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Partitioning Complete Multipartite Graphs by Monochromatic Trees

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Abstract

The tree partition number of an r-edge-colored graph G, denoted by $t_r(G)$, is the minimum number k such that whenever the edges of G are colored with r colors, the vertices of G can be covered by at most k vertex-disjoint monochromatic trees. We determine $t_2(K(n_1, n_2, \dots, n_k))$ of the complete k-partite graph $K(n_1, n_2, \dots, n_k)$. In particular, we prove that $t_2(K(n, m)) = \lfloor (m-2)/2^n \rfloor + 2$, where $1 \le n \le m$.

1 Introduction

We consider finite graphs without loops or multiple edges. The complete k-partite graph $K(n_1, n_2, \dots, n_k)$ has the vertex set $V_1 \cup V_2 \cup \cdots \cup V_k$ such that $V_i \cap V_j = \emptyset$ and $|V_i| = n_i$ for every $1 \le i < j \le k$, and the edge set $\{x_i x_j \mid x_i \in V_i,\ x_j \in V_j,\ 1 \le i < j \le k\}$. We often denote by $K(n, m)$ the complete bipartite graph with partite sets X and Y, where $|X| = n$ and $|Y| = m$.

The tree partition number of an r-edge-colorings of a graph G, denoted by $t_r(G)$, which was introduced by Erdös, Gyárfás and Pyber [1], is the minimum k such that whenever the edges of G are colored with at most r colors, the vertices of G can be covered by at most k vertex-disjoint monochromatic trees. Moreover it was conjectured in [1] that the tree partition number of an r-edge-colored complete graph is $r - 1$.

It is well-known that either a graph or its complement is connected. This is equivalent to the fact that every 2-edge-colored complete graph has a monochromatic spanning tree, which implies that $t_2(K_n) = 1$, and thus the conjecture is true for $r = 2$. The conjecture was proved for $r = 3$ in [1]. Recently Haxell and Kohayakawa [2] obtained the following Theorem A, which implies that $t_r(K_n) \le r$ if n is sufficiently large.

Theorem A ([2]). *The vertex set of the complete graph K_n whose edges are colored with r colors can be covered by at most r vertex-disjoint monochromatic trees with different colors provided $n \geq 3r^4 r!(1-1/r)^{3(1-r)} \log r$.*

It is mentioned in [3] that the following theorem can be proved by similar arguments given in [2].

Theorem B ([3]). *If n is sufficiently large, $t_r(K(n,n)) \leq 2r$.*

For related results of Ramsey partitioning type, see [4],[5].

When we consider 2-edge-colored graphs, we always assume that all their edges are colored red or blue. So a monochromatic tree of 2-edge-colored graph can be called a red tree or a blue tree.

In this paper we first consider $t_2(K(n_1, N_2, \dots, n_k))$, and prove the following theorem in section 2.

Theorem 1. *Let $n_1, n_2, \dots, n_k$ $(2 \leq k)$ be integers such that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$, and let $n = n_1 + n_2 + \cdots + n_{k-1}$ and $m = n_k$. Then*

$$t_2(K(n_1, n_2, \dots, n_k)) = \left\lfloor \frac{m-2}{2^n} \right\rfloor + 2. \tag{1}$$

Note that $K(1,1,\dots,1)$ (i.e. $n_k = 1$) is isomorphic to a complete graph, and so it has a monochromatic spanning tree. On the other hand, the right side of (1) is equal to one as $\lfloor (m-2)/2^n \rfloor = -1$.

Next we consider a related problem. Every complete multipartite graph that is not a complete graph has two non-adjacent vertices x and y. If all the edges incident with x are colored red and all the edges incident with y are colored blue, then such a 2-edge-colored complete multipartite graph has no monochromatic spanning trees. Therefore, in order to guarantee the existence of monochromatic spanning tree in a 2-edge-colored complete multipartite graph G, it is necessary to assume that for every vertex v of G, at least one red edge and at least one blue edge are incident with v. However this is not sufficient. Consider a complete bipartite graph $K(n,m)$ $(2 \leq n \leq m)$ with partite sets X and Y, and choose two vertices $x \in X$ and $y \in Y$. If the edge xy and all the edges joining $X - x$ to $Y - y$ are colored red and all the other edges are colored blue, then $K(n,m)$ has no monochromatic spanning trees.

Our second theorem shows that the bipartite case is exceptional.

