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SEMI-BALANCED PARTITIONS OF TWO SETS OF POINTS AND EMBEDDINGS OF ROOTED FORESTS

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Let m be a positive integer and let R_1, R_2 and B be three disjoint sets of points in the plane such that no three points of $R_1 \cup R_2 \cup B$ lie on the same line and $|B| = (m-1)|R_1| + m|R_2|$. Put $g = |R_1 \cup R_2|$. Then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that (i) $|X_i \cap (R_1 \cup R_2)| = 1$ for all $1 \le i \le g$; and (ii) $|X_i \cap B| = m-1$ if $|X_i \cap R_1| = 1$, and $|X_i \cap B| = m$ if $|X_i \cap R_2| = 1$. This partition is called a semi-balanced partition, and our proof gives an $O(n^4)$ time algorithm for finding the above semi-balance partition, where $n = |R_1| + |R_2| + |B|$.

We next apply the above result to the following theorem: Let $T_1, \ldots, T_g$ be g disjoint rooted trees such that $|T_i| \in \{m, m+1\}$ and v_i is the root of T_i for all $1 \le i \le g$. Let P be a set of $|T_1| + \cdots + |T_g|$ points in the plane in general position that contains g specified points $p_1, \ldots, p_g$. Then the rooted forest $T_1 \cup \cdots \cup T_g$ can be straight-line embedded onto P so that each v_i corresponds to p_i for every $1 \le i \le g$.

Keywords: Semi-balanced partition; red and blue points; line embedding.

1. Introduction

Let $G = (V, E)$ be a planar graph with vertex set V and edge set E, and P a set of points in the plane in general position (i.e., no three points of P are collinear). We denote by $|G|$ the order of G and by conv(P) the *convex hull* of P, which is the smallest convex set containing P. A graph drawn in the plane is called a *geometric graph* if every edge is a straight-line segment.

We say that G is *straight-line embedded onto* P or briefly *line embedded onto* P if there exists a one-to-one and onto mapping $\phi : V \to P$ which induces a crossing-free geometric graph. This mapping ϕ is called a *line embedding* of G onto P. Furthermore, suppose that G has n specified vertices $v_1, v_2, \ldots, v_n$, and P contains n specified points $p_1, p_2, \ldots, p_n$. Then we say that G is *strongly line embedded onto*

P if G can be line embedded onto P so that each v_i corresponds to p_i for all $1 \le i \le n$, that is, a line embedding ϕ of G onto P is called a *strong line embedding* if $\phi(v_i) = p_i$ for all $1 \le i \le n$.

Let $T_1, T_2, \ldots, T_n$ be n disjoint rooted trees with roots $v_1, v_2, \ldots, v_n$, respectively. Then the union $T_1 \cup T_2 \cup \cdots \cup T_n$ is called a *rooted forest* with roots $v_1, v_2, \ldots, v_n$. Of course, the roots $v_1, v_2, \ldots, v_n$ are its specified vertices. In this paper we consider strong line embeddings of rooted forests.

We begin with some known results related to our theorem. The following Theorem 1, which was proved by Ikebe et al., was originally posed by Perles[7] at the 1990 DIMACS workshop on arrangements, and partially solved by Pach and Töröcski [6]. Another simpler proof of this theorem can be found in Bose et al. [1] and Tokunaga [8].

Theorem 1 (Ikebe, Perles, Tamura, and Tokunaga [2]). *A rooted tree T can be strongly line embedded onto any set of $|T|$ points in the plane in general position containing a specified point.*

Notice that Bose, McAllister and Snoeyink [1] presented an optimal $\Theta(n(\log n))$ time algorithm for finding a strong line embedding of a rooted tree T with order $n = |T|$.

Theorem 2. [3,4] *The following two statements hold:*

(1) A rooted forest F consisting of two disjoint rooted trees can be strongly line embedded onto any set of $|F|$ points in the plane in general position containing two specified points.

(2) A rooted forest F consisting of n disjoint rooted stars can be strongly line embedded onto any set of $|F|$ points in the plane in general position containing n specified points, where a rooted star is a rooted tree isomorphic to a star and its root is not necessary to be its center.

In this paper, we prove the following theorem.

