DRAFT

Path Coverings of Two Sets of Points in the Plane

Atsushi Kaneko[1], M. Kano[2] and Kazuhiro Suzuki[1]

[1]Department of Computer Science and Communication Engineering
Kogakuin University
Nishi-Shinjuku, Shinjuku-ku, Tokyo 163-8677 Japan
e-mail: kaneko@ee.kogakuin.ac.jp

[2]Department of Computer and Information Sciences
Ibaraki University, Hitachi 316-8511 Japan
e-mail: kano@cis.ibaraki.ac.jp

Abstract

We consider the following problem: For given two sets of red points and blue points in the plane respectively, we want to cover all these points with disjoint non-crossing alternating geometric paths of the same length. Determine the length of a path for which the above covering always exists under a trivial necessary condition on the numbers of red points and blue points. We give a complete solution to this problem.

1 Introduction

A graph drawn in the plane is called a *geometric graph* if every edge is a straight-line segment, and said to be *non-crossing* if it has no crossings. It is well-known ([8]) that for given k red points and k blue points in the plane in general position, there exist a non-crossing geometric alternating perfect matching on these red and blue points, that is, there exist k disjoint straight-line segments that connect red points and blue points and have no crossings. Note that red and blue points are said to be *in general position* if no three their points lie on the same line.

We generalize the above problem by considering paths since a matching is a path of length one. A path with order n and length $n-1$ is denote by P_n, and a path drawn in the plane is called an *alternating path* if it passes through alternately red points and blue points. We consider the following problem: For any given red and blue points in the plane in general position, do there exist disjoint non-crossing geometric alternating paths P_n's that cover all the red and blue points under a trivial necessary condition on the numbers of red points and blue points (Figure 1 (b)) ? For convenience, we briefly say that there exists a P_n-*covering* if there exist such disjoint paths P_n's. In this paper, we prove the following theorem, which gives a complete solution to the above problem.

Theorem 1 *Let g and h denote non-negative integers. If n is an even integer such that $2 \le n \le 14$, then for any given $(n/2)g$ red points and $(n/2)g$ blue points in the plane in general position, there exists a P_n-covering. If n is an odd integer such that $3 \le n \le 11$, then for any given $\lfloor n/2 \rfloor g + \lceil n/2 \rceil h$ red points and $\lceil n/2 \rceil g + \lfloor n/2 \rfloor h$ blue points in the plane in general position, there exists a P_n-covering.*

Moreover, for any integer n such that $n = 13$ or $n \ge 15$, there exists a configuration with $\lfloor n/2 \rfloor$ red points and $\lceil n/2 \rceil$ blue points for which there exists no P_n-covering.

In order to prove the above Theorem 1, we prove the next theorem, which is a new balanced subdivision theorem of two sets of points in the plane.

Theorem 2 *Let $m \ge 1$, $g \ge 0$ and $h \ge 0$ be integers such that $g + h \ge 1$. Let R be a set of $mg + (m+1)h$ red points and B a set of $(m+1)g + mh$ blue points in the plane such that no three points of $R \cup B$ lie on the same line. Then there exists a subdivision $X_1 \cup \cdots \cup X_g \cup Y_1 \cup \cdots \cup Y_h$ of the plane into $g + h$ disjoint convex polygons such that every X_i $(1 \le i \le g)$ contains exactly m red points and $m + 1$ blue points and every Y_j $(1 \le j \le h)$ contains exactly $m + 1$ red points and m blue points (Figure 1 (a)).*

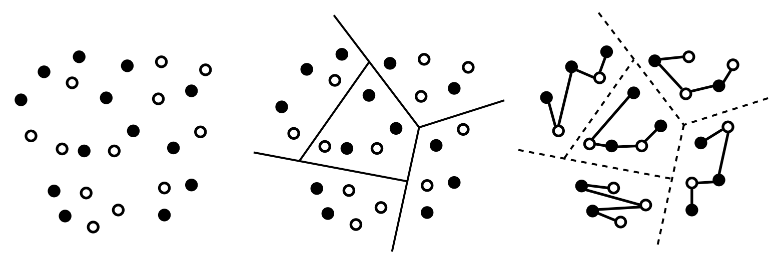

Figure 1: A subdivision given in Theorem 2 with $m = 2, g = 2, h = 3$, and a P_5-covering.

It should be remarked that the above theorem 2 cannot be generalized to partitions for two positive integers m and k with $k \ge m + 2$. Namely, if $kt + mt$ red points and $mt + kt$ blue points alternately lie on a circle in the plane for any integer $t \ge 1$, then we cannot subdivide the plane into $g+h$ disjoint convex polygons $X_1 \cup \cdots \cup X_g \cup Y_1 \cup \cdots \cup Y_h$ so that every X_i $(1 \le i \le g)$ contains exactly m red points and k blue points and every Y_j $(1 \le j \le h)$ contains exactly k red points and m blue points

We now explain a sketch of the proof of Theorem 1. Suppose first n is even. If there exists a P_n-covering of given red and blue points in the plane, then the number of red points must be equal to that of blue points, and its number is expressed as $(n/2)g$ for some integer $g \ge 1$. Conversely, if $(n/2)g$ red points and $(n/2)g$ blue points are given for some integer $g \geq 1$, then by Theorem 3, which will given in the next section, we can divide the plane into g convex polygons so that each polygon contains exactly $n/2$ red points and $n/2$ blue points. Thus if we can show that for every arrangement of $n/2$ red points and $n/2$ blue points in the plane in general position, there exists a P_n-covering, then we can say that there exist a P_n-covering of the given red and blue points, and the problem is affirmatively solved.

