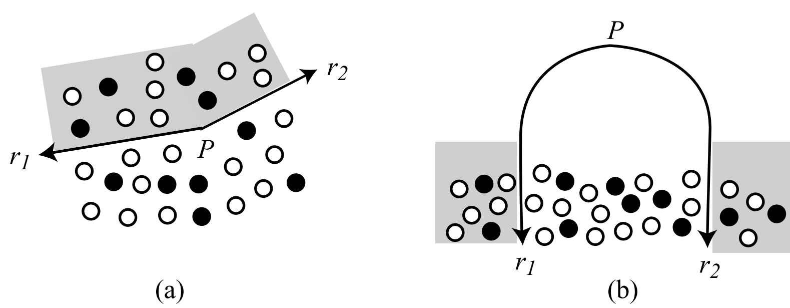

Fig. 2. (a): An open region containing 4 red points and 8 blue points; (b): A disconnected region containing 4 red points and 8 blue points when P is far away.

2 Proof

For a configuration with red and blue points on a line, we denote by R and B the sets of red points and blue points, respectively. A configuration X with n red points and m blue points on a line is expressed as

$$\{x_1\} \cup \{x_2\} \cup \cdots \cup \{x_{n+m}\},$$

where each x_i denotes a red point or a blue point ordered from left to right. The configuration X is also expressed as

$$R(1) \cup B(1) \cup \cdots \cup R(s) \cup B(s),$$

where $R(i)$ and $B(i)$ denote disjoint subsets of R and B, respectively, and some of them may be empty sets. For a set Y, we denote by $|Y|$ the cardinality of Y.

We shall prove the following five lemmas, where n, m, k and h denote integers given in the theorem, that is, they satisfy $1 \le n \le m$, $1 \le k \le n$ and $1 \le h \le m$.

Lemma 1. *If*

$$m \le \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)(h-1), \tag{2}$$

then there exists a configuration with n red points and m blue points that has no interval containing exactly k red points and h blue points.

Proof. Let $t = \lfloor \frac{n}{k+1} \rfloor$. Then $m \le (t+1)(h-1)$ by (2). Hence we can construct a configuration with n red points and m blue points as follows:

$$B(1) \cup R(1) \cup B(2) \cup R(2) \cup \cdots \cup B(t+1) \cup R(t+1),$$

where $|B(i)| \le h-1$ for every $1 \le i \le t+1$, $|B(1) \cup \cdots \cup B(t+1)| = m$, $|R(i)| = k+1$ for every $1 \le i \le t$, $|R(t+1)| = n-(k+1)t \ge 0$ and $|R(1) \cup \cdots \cup R(t+1)| = n$. Then this configuration obviously has no interval containing exactly k red points and h blue points since every interval containing h blue points must include $R(j)$ for some $1 \le j \le t$, which implies the interval contains at least $k+1$ red points.

Lemma 2. *If*

$$m > \left(\left\lfloor \frac{n}{k+1} \right\rfloor + 1 \right)(h-1), \tag{3}$$

then every configuration with n red points and m blue points has an interval containing exactly k red points and at least h blue points.

Proof. Let $t = \lfloor \frac{n}{k+1} \rfloor$. Let X be a configuration with n red points and m blue points. Suppose that X has no desired interval. Namely, we assume that every interval containing exactly k red points has at most $h-1$ blue points.

Let $r_1, r_2, \cdots, r_n$ be the red points of X ordered from left to right. For integers $1 \le i < j \le n$, let $I(i,j)$ denote an open interval (r_i, r_j), and let $B(i,j)$ denote the set of blue points contained in $I(i,j)$. Furthermore, $B(-\infty, i)$ denotes the set of blue points contained in the open interval $(-\infty, r_i)$, and $B(i, \infty)$ is defined analogously. Then for any integer $1 \le s \le t-1$, $I(s(k+1), (s+1)(k+1))$ contains exactly k red points $\{r_j \mid s(k+1)+1 \le j \le (s+1)(k+1)-1\}$, and thus $|B(s(k+1), (s+1)(k+1))| \le h-1$ by our assumption. Similarly, an open interval $(-\infty, r_{k+1})$ contains exactly k red points, and thus $|B(-\infty, k+1)| \le h-1$. Moreover, since $n < (t+1)(k+1)$, $I(t(k+1), \infty)$ has at most k red points, and thus $B(t(k+1), \infty) \le h-1$. Therefore

$$\begin{aligned} |B| &\le |B(-\infty, k+1) \cup B(k+1, 2(k+1)) \cup \cdots \cup B(t(k+1), \infty)| \\ &\le (t+1)(h-1). \end{aligned}$$

This contradicts (3). Consequently the lemma is proved.