Theorem 3. *Let m be a positive integer, and let $T_1, T_2, \ldots, T_g$ be g disjoint rooted trees with roots $v_1, v_2, \ldots, v_g$, respectively, such that $|T_i| \in \{m, m+1\}$ for all $1 \le i \le g$. Let P be a set of $|T_1| + |T_2| + \cdots + |T_g|$ points in the plane in general position that contains g specified points $p_1, p_2, \ldots, p_g$. Then the rooted forest $T_1 \cup T_2 \cup \ldots \cup T_g$ can be strongly line embedded onto P so that each v_i corresponds to p_i for all $1 \le i \le g$ (see Fig. 1).*

Notice that our proof of Theorem 3 gives $O(n^4)$ time algorithm for finding a strong line embedding of the rooted forest $T_1 \cup T_2 \cup \cdots \cup T_g$ of order $n = |T_1| + |T_2| + \cdots + |T_g|$.

Theorem 3 follows immediately from the following Theorem 4 and the previous Theorem 1.

Theorem 4. *Let m be a positive integer, and let $R_1 \cup R_2$ and B be disjoint sets of red points and blue points in the plane, respectively, such that $R_1 \cap R_2 = \emptyset$, no* *three points of* $R_1 \cup R_2 \cup B$ *are collinear, and* $|B| = (m-1)|R_1| + m|R_2|$. *Put* $g = |R_1 \cup R_2|$. *Then there exists a subdivision* $X_1 \cup X_2 \cup \cdots \cup X_g$ *of the plane into* g *disjoint convex polygons such that (i)* $|X_i \cap (R_1 \cup R_2)| = 1$ *for all* $1 \le i \le g$; *and (ii)* $|X_i \cap B| = m-1$ *if* $|X_i \cap R_1| = 1$, *and* $|X_i \cap B| = m$ *if* $|X_i \cap R_2| = 1$ *(see Fig. 1).*

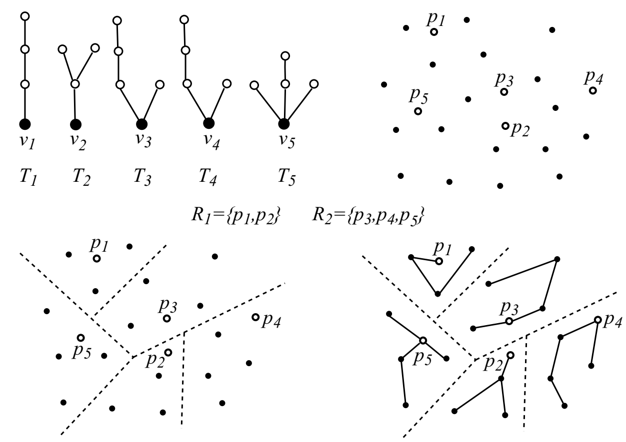

Fig. 1. A semi-balanced partition and a strong line embedding.

The partition given in the above Theorem 4 is called a *semi-balanced partition* of $R_1 \cup R_2$ and B. Notice that if $R_1 = \emptyset$ or $R_2 = \emptyset$, then this partition is called a *balanced partition*[5].

Theorem 3 follows from Theorems 1 and 4 because each rooted tree T_i can be strongly line embedded onto $X_i \cap (R_1 \cup R_2 \cup B)$ with a specified point $X_i \cap (R_1 \cup R_2)$. Some other results related to Theorems 3 and 4 can be found in a survey [5].

We conclude this section by showing that Theorem 4 can not be extended to the following statement. Let $R_1,\ R_2,\ R_3$ and B be four disjoint sets of points in the plane such that no three points of $R_1 \cup R_2 \cup R_3 \cup B$ are collinear and $|B| = (m-2)|R_1| + (m-1)|R_2| + m|R_3|$. Then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that (i) $|X_i \cap (R_1 \cup R_2 \cup R_3)| = 1$ for all $1 \le i \le g$; and (ii) $|X_i \cap B| = m-2$ if $|X_i \cap R_1| = 1$, $|X_i \cap B| = m-1$ if $|X_i \cap R_2| = 1$, and $|X_i \cap B| = m$ if $|X_i \cap R_3| = 1$.