Similarly, if n is odd, then a trivial necessary condition for the existence of P_n-covering is that the number of red points and that of blue points are expressed as $\lfloor n/2 \rfloor g + \lceil n/2 \rceil h$ and $\lceil n/2 \rceil g + \lfloor n/2 \rfloor h$, respectively, for some non-negative integers g and h. Conversely, if such numbers of red points and blue points are given in the plane in general position, then by Theorem 2, we can divide the plane into $g + h$ convex polygons so that each polygon contains either $\lfloor n/2 \rfloor$ red points and $\lceil n/2 \rceil$ blue points or $\lceil n/2 \rceil$ red points and $\lfloor n/2 \rfloor$ blue points. Therefore if we can show for every arrangement of $\lceil n/2 \rceil$ red points and $\lfloor n/2 \rfloor$ blue points in the plane in general position, there exists a P_n-covering, then the problem is affirmatively solved.

However, when $n = 13$ or $n \geq 15$, there exist configurations of $\lceil n/2 \rceil$ red points and $\lfloor n/2 \rfloor$blue points for which there exists no P_n-covering, and these configurations are shown in Figure 2 (a), (c) and (d).

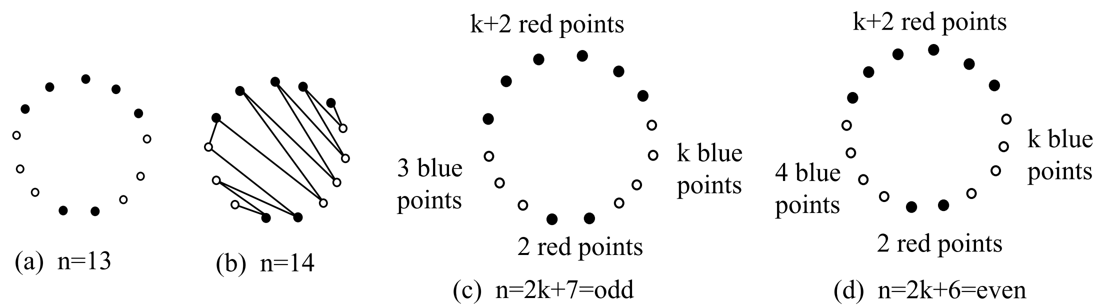

Figure 2: (a) A configuration of 13 points having no P_{13}-covering; (b) A configuration of 14 points having a P_{14}-covering; (c)+(d) Configurations of 15 or more points having no P_n-covering.

2 Proofs of Theorems

For convenience, we call a region in the plane whose boundary consists of straight-line segments a *polygon* even if it is an infinite region. For example, Figure 2 (a) illustrates a subdivision of the plane into five convex polygons.

The following Theorem 3, which was conjectured in [5] and proved for $n = 1, 2$ in [5] and [6], was recently completely proved by Bespamyatnikh, Kirkpatrick and Snoeyink [2], Sakai [9] and by Ito, Uehara and Yokoyama [4] independently. Note that this theorem with $g = 2$ is equivalent to the famous Ham-sandwich Theorem for the plane. Moreover, interesting results related to the next theorem can be found in [1].

Theorem 3 ([2], [9], [4]) *Let $a \geq 1$, $b \geq 1$ and $g \geq 2$ be positive integers. Let R be a set of ag red points and B a set of bg blue points in the plane such that $R \cup B$ consists of points in general position. Then there exists a subdivision $X_1 \cup X_2 \cup \cdots \cup X_g$ of the plane into g disjoint convex polygons such that every X_i contains exactly a red points and b blue points.*

Before giving proofs, we introduce some definitions and notation. We deal only with *directed lines* in order to define the right side of a line and the left side of it. Thus a *line* means a directed line. A line l dissects the plane into three pieces: l and two open half-planes $right(l)$ and $left(l)$, where $right(l)$ and $left(l)$ denote the *open half-planes* which are on the right side and the left side of l, respectively. Let r_1 and r_2 be two rays emanating from the same point p. Then we denote by $right(r_1) \cap left(r_2)$ the open region that is swept by the ray being rotated clockwise around p from r_1 to r_2, and does not contain the point p (see Figure 3). Similarly the open region $left(r_1) \cap right(r_2)$ can be defined, and $r_1 \cup r_2$ dissects the plane into three pieces: $r_1 \cup r_2$ and two open regions $right(r_1) \cap left(r_2)$ and $left(r_1) \cap right(r_2)$. If the internal angle $\angle r_1 p r_2$ is less than π, then we call $right(r_1) \cap left(r_2)$ the *wedge* defined by r_1 and r_2, and denote it by $wedge(r_1 r_2)$ or $wedge(r_2 r_1)$. For a line l_i with suffix i, we define l_i^* as the line lying on l_i and having the opposite direction of l_i.

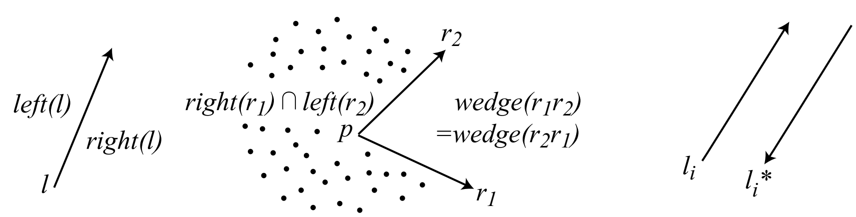

Figure 3: Open regions $right(l)$, $left(l)$ and $left(r_1) \cap right(r_2)$, and a wedge $wedge(r_1 r_2) = wedge(r_2 r_1)$.