Theorem 2. *Let $k \geq 3$ be an integer. If the edges of the complete k-partite graph $K(n_1, n_2, \dots, n_k)$ are colored red or blue in such a way that for every vertex v, at least one red edge and at least one blue edge are incident with v, then $K(n_1, n_2, \dots, n_k)$ has a monochromatic spanning tree.*

We prove Theorem 2 in section 3.

2 Proof of Theorem 1

For a graph G, we denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. An edge joining a vertex x to a vertex y is denoted by xy or yx. For two disjoint sets X and Y of vertices of a graph G, we denote by $E(X,Y)$ the set of edges of G joining a vertex in X to a vertex in Y, in fact, in all our proofs we use $E(X,Y)$ to denote something stronger: the complete bipartite graph with vertex classes X and Y, which is in every case a subgraph of the graph we are considering. For a set X of vertices of G, we denote by $\langle X\rangle$ the subgraph of G induced by X. For a subgraph H and a set X of vertices of G, we say that H covers X or X is covered by H if $X \subseteq V(H)$.

In order to prove Theorem 1, we need three lemmas.

Lemma 1. *Let $k \geq 3$ be an integer, and $\{n_1, n_2, \ldots, n_k\}$ be a set of positive integers such that $n_k \geq n_i$ for all $1 \leq i \leq k-1$ and $n_1 + n_2 + \cdots + n_{k-1} > n_k \geq 2$. Then the set of indices $\{1, 2, \ldots, k\}$ can be partitioned into two disjoint subsets $I \cup J$ so that*

$$n \leq m \leq 2^n + 1, \quad \text{where } n = \sum_{i\in I} n_i \quad \text{and } m = \sum_{j \in J} n_j.$$

In particular, the complete k-partite graph $K(n_1, n_2, \ldots, n_k)$ contains a spanning complete bipartite graph $K(n,m)$ such that $n \leq m \leq 2^n + 1$.

Proof. Without loss of generality, we may assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. Take an integer t $(1 \leq t \leq k-2)$ such that

$$\begin{aligned} n_t + n_{t+1} + n_{t+2} + \cdots + n_{k-1} &> n_k + n_1 + n_2 + \cdots + n_{t-1} \quad \text{and} \\ n_{t+1} + n_{t+2} + \cdots + n_{k-1} &\leq n_k + n_1 + n_2 + \cdots + n_{t-1} + n_t. \end{aligned}$$

Let $a = n_t$, $\;b = n_{t+1} + \cdots + n_{k-1}$ and $c = n_k + n_1 + n_2 + \cdots + n_{t-1}$. Then

$$a \leq b, \quad 2 \leq n_k \leq c < a + b \quad \text{and} \quad b \leq a + c.$$

It suffices to show that $b \leq a + c \leq 2^b + 1$ or $c < a + b \leq 2^c + 1$ by letting either $n = b$ and $m = a + c$ or $n = c$ and $m = a + b$.

If $b \leq 2$, then $a + b \leq b + b \leq 4 \leq 2^c + 1$. Therefore we may assume that $b \geq 3$. It is easy to prove that $2^x + 1 \geq 3x$ for $x \geq 3$. Hence $2^b + 1 \geq 3b \geq a + a + b > a + c$. Consequently the lemma is proved. ∎

The following Lemmas 2 and 3, which determine $t_2(K(n,m))$, are essential parts of the proof of Theorem 1.

Lemma 2. *Let n and m be integers such that $1 \leq n \leq m$. Then the vertices of 2-edge-colored complete bipartite graph $K(n,m)$ can be covered by at most*

$$\left\lfloor \frac{m-2}{2^n} \right\rfloor + 2 \tag{2}$$

vertex-disjoint monochromatic trees.

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$ be the partite sets of $K(n,m)$. Suppose that all the edges of $K(n,m)$ are colored red or blue. For an n-tuple $(a_1, \ldots, a_n)$ such that $a_i \in \{r, b\}$ for every i, we define

$$Y(a_1, a_2, \ldots, a_n) = \{y \in Y \mid yx_i \text{ is red or blue according as } a_i = r \text{ or } a_i = b\}.$$

In particular, $Y(r, r, \ldots, r)$ and $Y(b, b, \ldots, b)$ denote the sets of vertices $y \in Y$ such that all the edges incident with y are red or blue, respectively. We abbreviate $Y(r, r, \ldots, r)$ to $Y(r)$ and $Y(b, b, \ldots, b)$ to $Y(b)$. Similarly, we denote by $X(r)$ and $X(b)$ the sets of vertices in $x \in X$ such that all the edges incident with x are red or blue, respectively.