Lemma 3. *If* $2 \le k$ *and*

$$m \ge \left\lfloor \frac{n-1}{k-1} \right\rfloor (h+1), \tag{4}$$

then there exists a configuration with n *red points and* m *blue points that has no interval containing exactly* k *red points and* h *blue points.*

Proof. Let $t = \lfloor \frac{n-1}{k-1} \rfloor$, which implies $t(k-1)+1 \le n \le (t+1)(k-1)$, and $m \ge t(h+1)$ by (4). Hence we can obtain the following configuration with n red points and m blue points:

$$R(1) \cup B(1) \cup \cdots \cup R(t+1) \cup B(t+1),$$

where $|R(i)| \le k-1$ for every $1 \le i \le t+1$, $|R(1) \cup \cdots \cup R(t+1)| = n$, $|B(i)| = h+1$ for every $1 \le i \le t$, $|B(t+1)| = m-(h+1)t \ge 0$ and $|B(1) \cup \cdots \cup B(t+1)| = m$. Then this configuration obviously has no interval containing exactly k red points and h blue points since every interval containing k red points must include $B(j)$ for some $1 \le j \le t$.

Lemma 4. *If* $2 \le k$ *and*

$$m < \left\lfloor \frac{n-1}{k-1} \right\rfloor (h+1), \tag{5}$$

then every configuration with n *red points and* m *blue points has an interval containing exactly* k *red points and at most* h *blue points.*

Proof. Let $t = \lfloor \frac{n-1}{k-1} \rfloor$. Then $t(k-1)+1 \le n$. Let X be a configuration with n red points and m blue points. Suppose that X has no desired interval. Namely, we assume that every interval containing exactly k red points has at least $h+1$ blue points.

Let $r_1, r_2, \cdots, r_n$ be the red points of X ordered from left to right. For integers $1 \le i < j \le n$, let $I[i,j]$, denote a closed interval $[r_i, r_j]$, and let $B'(i,j)$ denote the set of blue points contained in $I[i,j]$.

Then for any integer $0 \le s \le t-2$, $I[k+s(k-1), k+(s+1)(k-1)]$ contains exactly k red points $\{r_j \mid k+s(k-1) \le j \le k+(s+1)(k-1))\}$, and thus $|B'(k+s(k-1), k+(s+1)(k-1))| \ge h+1$ by our assumption. Similarly, we have $|B'(1,k)| \ge h+1$. Therefore

$$\begin{aligned} |B| &\ge |B'(1,k) \cup B'(k, k+(k-1)) \cup \cdots \\ &\quad \cup B'(k+(t-2)(k-1), k+(t-1)(k-1))| \\ &\ge t(h+1). \end{aligned}$$

This contradicts (5). Consequently the lemma is proved.

Lemma 5. *Consider a configuration with* n *red points and* m *blue points on a line. Suppose that there exists two intervals* I *and* J *such that both* I *and* J *contain exactly* k *red points respectively,* I *contains at most* h *blue points, and that* J *contains at least* h *blue points. Then there exists an interval that contains exactly* k *red points and* h *blue points.*

Proof. If the sets of red points contained in I and J, respectively, are the same, then the lemma immediately follows. Thus we may assume that $I \cap R \neq J \cap R$, where R denote the set of n red points. Without loss of generality, we may assume that the leftmost red point of I lies to the left of J.

We shall show that we can move I to J step by step in such a way that the number of red points is a constant k and the number of blue points changes ± 1 at each step. We first remove the blue points left to the leftmost red point of I one by one, and then add the consecutive blue points lying to the right of I one by one, and denote the resulting interval by I_1 (Fig. 3). We next simultaneously remove the leftmost red point of I_1 and add the red point lying to the right of I_1, and get an interval I_2, which also contains exactly k red points and whose blue points are the same as those in I_1 (Fig. 3). By repeating this procedure, we can get an interval whose red point set is equal to that of J. Therefore, we can move I to J in the desired way. Consequently, we can find the required interval, which contains exactly k red points and h blue points.

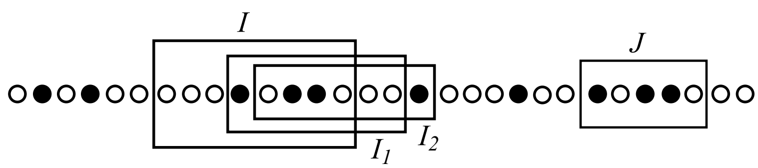

Fig. 3. Intervals $I,\ I_1,\ I_2,\ J$ containing exactly three red points.

Proof of Theorem It is obvious that every configuration with n red points and m blue points on the line has an interval containing exactly one red point and no blue point. Hence the conclusion of Lemma 4 always holds for $k = 1$. By Lemmas 1-5, the theorem follows immediately.

References

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