Hereafter, R and B always denote two disjoint sets of red points and blue points in the plane, respectively, such that no three points of $R \cup B$ lie on the same line.

Theorem 4 (The Ham-sandwich Theorem [3]) *For R and B, there exists a line l such that $|left(l) \cap R| = |right(l) \cap R|$, $|l \cap R| \leq 1$, $|left(l) \cap B| = |right(l) \cap B|$ and $|l \cap B| \leq 1$.*

The line l given in the above theorem is called a *bisector* of $R \cup B$, and we say that $R \cup B$ is bisected by l. It is clear that if both $|R|$ and $|B|$ are even, then the bisector l passes through no red point and no blue point. The following Lemma 5 is known, and its distinct proofs are found in [5] and [2].

Lemma 5 *For R and B, if there exist two lines l_1 and l_2 such that $|left(l_1) \cap R| = |left(l_2) \cap R|$ and $|left(l_1) \cap B| \leq k \leq |left(l_2) \cap B|$, then there exists a line l_3 such that $|left(l_3) \cap R| = |left(l_1) \cap R|$, $|left(l_3) \cap B| = k$ and l_3 passes through no point in $R \cup B$.*

The following theorem, called the 3-cutting Theorem, plays an important role. This theorem was proved by Bespamyatnikh, Kirkpatrick and Snoeyink [2] under the assumption that

$$\frac{g_1}{h_1} = \frac{g_2}{h_2} = \frac{g_3}{h_3}.$$

However this condition can be removed without changing the arguments in the proof given in [2]. This relaxation is necessary to prove our Theorem 2. Note that similar results, which seems to be essentially equivalent to the original 3-cutting Theorem, were obtained in [9] and [4], respectively.

Theorem 6 (The 3-cutting Theorem [2]) *Let $g_1, g_2, g_3, h_1, h_2, h_3$ be positive integers such that $|R| = g_1 + g_2 + g_3$ and $|B| = h_1 + h_2 + h_3$. Suppose that one of the following statements (i) or (ii) is true:*

(i) For every integer $i \in \{1, 2, 3\}$ and for every line l such that $|left(l) \cap R| = g_i$, we have $|left(l) \cap B| < h_i$ (Figure 4 (a)).

(ii) For every integer $i \in \{1, 2, 3\}$ and for every line l such that $|left(l) \cap R| = g_i$, we have $|left(l) \cap B| > h_i$.

Then there exists three rays emanating from a certain same point such that the three open polygon W_i $(1 \leq i \leq 3)$ defined by these three rays are convex, and each W_i $(1 \leq i \leq 3)$ contains exactly g_i red points and h_i blue points (Figure 4 (b)). Moreover, one of the three rays can be chosen to be a vertically downward ray.

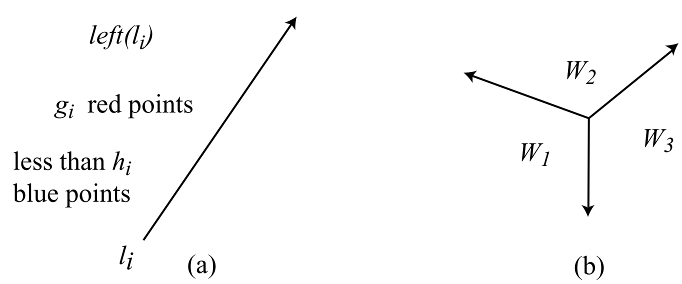

Figure 4: (a) The condition (i) ; (b) A 3-cutting.

Proof of Theorem 2. Suppose that $|R| = ag + (a+1)h$ and $|B| = (a+1)g + ah$. We prove the theorem by induction on $g + h$. In the proof, a line means a line that passes through no points in $R \cup B$, and when a line passes through some points in $R \cup B$, it is explicitly written.

If $g = 0$, then $|R| = (a+1)h$ and $|B| = ah$, and so we can get the desired subdivision by Theorem 3. Hence we may assume that $g \geq 1$, and similarly $h \geq 1$.

Assume that there exists a line l such that $left(l)$ contains exactly $as + (a+1)t$ red points and $(a+1)s + at$ blue points for some integers $0 \le s \le g$ and $0 \le t \le h$ such that $1 \le s+t \le g+h-1$. Then by applying the inductive hypotheses to $left(l)$ and $right(l)$ respectively, we can obtain the desired subdivision of the plane. Hence we may assume that there exists no such a line l. By Lemma 5 and by this fact, for every pair (i,j) of integers $0 \le i \le g$ and $0 \le j \le h$ such that $1 \le i+j \le g+h-1$, we can define $sign(i,j)$ as follows:

$sign(i,j) = +$ if $|left(l) \cap B| > (a+1)i + aj$ for every line l with $|left(l) \cap R| = ai + (a+1)j$; and

$sign(i,j) = -$ if $|left(l) \cap B| < (a+1)i + aj$ for every line l with $|left(l) \cap R| = ai + (a+1)j$.

Since $|left(l) \cap R| = as + (a+1)t$ implies $|left(l^*) \cap R| = a(g-s) + (a+1)(h-t)$ and since $|left(l) \cap B| + |left(l^*) \cap B| = |B|$, we obtain

$$sign(g-s, h-t) = -sign(s,t).$$

Claim 1 *We may assume* $sign(1,0) = sign(0,1) = -$.

