Perfect n-Partitions of Convex Sets in the Plane

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Abstract

For a region X in the plane, we denote by area(X) the area of X and by $\ell(\partial(X))$ the length of the boundary of X. Let S be a convex set in the plane, $n \geq 2$ an integer, and $\alpha_1,\ \alpha_2, \ldots, \alpha_n$ positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $0 < \alpha_i \leq 1/2$ for all $1 \leq i \leq n$. Then we shall show that S can be partitioned into n disjoint convex subsets $T_1,\ T_2, \ldots, T_n$ so that each T_i satisfies the following three conditions: (i) area$(T_i) = \alpha_i \times$ area(S); (ii) $\ell(T_i \cap \partial(S)) = \alpha_i \times \ell(\partial(S))$; and (iii) $T_i \cap \partial(S)$ consists of exactly one continuous curve.

Key words: Convex sets, Perfect partition.

1 Introduction

We begin with a motivation of the original problem related to our results. Some children attend a birthday party, and there is a big non-circular birthday cake. We want to divide the cake among all the children in such a way that each child gets the same amount of cake and the same amount of icing (exposed area) and holds it easily (i.e., each cake is convex and has exactly one icing side.)[1]. If the height of the cake is constant, then the above problem can be said as follows. Let S be a convex set in the plane, which corresponds to the base of the cake. Then is it possible to partition S into n convex subsets so that each subset has the same area and has exactly one continuous part of the boundary of S with the same length (see Figure 1)? If such a partition exists, we say that S can be *perfectly partitioned* into n convex subsets, and call this partition a *perfect n-partition.*

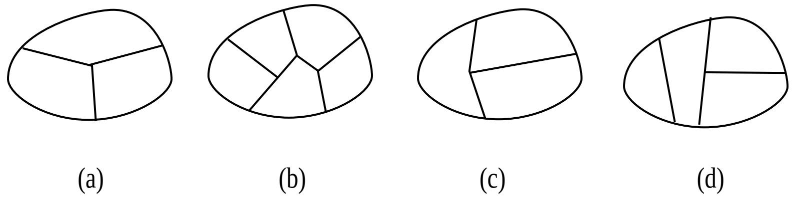

Fig. 1. Perfect partitions (a), (b); and non-perfect partitions (c), (d).

It was proved in [2] that a perfect partition always exists for $n = 3$.

Theorem 1 *Every convex set S in the plane can be perfectly partitioned into three convex subsets by three rays emanating from a point in S (see Figure 1 (a)).*

In this paper, we shall show that a perfect partition always exists for every $n \geq 3$, and obtain more general results in Theorem 2, which is our main theorem.

Note that an $a \times b$ rectangle with $b > 4a$ cannot be perfectly partitioned into n convex subsets by n rays emanating from the same point for every $n \geq 5$ (see Figure 2)[2].

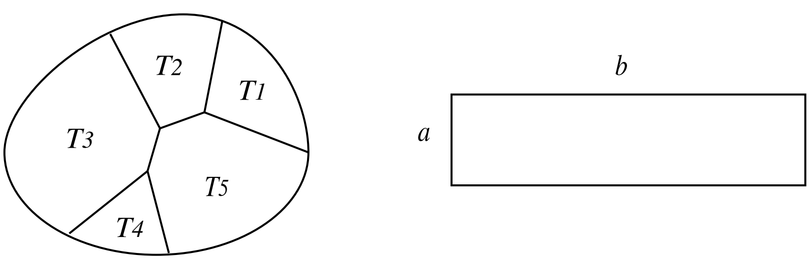

Fig. 2. A partition given in Theorem 2 and an $a \times b$ rectangle with $b > 4a$.

For a domain X in the plane, we denote by area(X) the area of X and by $\partial(X)$ the boundary of X. For a curve C in the plane, $\ell(C)$ denote the length of C. In particular, $\ell(\partial(X))$ denotes the length of the boundary of X. The purpose of this paper is to prove the following theorem, which obviously guarantees the existence of a perfect n-partition.

Theorem 2 *Let S be a convex set in the plane, $n \geq 2$ an integer, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $0 < \alpha_i \leq 1/2$ for all $1 \leq i \leq n$. Then S can be partitioned into n convex subsets $T_1, T_2, \ldots, T_n$ so that each T_i satisfies the following three conditions:*

(i) $\text{area}(T_i) = \alpha_i \times \text{area}(S)$*; (ii)* $\ell(T_i \cap \partial(S)) = \alpha_i \times \ell(\partial(S))$*; and (iii)* $T_i \cap \partial(S)$ *consists of exactly one continuous curve (see Figure 2).*

If $1/2 < \alpha_1$, $0 < \alpha_2$ and $\alpha_1 + \alpha_2 = 1$, then it is impossible to partition a circle C into two subsets satisfying the conditions of Theorem 2 since the area of a convex subset T_1 with $\ell(T_1 \cap \partial(C)) = \alpha_1 \times \ell(\partial(C))$ is always greater than $\alpha_1 \times \text{area}(C)$. Hence we need the condition that $\alpha_i \leq 1/2$ for all i.