We begin with an observation on $(a_1, a_2, \ldots, a_n)$, which will be often used. If there exists an n-tuple $(a_1, a_2, \ldots, a_n)$ such that $Y(a_1, a_2, \ldots, a_n) = \emptyset$, $a_s = r$ and $a_t = b$ for some $s \neq t$, then by letting $X_r = \{x_i \in X \mid a_i = r\} \ni x_s$ and $X_b = \{x_i \in X \mid a_i = b\} \ni x_t$, Y can be partitioned into

$$Y = Y_r \cup Y_b, \qquad Y_r \cap Y_b = \emptyset, \tag{3}$$

where the two subsets Y_r and Y_b are defined as

$$Y_r = \{y \in Y \mid yx_i \text{ is red for some } x_i \in X_b\},$$
$$Y_b = \{y \in Y \mid yx_i \text{ is blue for all } x_i \in X_b, \text{ and } yx_j \text{ is blue for some } x_j \in X_r\}.$$

It is clear that $Y_r \cap Y_b = \emptyset$ and $Y - (Y_r \cup Y_b) = Y(a_1, a_2, \ldots, a_r) = \emptyset$, and thus the partition given in (3) is well-defined. (see Figure 1)

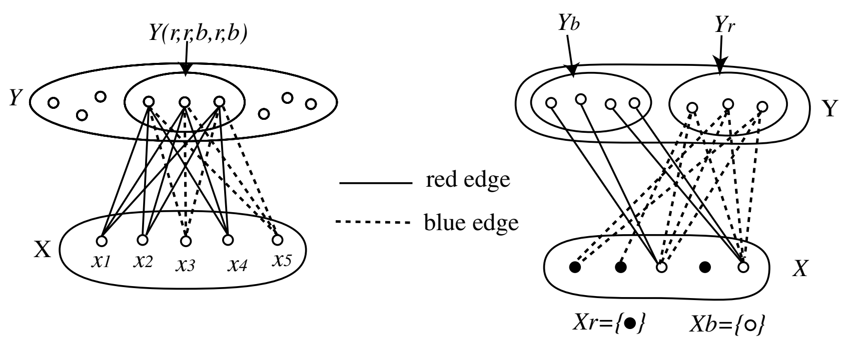

Figure 1: $Y(a_1, \ldots, a_5)$, Y_r and Y_b for $(a_1, \ldots, a_5) = (r, r, b, r, b)$

We consider four cases.

Case 1. $Y(r) \neq \emptyset$, $Y(b) \neq \emptyset$, and for some other n-tuple $(a_1, a_2, \ldots, a_n)$, $Y(a_1, a_2, \ldots, a_n) = \emptyset$.

For the n-tuple $(a_1, a_2, \ldots, a_n)$ with $Y(a_1, a_2, \ldots, a_n) = \emptyset$, we get the partition (3). Then $Y(r) \subseteq Y_r$ and $Y(b) \subseteq Y_b$, and the induced subgraph $\langle Y_r \cup X_b \rangle$ contains a spanning red tree Tr, and the induced subgraph $\langle Y_b \cup X_r \rangle$ contains a spanning blue tree T_b. Hence the set of vertices of $K(n,m)$ is covered by two monochromatic trees T_r and T_b.

Case 2. Either $Y(r) = \emptyset$ and $Y(b) \neq \emptyset$, or $Y(r) \neq \emptyset$ and $Y(b) = \emptyset$.

Without loss of generality, we may assume that $Y(r) = \emptyset$ and $Y(b) \neq \emptyset$. Since $Y(r) = \emptyset$, it follows that for every vertex $y \in Y$, at least one blue edge is incident with y. By $Y(b) \neq \emptyset$, it is immediate that $K(n, m)$ has a blue spanning tree.

Case 3. $Y(r) = Y(b) = \emptyset$.

In this case, for every vertex $y \in Y$, at least one red edge and at least one blue edge are incident with y. We first assume $X(r) = X(b) = \emptyset$, that is, we assume that for every vertex $x \in X$, at least one red edge and at least one blue edge are incident with x. Let H be the subgraph of $K(n, m)$ induced by the red edges of $K(n, m)$. Since $V(H) = V(K(n, m))$, we may assume that H is not connected. Let C be a component of H. Then C covers neither X nor Y since otherwise H must contain a spanning tree of $K(n, m)$, which is a contradiction.