Proof. Assume first $sign(1,0) = -$. Let l_1 be a line with $|left(l_1) \cap R| = a+1$. Let l_2 be a line which passes through one red point and satisfies the following:

$$|left(l_2) \cap R| = a \qquad \text{and} \qquad left(l_2) \cap (R \cup B) \subseteq left(l_1) \cap (R \cup B).$$

Then $|left(l_2) \cap B| < a+1$ as $sign(1,0) = -$. If $|left(l_1) \cap B| \ge a$, then there exists a line l_3 between l_2 and l_3 such that $|left(l_3) \cap R| = a+1$ and $|left(l_3) \cap B| = a$, which contradicts the fact mentioned above ($s = 0,\ t = 1$). Hence $|left(l_1) \cap B| < a$, which implies $sing(0,1) = -$.

Next assume $sign(1,0) = +$. By changing the colors red and blue, we have $sign(0,1) = -$. By the same argument given above, we can show that $sing(0,1) = -$ implies $sign(1,0) = -$. Therefore we may assume that Claim 1 holds. □

Claim 2 *We may assume* $sign(1,0) = \cdots = sign(g,0) = -$ *and* $sing(0,1) = \cdots = sign(0,h) = -$.

Proof. Suppose that there exists an integer k $(2 \le k \le g)$ such that $sign(1,0) = \cdots = sign(k-1,0) = -$ and $sign(k,0) = +$. Since $sign(k,0) = +$, we have $sign(g-k,h) = -$. Then

$$sign(g-k,h) = sign(k-1,0) = sign(1,0) = -, \qquad (1)$$

and thus by the 3-cutting Theorem, we can obtain a subdivision $W_1 \cup W_2 \cup W_3$ of the plane into three wedges, where W_1 contains $a(g-k)+(a+1)h$ red points and $(a+1)(g-k)+ah$ blue points, W_2 contains $a(k-1)$red points and $(a+1)k$ blue points, and W_3 contains a red points and $a+1$ blue points. By applying inductive hypotheses to each W_i, we can obtain the desired subdivision of the plane. Hence we may assume that $sign(1,0) = \cdots = sign(g,0) = -$, and similarly we may assume $sing(0,1) = \cdots = sign(0,h) = -$ by Claim 1. □

By Claim 2, we have $sign(g, 0) = -$, which implies $sign(0, h) = +$ by (1). However, this contradicts Claim 2. Consequently Theorem 2 is proved. □

Proof of Theorem 1. As we stated in the introduction, in order to prove Theorem 1, it suffices to show the next Theorem 7.

Theorem 7 *Let n be an integer such that $2 \leq n \leq 12$ or $n = 14$, and let R be a set of $\lceil n/2 \rceil$ red points and B be a set of $\lfloor n/2 \rfloor$ bule points in the plane such that no three points of $R \cup B$ lie on the same line. Then there exists a P_n-covering of $R \cup B$.*

In order to prove the above Theorem 7, we need some definitions and lemmas. For a set X of points in the plane in general position, we denote by $conv(X)$ the convex hull of X. For two points $s \notin conv(X)$ and $t \in X$, we say that *a vertex t of $conv(X)$ is visible from s* if the straight-line segment st intersects $conv(X)$ in exactly one point t, which implies that t must be a vertex of $conv(X)$. Let R_i and B_i always denote subsets of R and B, respectively. Note the following simple lemma.

Lemma 8 *Let R be a set of two red points and B a set of two blue points in the plane. Then for any vertex z of $conv(R \cup B)$, there exists a P_4-covering of $R \cup B$ starting with z.*

The following lemma is an easy consequence of Lemma 8.

Lemma 9 *Let R be a set of three red points and B a set of two blue points in the plane, and let x be a red vertex of $conv(R \cup B)$. If a blue vertex of $conv(R \cup B - \{x\})$ is visible from x, then there exists a P_5-covering of $R \cup B$ starting with x (Figure 5 (a)).*

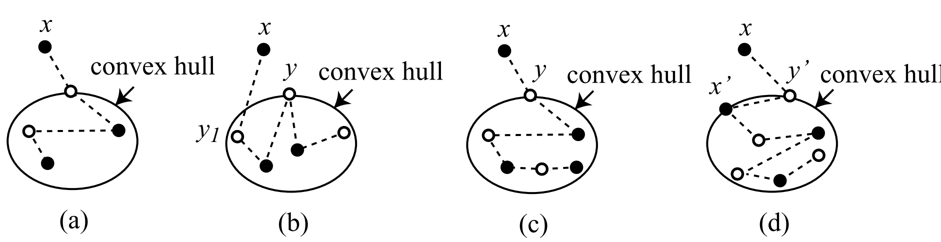

Figure 5: (a) A P_5-covering; (b) A P_6-covering; (c) A P_7-covering; (d) A P_8-covering.

Lemma 10 *Let R be a set of three red points and B a set of three blue points in the plane, and let x be a red vertex of $conv(R \cup B)$. If a blue vertex of $conv(R \cup B - \{x\})$ is visible from x, then there exists a P_6-covering of $R \cup B$ starting with x (Figure 5 (b)).*

Proof. Let y be a blue vertex of $conv(R \cup B - \{x\})$ that is visible from x. If a red vertex of $conv(R \cup B - \{x, y\})$ is visible from y, then by Lemma 9, there exists a P_5-covering of $R \cup B - \{x\}$ starting with y, which implies the existence of the desired P_6-covering of $R \cup B$. So we may assume that all the vertices of $conv(R \cup B - \{x, y\})$ visible from y are blue points. Then there are exactly two such blue vertices, and at least one of them, say y_1, is visible from x, and at least one red vertex of $conv(R \cup B - \{x, y_1\})$ is visible from y_1 (Figure 5 (b)). Then by Lemma 9, there exists a P_6-covering of $R \cup B$ starting with x. □