We now explain the relationship between a perfect n-partition and the result on balanced partitions of two sets of points in the plane. The following theorem was conjectured and proved for $n = 1,\ 2$ in [5] and [6], and was recently proved for every $n \geq 1$ independently by [3], [4] and [7].

Theorem 3 *Let* $m \geq 1, n \geq 1$ *and* $k \geq 2$ *be positive integers. Let* R *be a set of* mk *red points and* B *a set of* nk *blue points in the plane such that no three points of* $R \cup B$ *lie on the same line. Then* $R \cup B$ *can be partitioned into* k *disjoint subsets* $X_1, X_2, \dots, X_k$ *so that* X_i *contains exactly* m *red points and* n *blue points for every* $1 \leq i \leq k$ *and* $\text{conv}(X_i) \cap \text{conv}(X_j) = \emptyset$ *for all* $i \neq j$*, where* $\text{conv}(X_i)$ *denotes the convex hull of* X_i*.*

For a given convex set S in the plane, if we uniformly put a lot of red points on $\partial(S)$ and a lot of blue points on S, then by the above Theorem 3, we can partition S into k convex subsets $\{X_i\}$ so that each X_i contains the same number of red points and the same number of blue points, that is, the length of $X_i \cap \partial(S)$ is constant and the area of X_i is also constant. However we cannot say that $X_i \cap \partial(S)$ consists of exactly one continuous curve (see Figure 1 (d)). Thus even a perfect n-partition cannot be obtained directly from Theorem 3. Important parts of the proof of our theorem, which are given in lemmas 8 and 9, are devoted to guarantee that $X_i \cap \partial(S)$ consists of exactly one continuous curve.

We conclude this section with two remarks on our theorems. When we consider convex polygons in the plane instead of convex sets, we can also obtain their partitions similar to the partitions of convex sets given in Theoerm 2 but under much weaker conditions than those in Theorem 2. This partition is stated in Theorem 5 at the beginning of the next section. The other is to propose the following conjecture, which is a generalization of Theorem 3 and obtained by considering Theorem 2.

Conjecture 4 *Let* $m_1 \geq m_2 \geq \ldots \geq m_k \geq 1,\ n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$ *and* k *be positive integers such that* $m_1 \leq (m_1 + m_2 + \ldots + m_k)/2$ *and*

$$\frac{m_1}{n_1} = \frac{m_2}{n_2} = \cdots = \frac{m_k}{n_k}.$$

Let R *be a set of* $m_1 + m_2 + \ldots + m_k$ *red points and* B *a set of* $n_1 + n_2 + \ldots n_k$ *blue points in the plane such that no three points of* $R \cup B$ *lie on the same line. Then*

$R \cup B$ *can be partitioned into* k *disjoint subsets* $X_1, X_2, \ldots, X_k$ *so that every* X_i *contains exactly* m_i *red points and* n_i *blue points and* $\mathrm{conv}(X_i) \cap \mathrm{conv}(X_j) = \emptyset$ *for all* $i \neq j$.

2 Proof of Theorem 2

We define some notations. For two points X and Y in the plane, we denote by XY the *straight-line segment* joining X to Y and by $|XY|$ the length of XY, which is equal to the distance between X and Y. For two points P and Q on the boundary of a convex set S, the boundary of S is divided into two arcs by P and Q, and $\mathrm{arc}(PQ)$ denotes one of the arcs between P and Q that is easily determined from the context and is the shorter one in almost every case. If it is not easily determined, we explain it more precisely. A quadrilateral with consecutive vertices (P_1, P_2, P_3, P_4) and a hexagon with consecutive vertices $(Q_1, Q_2, \ldots, Q_6)$ are denoted by $\mathrm{quad}(P_1P_2P_3P_4)$ and $\mathrm{hex}(Q_1Q_2 \cdots Q_6)$, respectively.

We begin with a theorem on partitions of convex polygons, which will be used in the proof of Theorem 2.