Put

$$A = X \cap V(C),\ P = X - A,\ B = Y \cap V(C),\ \ Q = Y - B.$$

Then each of the above four vertex sets is a non-empty set, and all the edges in $E(A, Q) \cup E(P, B)$ are blue, and thus the vertices of $K(n, m)$ can be covered by two blue trees.

If either $X(r) = \emptyset$ and $X(b) \neq \emptyset$, or $X(r) \neq \emptyset$ and $X(b) = \emptyset$, then we can find a blue spanning tree or a red spanning tree of $K(n, m)$, respectively.

We finally assume $X(r) \neq \emptyset$ and $X(b) \neq \emptyset$. For an m-tuple $(a_1, a_2, \ldots, a_m)$ ($a_i \in \{r, b\}$), we define

$$X(a_1, a_2, \ldots, a_m) = \{x \in X \mid xy_i \text{ is red or blue according as } a_i = r \text{ or } a_i = b\}.$$

Then since $n \leq m < 2^m$, for some m-tuple $(a_1, \ldots, a_m)$, $X(a_1, \ldots, a_m) = \emptyset$. Hence by the same argument as in the proof of Case 1, we can show that the vertices of $K(n, m)$ can be covered by a red tree and a blue tree which are vertex-disjoint.

Case 4. For every n-tuple $(a_1, \ldots, a_n)$, $Y(a_1, \ldots, a_n) \neq \emptyset$.

Let Z be the set of all $Y(a_1, \ldots, a_n)$, $t' = \min\{|V| \mid V \in Z \setminus \{Y(r), Y(b)\}\}$, and $t = \min\{|Y(r)| - 1, |Y(b)| - 1, t'\}$. Since $|Z| = 2^n$ and the union of all the sets $Y(a_1, \ldots, a_n)$ is equal to Y, we have

$$t \leq \left\lfloor \frac{m-2}{2^n} \right\rfloor. \tag{4}$$

If $t = |Y(r)| - 1$, then $K(n, m) - Y(r)$ has a blue spanning tree since $Y(b) \neq \emptyset$ and for every vertex $y \in Y - Y(r)$, there is at least one blue edge incident with y. Therefore $V(K(n, m))$ can be covered by $t + 1$ isolated vertices in $Y(r)$ and one blue tree, which implies the lemma holds. Thus we may assume that $t \neq |Y(r)| - 1, |Y(b)| - 1$.

Suppose that $t = |Y(a_1, \ldots, a_n)|$, where $(a_1, \ldots, a_n) \neq (r, r, \ldots, r), (b, b, \ldots, b)$. Then $K(n, m) - Y(a_1, \ldots, a_n)$ satisfies the condition of Case 1, and hence the vertices of

$K(n,m) - Y(a_1,\dots,a_n)$ are covered by two vertex-disjoint monochromatic trees. Therefore $V(K(n,m))$ can be covered by t vertex-disjoint isolated vertices in $Y(a_1,\dots,a_n)$ and two monochromatic trees of $K(n,m) - Y(a_1,\dots,a_n)$, which implies the lemma holds. Consequently the lemma is proved. ∎

Lemma 3. *We can color all the edges of the complete bipartite graph $K(n,m)$ $(1 \le n \le m)$ red or blue so that the vertices of the resulting 2-edge-colored $K(n,m)$ cannot be covered by fewer than $\lfloor (m-2)/2^n \rfloor + 2$ vertex-disjoint monochromatic trees.*

Proof. If $n = 1$, then $K(1,m)$ is a star, and $\lfloor (m-2)/2^n \rfloor + 2 = \lfloor m/2 \rfloor + 1$. We can color $\lfloor m/2 \rfloor$ edges red and $\lceil m/2 \rceil$ edges blue, which implies the lemma holds. Thus we may assume $2 \le n$.

Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_m\}$ be the bipartite sets of $K(n,m)$. Let $k \ge 0$ be an integer such that $k2^n + 2 \le m < (k+1)2^n + 2$. Then it suffices to show that there exists an edge-coloring of $K(n,m)$ for which the vertices of $K(n,m)$ cannot be covered by $k+1$ vertex-disjoint monochromatic trees.