Lemma 11 *Let R be a set of four red points and B a set of three blue points in the plane, and let x be a red vertex of $conv(R \cup B)$. If a blue vertex of $conv(R \cup B - \{x\})$ is visible from x, then there exists a P_7-covering of $R \cup B$ starting with x (Figure 5 (c)).*

Proof. Let y be a blue vertex of $conv(R \cup B - \{x\})$ that is visible from x. If a red vertex of $conv(R \cup B - \{x, y\})$ is visible from y, then by applying Lemma 10 to $R \cup B - \{x\}$ and y, we can obtain the desired P_7-covering of $R \cup B$ starting with x. So we may assume that all the vertices of $conv(R \cup B - \{x, y\})$ that is visible from y are blue points. Then there exist exactly two such blue vertices, and at least one of them, say y_1, is visible from x, and at least one red vertex of $conv(R \cup B - \{x, y_1\})$ is visible from y_1. Then by Lemma 10, there exists a P_7-covering of $R \cup B$ starting with x. □

Lemma 12 *Let R be a set of four red points and B a set of four blue points in the plane, and let x be a red vertex of $conv(R \cup B)$. If a red vertex and a blue vertex of $conv(R \cup B - \{x\})$ are both visible from x, then there exists a P_8-covering of $R \cup B$ starting with x (Figure 5 (d)).*

Proof. There exist a red vertex x_1 and a blue vertex y_1 of $conv(R \cup B - \{x\})$ such that both of them are visible from x and $x_1 y_1$ is an edge of $conv(R \cup B - \{x\})$. It is obvious that x_1 is a red vertex of $conv(R \cup B - \{x, y_1\})$ which is visible from y_1. Hence by Lemma 11, there exists the required P_8-covering of $R \cup B$ starting with x. □

Proof of Theorem 7. Suppose that $|R| = \lceil n/2 \rceil$ and $|B| = \lfloor n/2 \rfloor$. If $2 \leq n \leq 6$, then we can easily show the existence of the required P_n-covering of $R \cup B$ by similar arguments in the case of $n = 7$, which is given below. Hence we may assume that $7 \leq n \leq 12$ or $n = 14$. We consider several cases corresponding to the value of n.

Case 1. $n = 7$.

By the Ham-Sandwich Theorem, there exists a bisector l such that l passes through exactly one blue point, say y, and each of $left(l)$ and $right(l)$ contains exactly two red points and one blue point. Let $R_1 \cup B_1 = (R \cup B) \cap left(l)$. Since y is a vertex of $conv(R_1 \cup B_1 \cup \{y\})$, by Lemma 8 there exists a P_4-covering of $R_1 \cup B_1 \cup \{y\}$ starting with y. Similarly, there exists a P_4-covering of $((R \cup B) \cap right(l)) \cup \{y\}$ starting with y. Hence there exists the desired P_7-covering of $R \cup B$.

Case 2. $n = 8$.

Suppose that $R \cup B$ is bisected by a line l so that a red vertex x of $conv((R \cup B) \cap left(l))$ and a blue vertex y of $conv((R \cup B) \cap right(l))$ are visible from each other (Figure 6 (a)). Then by Lemma 8, there exist a P_4-covering of $(R \cup B) \cap left(l)$ starting with x and a P_4-covering of $(R \cup B) \cap right(l)$ starting with y. By connecting these two paths by an edge xy, we obtain the desired P_8-covering of $R \cup B$. Hence we may assume that there exists no such a bisector l of $R \cup B$.

Let l_1 be a bisector, and let $R_1 \cup B_1 = (R \cup B) \cap left(l_1)$ and $R_2 \cup B_2 = (R \cup B) \cap right(l_1)$. By the above assumption, if a vertex of $conv(R_1 \cup B_1)$ and a vertex of $conv(R_2 \cup B_2)$ are visible from each other, then they must have the same color. So, without loss generality, we may assume that these vertices are red. Take a tangent line to $conv(R_1 \cup B_1)$ and $conv(R_2 \cup B_2)$, which passes through two red vertices, and rotate it slightly, then we obtain a new bisector l_2 for which the partition $R \cup B = ((R \cup B) \cap left(l_2)) \cup ((R \cup B) \cap right(l_2))$ does not satisfy the above assumption (Figure 6 (b)). Therefore the case is proved.

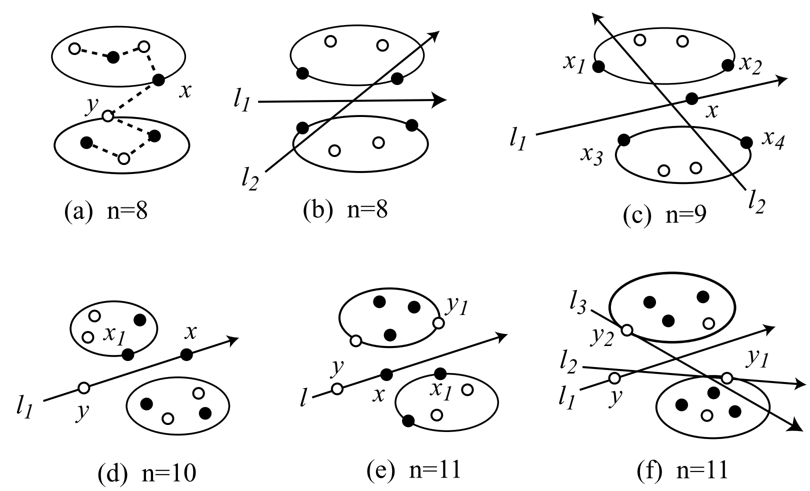

Figure 6: (a) A P_8-covering; (b) A bisector l_2; (c) A P_9-covering; (d) A P_{10}-covering; (e) A P_{11}-covering in Subcase 5.1; (f) Bisectors l, l_1, l_2 in Subcase 5.2.