A counterexample to this statement is given in Fig. 2, in which $R_1 = \{r_1\},\ R_2 = \{r_2\},\ R_3 = \{r_3\}$, $B = B_1 \cup B_2 \cup B_3$, $|B_1| = |B_2| = |B_3| = m-1$, and $|B| = 3(m-1)$. We can easily see that there exists no desired subdivision $X_1 \cup X_2 \cup \cup X_3$ of the plane into three disjoint convex polygons that satisfy the above conditions.

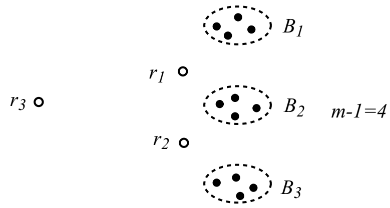

Fig. 2. Three red points r_1, r_2, r_3 and a set $B = B_1 \cup B_2 \cup B_3$ of blue points.

2. Proof of Theorem 4

In this paper, we deal with only *directed lines* in order to be able to refer to the right side of a line and the left side of it. Thus a *line* means a directed line. A line l dissects the plane into three pieces: l and the two open half-planes $right(l)$ and $left(l)$ that are bounded to the left and to the right of l, respectively (see Fig. 3). For two lines l_1 and l_2 intersecting at a point x, we denote by $\measuredangle l_1 l_2$ the angle of two lines at x such that $0 < \measuredangle l_1 l_2 < \pi$, and call it the *angle* of l_1 and l_2. For two parallel lines l and l', we define $\measuredangle ll' = 0$ if l and l' have the same direction, and $\measuredangle ll' = \pi$ if l and l' have the opposite directions. For two parallel lines l and l', we also call $\measuredangle ll'$ its angle.

We define the function f of a region W in the plane by

$$f(W) = (m-1)|W \cap R_1| + m|W \cap R_2| - |W \cap B|.$$

Then for two disjoint regions W_1 and W_2, we have $f(W_1 \cup W_2) = f(W_1) + f(W_2)$. A region W is said to be *balanced* if $f(W) = 0$ and W contains at least one point of $R_1 \cup R_2 \cup B$.

Let $R = R_1 \cup R_2$ denote the set of red points. We shall prove Theorem 4 by induction on $|R|$. It is clear that we may assume $|R| \geq 2$. Unless otherwise stated, except when it moves, we always consider a line that passes through no point of $R \cup B$.

Proof. We start with some claims.

Claim 1. *If there exist two lines l_1 and l_2 such that $|right(l_1) \cap R| = |right(l_2) \cap R|$ and $f(right(l_1)) < f(right(l_2))$, then for every integer n, $f(right(l_1)) < n < f(right(l_2))$, there exists a line l_3 such that $|right(l_3) \cap R| = |right(l_1) \cap R|$ and $f(right(l_3)) = n$.*

We first assume that $right(l_1) \cap R = right(l_2) \cap R$ (see the left of Fig. 3). Then we can continuously move a line l from l_1 to l_2 in such a way that each line l passes through at most one blue point but no red point. Then $right(l) \cap R = right(l_1) \cap R$, and the value $f(right(l))$ decreases by 1 when a blue point comes in $right(l)$, and increases by 1 when a blue point goes out of $right(l)$. Therefore we can find the desired line l_3 .

We next assume $right(l_1) \cap R \neq right(l_2) \cap R$ (see the right of Fig. 3). Consider two convex hulls $\mathrm{conv}(R \cap right(l_1))$ and $\mathrm{conv}(R \cap left(l_1))$. Then we can find two vertices $x \in \mathrm{conv}(R \cap right(l_1))$ and $y \in \mathrm{conv}(R \cap left(l_1))$ such that a line l_4 passing through x and y satisfies $\measuredangle l_2 l_4 < \measuredangle l_2 l_1$, that is, in Fig. 3, we can obtain l_4 from l_1 by rotating it counterclockwise. Let l'_4 denote a line very closed to l_4 such that $right(l'_4)$ contains x but not y, and l''_4 denote a line very closed to l_4 such that $right(l''_4)$ contains y but not x. We may assume that no point of $(R \cup B) - \{x, y\}$ lies between l'_4 and l''_4. We can continuously move a line l from l_1 to l'_4 in such a way that l passes through no red point and the value $f(right(l))$ changes by ± 1 when l passes through blue points. Moreover $f(right(l))$ does not change or changes by ± 1 when l moves from l'_4 to l''_4 since $right(l''_4) \cap B = right(l'_4) \cap B$ and $right(l''_4) \cap R = (right(l'_4) \cap R) - x + y$.