We color all the edges of $K(n,m)$ red or blue so that $k+1 \le |Y(r,r,\dots,r)| \le |Y(b,b,\dots,b)| \le k+2$, and for every $(a_1,\dots,a_n) \ne (r,r,\dots,r),(b,b,\dots,b)$, $k \le |Y(a_1,\dots,a_n)| \le k+1$. Note that the number of n-tuples $(a_1,\dots,a_n)$ is 2^n, and thus the above edge-coloring exists.

Assume that $V(K(n,m))$ can be covered by $k+1$ vertex-disjoint monochromatic trees $T_1, T_2, \dots, T_{k+1}$, where $T_1, \dots, T_s$ are trees of order one (i.e., T_i consists of one vertex and no edge) and $T_{s+1}, \dots, T_{k+1}$ are trees of order at least two. Set $F = \{T_1, T_2, \dots, T_{k+1}\}$ and $F' = \{T_{s+1}, \dots, T_{k+1}\}$.

Let $S = V(T_1) \cup V(T_2) \cup \dots \cup V(T_s)$, $S_x = S \cap X$ and $S_y = S \cap Y$, where $|S| = s$. Then F' is a set of $k+1-s$ vertex disjoint monochromatic trees, which cover $V(K(n,m)) - S$. Let V_r and V_b denote the subsets of $X - S_x$ covered by the red trees and by the blue trees of F', respectively. Define the n-tuple $(a_1, a_2, \dots, a_n)$ by $a_i = r$ if $x_i \in V_b \cup S_x$ and $a_i = b$ otherwise. It is easy to see that $Y(a_1, a_2, \dots, a_n)$ must be contained in S_y, and so $k \le |Y(a_1, a_2, \dots, a_n)| \le |S_y|$.

Hence $|F'| + |S_x| = |F| - |S_y| = k+1-|S_y| \le k+1-k = 1$. This inequality implies that $V_r = \emptyset$ or $V_b = \emptyset$. Without loss of generality, we may assume $V_r = \emptyset$. Then $(a_1, a_2, \dots, a_n) = (r,r,\dots,r)$, $Y(r,r,\dots,r) \subseteq S_y$ and $k+1 \le |Y(r,r,\dots,r)|$. This implies $|F'| + |S_x| = |F| - |S_y| = k+1-|S_y| \le k+1-(k+1) = 0$, and hence $|F| = |S_y|$. This is a contradiction. ∎

We are now ready to prove Theorem 1.

Proof of Theorem 1. By Lemmas 2 and 3, we may restrict ourselves to complete multipartite graphs $K(n_1, n_2, \dots, n_k)$ with $k \ge 3$. Let $n = n_1 + n_2 + \dots + n_{k-1}$ and $m = n_k$.

If $n_k = 1$, then $\lfloor (m-2)/2^n \rfloor + 2 = 1$. $K(n_1, n_2, \dots, n_k) = K(1,1,\dots,1)$ is isomorphic to a complete graph, and so it has a monochromatic spanning tree. Thus we may assume $2 \le n_k$.

We first prove that the set of vertices of 2-edge-colored $K(n_1, n_2, \ldots, n_k)$ is covered by at most $\lfloor (m-2)/2^n \rfloor + 2$ vertex-disjoint monochromatic trees.

Suppose that $n_1 + n_2 + \cdots + n_{k-1} > n_k \geq 2$. Then $\lfloor (m-2)/2^n \rfloor + 2 = 2$. By Lemma 1, $K(n_1, n_2, \ldots, n_k)$ has a spanning complete bipartite graph $K(n', m')$ such that $n' \leq m' \leq 2^{n'} + 1$. Then by Lemma 2, the vertices of $K(n', m')$ can be covered by at most two vertex-disjoint monochromatic trees. Hence we may assume that $n = n_1 + n_2 + \cdots + n_{k-1} \leq m = n_k$.

It is obvious that $K(n_1, n_2, \ldots, n_k)$ has a spanning complete bipartite graph $K(n, m)$. By Lemma 2, the set of vertices of $K(n, m)$ can be covered by at most $\lfloor (m-2)/2^n \rfloor + 2$ vertex-disjoint monochromatic trees, and so can $K(n_1, n_2, \ldots, n_k)$.

We next prove that there exists a 2-edge-coloring of $K(n_1, n_2, \ldots, n_k)$ such that the set of vertices of $K(n_1, n_2, \ldots, n_k)$ cannot be covered by fewer than $\lfloor (m-2)/2^n \rfloor + 2$ vertex-disjoint monochromatic trees.