Theorem 5 *Let* n *and* m *be integers such that* $3 \leq n$ *and* $1 \leq m \leq n$. *Let* P *be a convex polygon in the plane with consecutive vertices* $(V_1, V_2, \ldots, V_n)$, *and* $\beta_1, \beta_2, \ldots, \beta_m$ *be positive real numbers such that* $\beta_1 + \beta_2 + \cdots + \beta_m = \mathrm{area}(P)$. *Then for given* m *edges* $e_1, e_2, \ldots, e_m$ *of* P, P *can be partitioned into* m *disjoint convex polygons* $Q_1, Q_2, \ldots, Q_m$ *so that each* Q_i $(1 \leq i \leq m)$ *contains the edge* e_i *and has area* β_i *(see Figure 3).*

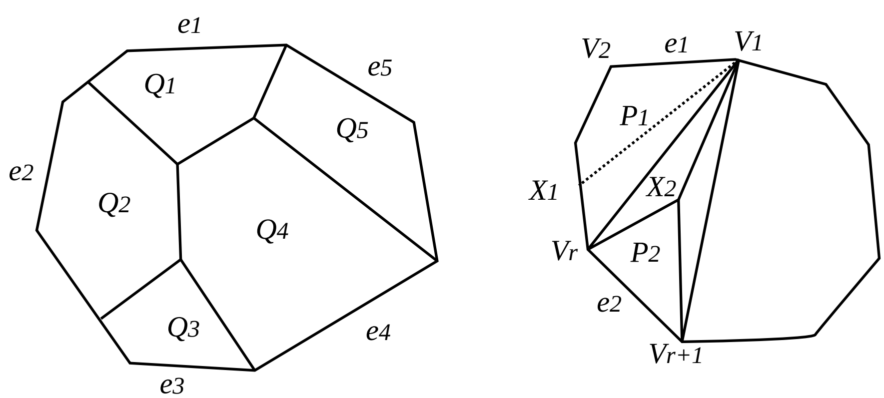

Fig. 3. A partition $\{Q_i\}$ of P; and the figure of proof.

Proof We prove the theorem by induction on m. If $m = 1$, then $Q_1 = P$ is the desired partition. If $m = 2$, then there exists a line that partition P into two sub-polygons R_1 and R_2 in such a way that each R_i contains e_i and has area β_i. So we may assume that $m \geq 3$.

By a new labeling of $\{V_i\}$ and $\{\beta_i\}$, we may assume that $e_1 = V_1V_2$, $e_2 = V_rV_{r+1}$ and no edges of P between e_1 and e_2 are chosen in $\{e_i\}$.

Let P_1 be the sub-polygon with vertex set $\{V_1, V_2, \ldots, V_r\}$, which is obtained from P by dividing by the diagonal V_1V_r (see Figure 3). If $\mathrm{area}(P_1) \geq \beta_1$, then we can find a point X_1 on the edges $V_2V_3 \cup \cdots \cup V_{r-1}V_r$ such that the area of the sub-polygon divided by V_1X_1 is equal to β_1. Then we can apply the inductive hypothesis to the remaining polygon. Therefore we may assume that $\mathrm{area}(P_1) < \beta_1$.

Let P_2 be the sub-polygon with vertex set $\{V_1, V_2, \ldots, V_r, V_{r+1}\}$. If $\mathrm{area}(P_2) \geq \beta_1 + \beta_2$, then we can easily find a point X_2 in $\triangle V_1V_rV_{r+1}$ such that

$$\mathrm{area}(\triangle X_2V_rV_{r+1}) = \beta_2 \qquad \text{and} \qquad \mathrm{area}(P_1 \cup \triangle V_1V_rX_2) = \beta_1$$

(see Figure 3). Then we can apply the inductive hypothesis to the remaining convex polygon, and get the desired partition of P. Hence we may assume that $\mathrm{area}(P_2) < \beta_1 + \beta_2$.

Put $\gamma = \beta_1 + \beta_2 - \mathrm{area}(P_2) > 0$. We consider the polygon $P - P_2$ together with the edges $\{V_1V_{r+1}, e_3, \ldots, e_m\}$ and the positive real numbers $\gamma, \beta_3, \ldots, \beta_m$. Then by the inductive hypothesis $P - P_2$ can be partitioned into $m-1$ convex subsets $R, Q_3, \ldots, Q_m$. It is easy to see that $R \cup P_2$ is a convex polygon with area $\beta_1 + \beta_2$, and can be partitioned into two convex polygons which contains e_1 and e_2, respectively, and have areas β_1 and β_2, respectively. Consequently, the lemma is proved. $\square$

In order to prove our theorem, we need some lemmas. The following lemma was proved in [2].

Lemma 6 *Let $\triangle ABC$ be a triangle in the plane, and S a convex set that is contained in $\triangle ABC$ and contains BC. If $\angle B \geq \angle C$, then for a point X on AB such that $|BX| + |XC| = \ell(\mathrm{arc}(BC))$, where $\mathrm{arc}(BC) = \partial(S) - BC$, it follows that $\mathrm{area}(\triangle XBC) \leq \mathrm{area}(S)$ (see Figure 4).*

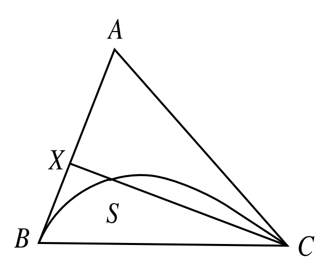

Fig. 4. A triangle $\triangle ABC$ and a convex set S.