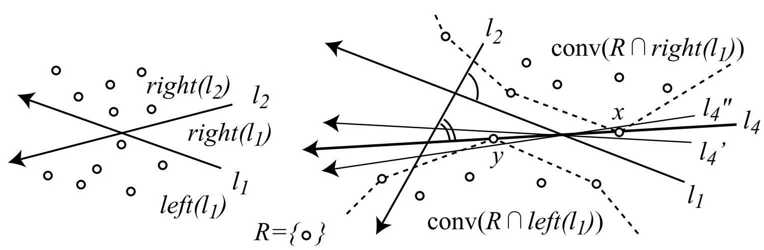

Fig. 3. Two cases of the proof of Claim 1.

We next consider l''_4 and l_2 instead of l_1 and l_2, and apply the same procedure to them. By repeating this procedure, we can move a line l from l_1 to l_2 in such a way that $f(right(l))$ changes by ± 1 and $|right(l) \cap R|$ is constant. Consequently we can find the desired line, and thus Claim 1 is proved.

Claim 2. *We may assume that* $|R|$ *is odd. In particular, we can put* $|R| = 2k + 1,\ k \geq 1$.

Suppose that $|R|$ is even. Let l_1 be a line that satisfies $|right(l_1) \cap R| = |left(l_1) \cap R| = |R|/2$. Let l_1^* denote the line obtained from l_1 by changing its direction oppositely (i.e., $\measuredangle l_1 l_1^* = \pi$). Then

$$|right(l_1^*) \cap R| = |right(l_1) \cap R| \quad \text{and} \quad f(right(l_1^*)) = -f(right(l_1)).$$

This is because $right(l_1^*) = left(l_1)$ and $f(right(l_1)) + f(left(l_1)) = 0$. By applying Claim 1 with $n = 0$ to l_1 and l_1^*, there exists a line l_2 such that $|right(l_2) \cap R| = |right(l_1) \cap R| = |R|/2$ and $f(right(l_2)) = 0$, which implies that both regions $right(l_2)$ and $left(l_2)$ are balanced. Hence we can obtain the required subdivision of the plane by applying the inductive hypothesis to $right(l_2)$ and $left(l_2)$.

By the same argument as in the proof above, if there exists a line l such that $f(right(l)) = 0$ and $1 \leq |right(l) \cap R| < |R|$, then both $right(l)$ and $left(l)$ are balanced, and thus we can obtain the desired subdivision of the plane by induction. Therefore we may assume that

$$f(right(l)) \neq 0 \quad \text{for every line } \ l \ \text{ with } \ 1 \leq |right(l) \cap R| < |R|. \tag{1}$$

Claim 3. *We may assume that if a line l satisfies $1 \leq |right(l) \cap R| \leq k$, then $f(right(l)) < 0$.*

Suppose that there exists a line l_1 such that $1 \leq |right(l_1) \cap R| \leq k$ and $f(right(l_1)) > 0$. We may assume that every line parallel to l_1 passes through at most one point of $R \cup B$, if necessary, by considering a very small rotation of l_1. Since $|right(l_1) \cap R| \leq k$ and $|left(l_1) \cap R| \geq k+1$, there exists a line l_2 in $left(l_1)$ such that and $|right(l_2) \cap R| = |right(l_1) \cap R|$ and $\measuredangle l_1 l_2 = \pi$ (see Fig. 4).

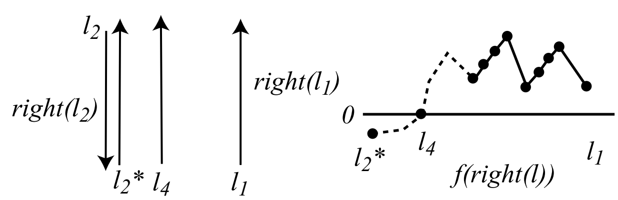

Fig. 4. Lines l_1, l_2, l_2^* and l_4, and the value $f(right(l))$ in the proof of Claim 3.