Since $n_k \geq 2$, we can easily color all the edges of $K(n_1, n_2, \ldots, n_k)$ red or blue so that the resulting graph has no monochromatic spanning trees. Thus the required statement mentioned above holds if $\lfloor (m-2)/2^n \rfloor = 0$. Hence we may assume that $\lfloor (m-2)/2^n \rfloor \geq 1$.

It is obvious that $K(n_1, n_2, \ldots, n_k)$ has a spanning complete bipartite graph $K(n, m)$ with partite sets $V_1 \cup \ldots \cup V_{k-1} = X$ and $V_k = Y$. We color all the edges of $K(n, m)$ in the same way as given in the proof of Lemma 3, and all the edges of $K(n_1, n_2, \ldots, n_k)$ not contained in $K(n, m)$ red. Then we apply the same arguments as in the proof of Lemma 3. The condition that there are no edges in X is not necessary for the proof of Lemma 3. Consequently the set of vertices of the 2-edge-colored $K(n_1, n_2, \ldots, n_k)$ cannot be covered by fewer than $\lfloor (m-2)/2^n \rfloor + 2$ vertex-disjoint monochromatic trees, and hence the theorem is proved. ∎

3 Proof of Theorem 2

We simplify notation by denoting by G the complete k-partite graph $K(n_1, n_2, \ldots, n_k)$ with partite sets $V_1, V_2, \ldots, V_k$, where $k \geq 3$. Suppose that all the edges of G are colored red or blue in such a way that for every vertex v, at least one red edge and at least one blue edge are incident with v.

Let H be the subgraph of G induced by the set of red edges of G. Then H is a spanning subgraph of G, and we may assume that H is not connected since otherwise H contains a red spanning tree of G. Let D be a component of H. Then there exist at least two partite sets, say V_a and V_b, which are not covered by D since if $k-1$ partite sets are covered by D, then every edge incident with a vertex not contained in $V(D)$ must be blue, which contradicts the assumption that for every vertex, at least one red edge is incident with it. Let

$$A = V_a \cap V(D) \neq V_a, \; S = V_a - A \neq \emptyset, \quad B = V_b \cap V(D) \neq V_b$$
$$T = V_b - B \neq \emptyset, \; X = V(D) - (A \cup B), \quad Y = V(G) - (V_a \cup V_b \cup V(D)).$$
$$(see\ Figure\ 2)$$

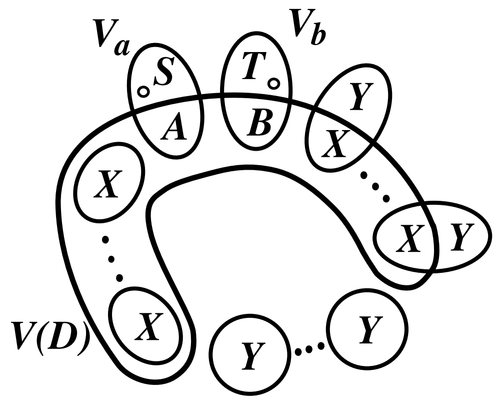

Figure 2: Covering by D

Then since all the edges of G joining a vertex in $V(D)$ to a vertex in $V(G) - V(D)$ are blue, all the edges contained in

$$E(X, S \cup T) \cup E(A \cup B, Y) \cup E(B, S) \cup E(A, T)$$

are blue. Thus if $A \cup B \neq \emptyset$ and $X \neq \emptyset$, then G has a blue spanning tree. So we may assume that $A \cup B = \emptyset$ or $X = \emptyset$.

Suppose first $X = \emptyset$. Then since $V(D) = A \cup B \neq \emptyset$ and D has at least one edge, it follows that $A \neq \emptyset$ and $B \neq \emptyset$. Hence G has a blue spanning tree.

We next assume $X \neq \emptyset$. Then $A \cup B = \emptyset$. Let V_c be a partite set such that $V_c \cap V(D) \neq \emptyset$ and $V_c \neq V_a, V_b$. Let $C = V_c \cap V(D) \neq \emptyset$ and $U = V_c - C = V_c \cap Y$. Then all the edges in $E(V_a \cup V_b, X) \cup E(X - C, U) \cup E(C, Y - U)$ are blue. Thus if $X - C \neq \emptyset$, then G has a blue spanning tree, and so we may assume $X - C = \emptyset$, which implies $V(D) \subseteq V(V_c)$. This is a contradiction since D has no edges. Consequently the theorem is proved. ∎

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