Case 3. $n = 9$.

Let l_1 be a bisector, which passes through one red point, say x. Let $R_1 \cup B_1 = (R \cup B) \cap left(l_1)$ and $R_2 \cup B_2 = (R \cup B) \cap right(l_1)$. If a blue vertex of $conv(R_1 \cup B_1)$ and a blue vertex of $conv(R_2 \cup B_2)$ are both visible from x, then there exists a P_9-covering of $R \cup B$ by Lemma 9. Thus we may assume that every vertex of $conv(R_1 \cup B_1)$ visible from x is red. Hence every red point of $R_1 = \{x_1, x_2\}$ is a vertex of $conv(R_1 \cup B_1)$ and visible from x (Figure 6 (c)). If a blue vertex y of $conv(R_2 \cup B_2)$ is visible from x_1 or x_2, then at least one of yx_1 and yx_2 intersects $conv(R_1 \cup B_1 \cup \{x\})$ in exactly one point x_1 or x_2, and so by Lemmas 8 and 10, we can obtain the desired P_9-covering of $R \cup B$. Hence we may assume that every vertex of $conv(R_2 \cup B_2)$ that is visible from x_1 or x_2 is red, which implies that the two red points of R_2 are vertices of $conv(R_2 \cup B_2)$ and visible from x_1 or

x_2. Similarly, we may assume that every vertex of $conv(R_1 \cup B_1)$ visible from a red point of R_3 is red.

Let $R_3 = \{x_3, x_4\}$. Then there exists a bisector l_2 of $R \cup B$ such that $left(l_2) \cap R = \{x_1, x_3\}$ or $right(l_2) \cap R = \{x_2, x_4\}$ (Figure 6 (c)). By symmetry, we may assume that l_2 satisfies $left(l_2) \cap R = \{x_1, x_3\}$, which implies l_2 passes through exactly one point of $\{x, x_2, x_4\}$, say x'. Since a blue vertex of $conv((R \cup B) \cap left(l_2))$ is visible from x', we can obtain a P_9-covering of $R \cup B$ by the above same argument as above.

Case 4. $n = 10$.

Let l be a bisector of $R \cup B$. Then l passes through one red point, say x, and one blue point, say y. Let $R_1 \cup B_1 = (R \cup B) \cap left(l)$ and $R_2 \cup B_2 = (R \cup B) \cap right(l)$. Without loss of generality, a red vertex x_1 of $conv(R_1 \cup B_1)$ is visible from y since otherwise a blue vertex of $conv(R_1 \cup B_1)$ is visible from x. By Lemma 9, $R_1 \cup B_1 \cup \{y\}$ has a P_5-covering starting with y (Figure 6 (d)). Since x is a red vertex of $conv(R_2 \cup B_2 \cup \{x\})$ that is visible from y, by Lemma 10, $R_2 \cup B_2 \cup \{x, y\}$ has a P_6-covering starting with y. Consequently, $R \cup B$ has a P_{10}-covering.

Case 5. $n = 11$.

Subcase 5.1. There exists a line l such that l passes through one red point and one blue point, $(R \cup B) \cap left(l)$ consists of three red points and two blue points, and $(R \cup B) \cap right(l)$ consists of two red points and two blue points (Figure 6 (e)).

Let x and y be the red point and the blue point on the line l, respectively, and let $R_1 \cup B_1 = (R \cup B) \cap left(l)$ and $R_2 \cup B_2 = (R \cup B) \cap right(l)$. If a red vertex of $conv(R_1 \cup B_1)$ is visible from y, then by Lemma 10, $R_1 \cup B_1 \cup \{y\}$ has a P_6-covering starting with y. Moreover, by Lemma 10, $R_2 \cup B_2 \cup \{x, y\}$ has a P_6-covering starting with y, and so we can obtain the desired P_{11}-covering of $R \cup B$. Hence we may assume that every vertex of $conv(R_1 \cup B_1)$ visible from y is blue. Similarly, if a blue vertex of $conv(R_2 \cup B_2)$ is visible from x, then by Lemma 9, $R_2 \cup B_2 \cup \{x\}$ has a P_5-covering starting with x. Moreover, by Lemma 11, $R_1 \cup B_1 \cup \{x, y\}$ has a P_7-covering starting with x, and hence there exists the desired P_{11}-covering of $R \cup B$. Thus we may assume that every vertex of $conv(R_2 \cup B_2)$ visible from x is red. Therefore there exist $y_1 \in B_1$ and $x_1 \in R_2 \cup \{x\}$ such that $y_1 x_1$ intersects $conv(R_1 \cup B_1 \cup \{y\})$ in exactly one point y_1 and intersects $conv(R_2 \cup B_2 \cup \{x\})$ in exactly one point x_1 (Figure 6 (d)). Since a red vertex of $conv(R_1 \cup B_1 \cup \{y\})$ is visible from y_1, by Lemma 10, $R_1 \cup B_1 \cup \{y\}$ has a P_6-covering starting with y_1. Similarly, $R_2 \cup B_2 \cup \{x\}$ has a P_5-covering starting with x_1. Therefore $R \cup B$ has the desired P_{11}-covering.