If $f(right(l_2)) < 0$, then by Claim 1, there exists a line l_3 such that $|right(l_3) \cap R| = |right(l_1) \cap R|$ and $f(right(l_3)) = 0$, which contradicts (1). Hence $f(right(l_2)) > 0$, which implies $f(left(l_2)) < 0$. Let l_2^* denote a line obtained from l_2 by changing its direction oppositely. Then $\measuredangle l_1 l_2^* = 0$. We move a line l from l_1 to l_2^* in such a way that l hits at most one point of $R \cup B$ in each time. Then the value $f(right(l))$ decreases by 1 when l passes through a blue point, and increases by $m-1$ or m when l passes through a red point. Since $f(right(l_1)) > 0$ and $f(right(l_2^*)) = f(left(l_2)) < 0$, we can find a line l_4 between l_1 and l_2^* such that $f(right(l_4)) = 0$, which contradicts (1). Consequently the claim is proved.

Let l_1 be a line which passes through exactly one red point, say x, but no other point of $R \cup B$ and satisfies $right(l_1) \cap R = \emptyset$. By a suitable rotation of the plane, we may assume that l_1 is horizontal and goes from right to left (see Fig. 5). By considering a line l_1' lying very little below l_1, we have by Claim 3 that $f(right(l_1')) = m - 1 - |right(l_1) \cap B| < 0$ if $x \in R_1$, or $f(right(l_1')) = m - |right(l_1) \cap B| < 0$ if $x \in R_2$. For simplicity, we may assume that $x \in R_2$ since otherwise we can similarly prove the theorem. Hence we have

$$c := |right(l_1) \cap B| > m \quad \text{and} \quad x \in R_2. \tag{2}$$

Let l_2 be a line which passes through x, goes downward and satisfies $|right(l_2) \cap R| = |left(l_2) \cap R| = k$ (see Fig. 5). Then by Claim 3 and by $f(right(l_2)) +$

$f(left(l_2)) = -f(\{x\}) = -m$, we have

$$a := -f(right(l_2)) > 0, \quad b := -f(left(l_2)) > 0, \quad \text{and} \quad a + b = m. \qquad (3)$$

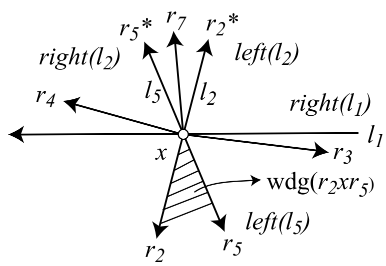

Fig. 5. Lines l_1, l_2 and rays r_3, r_4 and r_5. $right(l_1)$ contains no red points.

Hereafter we consider rays emanating from x, and so a *ray* means such a ray. For a line l_2, we define two rays r_2 and r_2^* lying on l_2 such that r_2 has the same direction as l_2, and r_2^* has an opposite direction of l_2 (see Fig. 5). For two rays r and r', we denote by $\text{wdg}(rxr')$ or $\text{wdg}(r'xr)$ the *open wedge* with apex x whose boundary is $r \cup r'$ and its internal angle is smaller than π. Notice that $x \notin \text{wdg}(rxr')$.

Claim 4 *We may assume that there exist two rays r_3 in $left(l_2)$ and r_4 in $right(l_2)$ such that both wedges wdg(r_2xr_3) and wdg(r_2xr_4) are balanced.*

We first notice that when we rotate a ray r counterclockwise around x from r_2 to r_2^*, the value $f(\text{wdg}(r_2xr))$ decreases by 1 when r passes through a blue point and increases $m-1$ or m when r passes through a red point. Thus if there exists a ray r such that $f(\text{wdg}(r_2xr)) \geq 0$ and $\text{wdg}(r_2xr)$ contains at least one point of $R \cup B$, then since $f(left(l_2)) < 0$, we can find a ray r' such that $\text{wdg}(r_2xr')$ is balanced.

Now we only show the existence of r_3 because of symmetry. Let r_5 be a ray in $left(l_2)$ such that $\text{wdg}(r_2xr_5)$ contains exactly m points of $R \cup B$. Fist assume that $\text{wdg}(r_2xr_5)$ contains at least one red point. Then $f(\text{wdg}(r_2xr_5)) \geq m - 1 - |\text{wdg}(r_2xr_5) \cap B| \geq 0$, and hence there exists a ray r_6 such that $f(\text{wdg}(r_2xr_6)) = 0$, which can be regarded as the desired ray r_3.