Subcase 5.2. There exists no line l such that l passes through one red point and one blue point, $(R \cup B) \cap left(l)$ consists of three red points and two blue points, and $(R \cup B) \cap right(l)$ consists of two red points and two blue points.

Let l_1 be a bisector, which passes through one blue point, say y. By the assumption of this subcase, when we rotate l_1 clockwise around y until it is tangent to $conv((R \cup B) \cap right(l_1))$ or $conv((R \cup B) \cap left(l_1))$, it must be tangent at a blue vertex. Without loss of generality, we may assume that it is tangent to $conv((R \cup B) \cap right(l_1))$ at a blue vertex, say y_1 (Figure 6 (f)). Then by a small rotation of the tangent line around y_1, we can obtain a new bisector l_2 such that $(R \cup B) \cap left(l_2) = (R \cup B) \cap left(l_1)$ and $(R \cup B) \cap right(l_2) = ((R \cup B) \cap right(l_1)) \cup \{y\} - \{y_1\}$ (Figure 6 (f)). We repeat the above procedure one more time or two more times until we can get a bisector l_3 that passes through a blue vertex y_2 of $conv((R \cup B) \cap left(l_2))$ (Figure 6 (f)). However, this bisector l_3 does not satisfy the assumption of this subcase, which implies that the proof of the subcase is complete.

Case 6. $n = 12$.

We consider two subcases.

Subcase 6.1. There exists a line l such that l passes through one red point and one blue point, $(R \cup B) \cap left(l)$ consists of two red points and three blue points, and $(R \cup B) \cap right(l)$ consists of three red points and two blue points (Figure 7 (a)).

Let x and y be the red point and the blue point on l, respectively, and let $R_1 \cup B_1 = (R \cup B) \cap left(l)$ and $R_2 \cup B_2 = (R \cup B) \cap right(l)$. If a blue vertex of $conv(R_1 \cup B_1)$ is visible from x, then by Lemma 10, $R_1 \cup B_1 \cup \{x\}$ has a P_6-covering starting with x. Moreover, by Lemma 11, $R_2 \cup B_2 \cup \{x, y\}$ has a P_7-covering starting with x, and so there exists the desired P_{12}-covering of $R \cup B$. Hence we may assume that every vertex of $conv(R_1 \cup B_1)$ visible from x is red. By symmetry, if a red vertex of $conv(R_2 \cup B_2)$ is visible from y, then we can obtain the desired P_{12}-covering of $R \cup B$. Hence we may assume that every vertex of $conv(R_2 \cup B_2)$ visible from y is blue. Therefore we can find two points $x_1 \in R_1$ and $y_1 \in B_2$ such that x_1y_1 intersects $conv(R_1 \cup B_1 \cup \{x\})$ in exactly one point x_1 and intersects $conv(R_2 \cup B_2 \cup \{y\})$ in exactly one point y_1. Since $R_1 \cup B_1 \cup \{x\}$ has a P_6-covering starting with x_1 and $R_2 \cup B_2 \cup \{y\}$ has a P_6-covering starting with y_1, we can obtain the desired P_{12}-covering of $R \cup B$ by connecting these paths by x_1y_1.

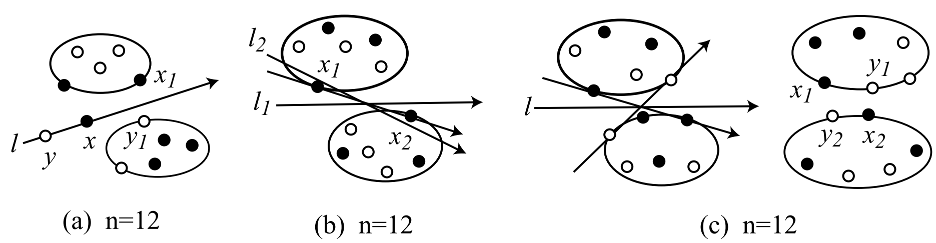

Figure 7: (a) A configuration of Subcase 6.1; (b) Bisectors l_1 and l_2; (c) A configuration of Subcase 6.2.

Subcase 6.2. There exists no line l such that l passes through one red point and one blue point, $(R \cup B) \cap left(l)$ consists of two red points and three blue points, and $(R \cup B) \cap right(l)$ consists of three red points and two blue points (Figure 7 (b)).

Let l_1 be a bisector, which passes through no red and blue points. If we rotate l_1 clockwise until it is tangent to both $conv((R \cup B) \cap left(l_1))$ and $conv((R \cup B) \cap$

$right(l_1)$), then the line passes through two vertices with the same colors since otherwise the assumption of the subcase does not hold for the tangent line. By a small rotation of the tangent line around its midpoint, we can obtain a new bisector l_2 (Figure 7 (b)). By repeating this procedure at most two more times, we can find a bisector l such that $(R \cup B) \cap left(l)$ and $(R \cup B) \cap right(l)$ have two common tangent lines that pass through two red vertices or two blue vertices each (Figure 7 (c)). Let $R_3 \cup B_3 = (R \cup B) \cap left(l)$ and $R_4 \cup B_4 = (R \cup B) \cap right(l)$. Then we can find four points $x_1 \in R_3, y_1 \in B_3, x_2 \in R_4,\ y_2 \in R_4$ such that x_1y_1 is an edge of $conv(R_3 \cup B_3)$, x_2y_2 is an edge of $conv(R_4 \cup B_4)$, and x_1y_2 intersects $conv(R_3 \cup B_3)$ and $conv(R_4 \cup B_4)$ in exactly one point x_1 and y_2, respectively. Then by Lemma11, $R_3 \cup B_3$ has a P_6-covering starting with x_1 and $R_4 \cup B_4$ has a P_6-covering starting with y_2, and thus $R \cup B$ has the desired P_{12}-covering.

Case 7. $n = 14$.

Let l be a bisector. Then l passes through one red point, say x, and one blue point, say y. Let $R_1 \cup B_1 = (R \cup B) \cap left(l)$ and $R_2 \cup B_2 = (R \cup B) \cap right(l)$. By symmetry, we may assume that a red vertex x_1 of $conv(R_1 \cup B_1)$ is visible from both x and y. If a blue vertex of $conv(R_2 \cup B_2)$ is visible from x, then by Lemma 11, both $R_1 \cup B_1 \cup \{y\}$ and $R_2 \cup B_2 \cup \{x\}$ have P_7-coverings starting with y and x, respectively, and so $R \cup B$ has a P_{14}-covering. Hence we may assume that every vertex of $conv(R_2 \cup B_2)$ visible from x is red, which implies that there exists a red vertex, say x_2, of $conv(R_2 \cup B_2)$ which is visible from both x and y. Since x_2 is visible from y, by the same argument as above, we can prove that we may assume every vertex of $conv(R_1 \cup B_1)$ visible from x is a red point. If a blue vertex of $conv(R_1 \cup B_1)$ is visible from y, then by applying Lemma 12 to $R_1 \cup B_1 \cup \{y\}$, $R_1 \cup B_1 \cup \{y\}$ has a P_8-covering starting with y. By Lemma 11, $R_2 \cup B_2 \cup \{y\}$ has a P_7-covering starting with y. Therefore $R \cup B$ has the desired P_{14}-covering. Thus we may assume that every vertex of $conv(R_1 \cup B_1)$ visible from y is red. By symmetry, we may also assume that every vertex of $conv(R_2 \cup B_2)$ visible from y is red.

We consider the two subcases.

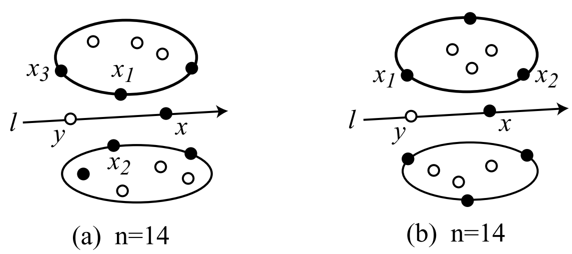

Figure 8: (a) A configuration of Subcase 7.1; (b) A configuration of Subcase 7.2; (c) A configuration of $((R \cup B) \cap left(l)) \cup \{x\}$.

Subcase 7.1. Three vertices of $conv(R_1 \cup B_1)$ are visible from x or y.

Without loss of generality, we may assume that the line l is horizontal and directed from left to right, and y lies to the left of x (Figure 7 (a)). Let x_3 be the left most vertex of $conv(R_1 \cup B_1)$ that is visible from y. Then x_3 is a red point, and a blue vertex of $conv((R_1 \cup B_1 \cup \{x\}) - \{x_3\})$ is visible from x_3. Thus by Lemma 11, $R_1 \cup B_1 \cup \{x\}$ has a P_7-covering starting with x_3. Similarly, $R_2 \cup B_2 \cup \{y\}$ has a P_7-covering starting with y. By connecting these two paths by x_3y, we obtain the desired P_{14}-covering of $R \cup B$.

Subcase 7.2. Exactly two vertices of $conv(R_1 \cup B_1)$ are visible from x or y.

It is shown as in the proof of the above subcase that we may assume that the two vertices of $conv(R_1 \cup B_1)$ visible from x or y are red points (Figure 7 (b)). Of course, these points visible from both x and y. We denote these red points by x_1 and x_2, and the remaining red point of R_1 by x_3. By the same argument as in the proof of the above subcase, we may assume that no blue vertex of $conv(R_1 \cup B_1 - \{x_i\})$ is visible from x_i for every $i \in \{1, 2\}$ since both x_1 and x_2 are visible from y. However, there exists no such a configuration. Consequently the proof is complete. □

References

[1] I. Bárány and J. Matoušek, Simultaneous partitions of measures by k-fans, *Discrete Comput. Geom.* **25** (2001)317–334.

[2] S. Bespamyatnikh, D. Kirkpatrick and J. Snoeyink, Generalizing ham sandwich cuts to equitable subdivisions, *Discrete Comput. Geom.* **24** (2000) 605–622.

[3] J. Goodman and J. O'Rourke, *Handbook of Discrete and Computational Geometry*, CRC Press, p.211 (1997)

[4] H. Ito, H. Uehara, and M. Yokoyama, 2-dimentinal ham-sandwich theorem for partitioning into three convex pieces, *Discrete and Computational Geometry* (LNCS **1763**), (2000) 129–157.

[5] A. Kaneko, and M. Kano, Balanced partitions of two sets of points in the plane, *Computational Geometry: Theory and Applications*, **13** (1999) 253-261.

[6] A. Kaneko, and M. Kano, A balanced partition of points in the plane and tree embedding problems, preprint

[7] A. Kaneko, and M. Kano, Generalized balanced partitions of two sets of points in the plane, *Discrete and Computational Geometry* (LNCS **2098**), (2001) 176-186.

[8] L.C. Larson, *Problem-Solving Through Problems*, (Springer, New York), (1983) 200–201.

[9] T. Sakai, Balanced Convex Partitions of Measures in R^2, to appear in *Graphs and Combinatorics*.