We next assume that $\text{wdg}(r_2xr_5)$ contains m blue points but no red points. Let l_5 be a line that contains r_5 and has the same direction as r_5. Then $f(left(l_5)) < 0$ by Claim 3. Moreover since $f(\text{wdg}(r_5xr_2^*)) = f(left(l_2)) + |\text{wdg}(r_2xr_5) \cap B| = -b + m > 0$ by (3), we can find a ray r_7 in $\text{wdg}(r_2^*xr_5^*)$ which satisfies $f(\text{wdg}(r_5xr_7)) = 0$, where r_5^* is a ray lying on l_5 and having an opposite direction of r_5 (see Fig. 5). Hence the plane can be partitioned into three disjoint balanced convex polygons $\text{wdg}(r_2xr_5) \cup \{x\}$, $\text{wdg}(r_5xr_7)$ and $\text{wdg}(r_7xr_2)$. Therefore in this case Theorem 4 is proved by applying the inductive hypothesis to each convex polygon. Consequently the claim is proved.

Claim 5 *Consider two rays r_3 and r_4 given in Claim 4. If we choose a ray r_3 so that $wdg(r_2xr_3)$ is balanced and contains as many points of $R \cup B$ as possible, then $wdg(r_3xr_2^*)$ contains exactly b blue points and no red points. Similar statement also holds for a ray r_4.*

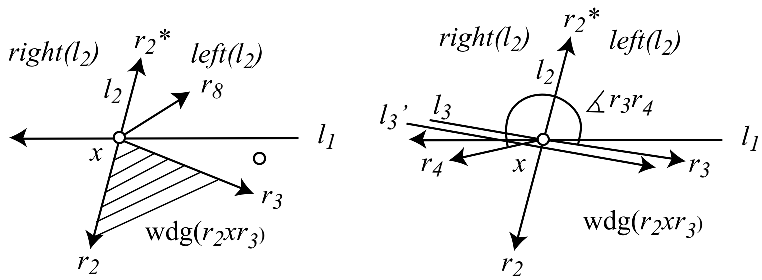

Fig. 6. Lines l_1, l_2, l_3, l_3' and rays $r_2, r_2^*, r_3, r_4, r_8$.

We choose a ray r_3 so that a balanced wedge $\text{wdg}(r_2xr_3)$ contains as many points of $R \cup B$ as possible. Suppose that $\text{wdg}(r_3xr_2^*)$ contains a red point (see the left of Fig. 6). Then $\text{wdg}(r_3xr_2^*)$ contains at least m points of $R \cup B$ since

$$0 > f(left(l_2)) = f(\text{wdg}(r_3xr_2^*)) \geq m - 1 - |\text{wdg}(r_3xr_2^*) \cap B|.$$

Hence we can take a ray r_8 such that a wedge $\text{wdg}(r_3xr_8)$ contains exactly m points of $R \cup B$. We argue in the same way as in the proof of Claim 4 for r_8 and r_3 instead of r_5 and r_2, namely, if $\text{wdg}(r_3xr_8)$ contains a red point then we can obtain a larger balanced wedge, which contradicts the choice of r_3; otherwise, we can obtain a balanced subdivision with three convex polygons of the plane, and prove the theorem by the inductive hypothesis. Notice that in the above case where $\text{wdg}(r_3xr_8)$ contains no red points, it follows that r_8 must lie below l_1 since $wdg(r_8xr_2^*)$ contains at least one red point and $right(l_1)$ contains no red points. Consequently we may assume that $\text{wdg}(r_3xr_2^*)$ contains no red points.

Since $\text{wdg}(r_2xr_3)$ is balanced and $\text{wdg}(r_3xr_2^*)$ contains no red points,

$$-|\text{wdg}(r_3xr_2^*) \cap B| = f(\text{wdg}(r_3xr_2^*)) = f(left(l_2)) = -b.$$

Hence $|\text{wdg}(r_3xr_2^*) \cap B| = b$. Consequently we may assume that Claim 5 holds.

By Claim 5, we can choose two rays r_3 in $left(l_2)$ and r_4 in $right(l_2)$ such that $\text{wdg}(r_2xr_3)$ and $\text{wdg}(r_2xr_4)$ are balanced, and that neither $\text{wdg}(r_3xr_2^*)$ nor $\text{wdg}(r_4xr_2^*)$ contains a red point. We shall show that the angle $\measuredangle r_3r_4$ defined by $\measuredangle r_3r_2^* + \measuredangle r_4r_2^*$ is smaller than or equal to π. Suppose that $\measuredangle r_3r_4 > \pi$ (see the right of Fig. 6). Let l_3 be a line containing r_3 and having the same direction as

r_3, and let l'_3 be a line lying very little below l_3 such that $left(l'_3)$ contains x and $left(l'_3) \cap B = left(l_3) \cap B$. Then $left(l_3) \subset \mathrm{wdg}(r_3xr_2^*) \cup \mathrm{wdg}(r_4xr_2^*)$, and so

$$|left(l_3) \cap B| \leq |\mathrm{wdg}(r_3xr_2^*) \cap B| + |\mathrm{wdg}(r_4xr_2^*) \cap B| = a + b = m.$$

On the other hand, by Claim 3, we have $f(left(l'_3)) = m - |left(l'_3) \cap B| < 0$, which implies $|left(l_3) \cap B| > m$. This is a contradiction. Therefore we have $\measuredangle r_3r_4 \leq \pi$. Consequently the plane can be partitioned into three disjoint convex polyongs $\mathrm{wdg}(r_2xr_3)$, $\mathrm{wdg}(r_2xr_4)$ and $\mathrm{wdg}(r_3xr_4) \cup \{x\}$, and thus by applying the inductive hypothesis to each polygon, we can obtain the desired subdivision of the plane. Consequently the proof is complete. □

We now analysis the time complexities of algorithms. We first consider an algorithm for finding a semi-balanced subdivision of the plane based on the proof of Theorem 4. Let $R = R_1 \cup R_2$, $g = |R|$ and $n = |R \cup B|$. Since there are $O(n^2)$ lines passing through two points of $R \cup B$, we can find a line l with $f(right(l)) = 0$, if any, in $O(n^3)$ time. Notice that for a line l passes through two points x and y of $R \cup B$, then we consider the four cases; (i) $x, y \in right(l)$, (ii) $x \in right(l)$ and $y \notin right(l)$, (iii) $x \notin right(l)$ and $y \in right(l)$; and (vi) $x, y \notin right(l)$. If there exists no such a line l, then $|R|$ must be odd by Claim 2, and for a red vertex x of $\mathrm{conv}(R)$, we can find a line l_2, for which both $left(l_2)$ and $right(l_2)$ contains exactly $(|R|-1)/2$ red points. We sort all the points of $R \cup B$ in $right(l_2)$ and those in $left(l_2)$ in clockwise and counterclockwise order, respectively, around x from r_2 in $n \log n$ time. Then we compute $f(r_2xr_y)$ in this order, where r_y denote a ray emanating from x and passing through a little below a point $y \in R \cup B$. By these values $f(r_2xr_y)$'s, we can find two rays r_3 and r_4 given in Claim 4. Then we can find a convex balanced subdivision of the plane given in the last stage of the proof of Theorem 4 or in the proof of Claim 4, which is defined by three rays emanating from x. We can find such a balanced convex subdivision in $O(n)$ time using the values $f(r_2xr_y)$'s and sorting of $R \cup B$ around x. Therefore

$$f(n) \leq f(n - m') + O(n^3) \quad \text{or} \quad f(n) \leq f(n_1 + n_2 + n_3) + O(n^3),$$

where $m' \in \{m, m+1\}$ and $n = n_1 + n_2 + n_3,\ m \leq n_1, n_2, n_3$ and $f(n)$ denotes the expected running time of our algorithm. Therefore we can find the desired semi-balanced subdivision of the plane in $O(n^4)$ time.

By the result[1], each rooted tree T_i of order $m-1$ or m can be strongly line embedded in $\Theta(m(\log m))$ time. Hence after obtaining a semi-balanced subdivision of the plane given in Theorem 4, we can find a strong line embedding of the rooted forest $T_1 \cup T_2 \cup \cdots \cup T_g$ with order $n = |T_1| + |T_2| + \cdots + |T_g|$ in $g\Theta(m(\log m)) \leq n \log n$ time. Consequently, we can find a strong line embedding of the rooted forest of order n in $O(n^4)$ time.

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