Lemma 7 *Let $\triangle ABC$, S and* arc(BC) *be the same as the above lemma 6. Let h denote the height of $\triangle ABC$ with base AB or AC (see Figure 5). Then*

$$\text{area}(S) < \frac{1}{2} h \times \ell(\text{arc}(BC)). \tag{1}$$

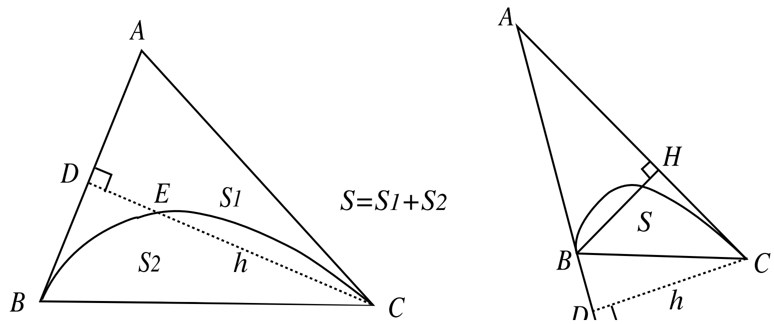

Fig. 5. Triangles $\triangle ABC$ and convex sets S.

Proof Without loss generality, we may assume that h is the height of $\triangle ABC$ with base AB. We first assume that $\angle B \leq \frac{\pi}{2}$, that is, $\angle B$ is acute. Let D be the foot of the perpendicular dropped from C to line AB, and let E, if any, be the intersection of CD and arc(BC) (see Figure 5). Then $h = |CD|$. If E does not exist, then S is contained in $\triangle DBC$, and so area$(S) <$ area$(\triangle DBC) \leq \frac{1}{2} h \times \ell(\text{arc}(BC))$. Thus we may assume that the intersection E exists. Then S is divided into two subset $S_1 = S \cap \triangle ADC$ and $S_2 = S \cap \triangle DBC$ by the line CD. We have

$$\text{area}(S_2) \leq \triangle DBC = \frac{1}{2} h \, |DB| < \frac{1}{2} h \times \ell(\text{arc}(BE)).$$

Since S_1 is clearly contained in the rectangle R one of whose edge is CD and whose height is $\ell(\text{arc}(CE))/2$, it follows that

$$\text{area}(S_1) < \text{area}(R) = \frac{1}{2} h \times \ell(\text{arc}(CE)).$$

Therefore we get the desired inequality in this case.

Next suppose that $\angle B > \frac{\pi}{2}$ (see Figure 5). Let H be the foot of the perpendicular dropped from B to line AC. In this case we can show that the following inequality holds by the same argument as above.

$$\text{area}(S) < \frac{1}{2} |BH| \times \ell(\text{arc}(BC)).$$

Since $h = |CD| > |BH|$, the above inequality implies the desired inequality (1) of the lemma. $\square$

For two points P and Q on the boundary of a convex set S in the plane, we denote by lune(PQ) the *disc segment* determined by the arc arc(PQ) and by the line segment PQ (see Figure 5).

Lemma 8 *Let $n \geq 3$ be an integer, and let S and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the same as Theorem 1. Let $P_1, P_2, \ldots, P_n$ be n points on $\partial(S)$ such that for every $1 \leq i \leq n$, $\ell(\text{arc}(P_iP_{i+1})) = \alpha_i \times \ell(\partial(S))$. Then* $\text{area}(\text{lune}(P_iP_{i+1})) < \alpha_i \times \text{area}(S)$ *for all $1 \leq i \leq n$ except at most one certain integer, where $P_{n+1} = P_1$ (see Figure 5).*

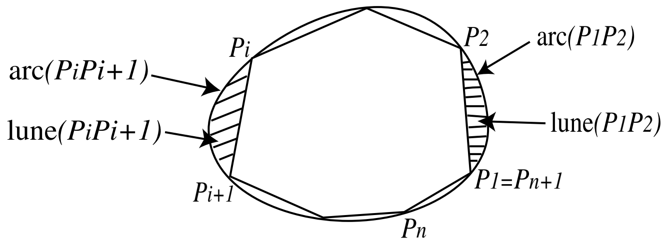

Fig. 6. $\text{lune}(P_1P_2), \cdots, \text{lune}(P_nP_1)$.

Proof Suppose that the lemma does not hold. By a new suitable labeling of $\{P_i\}$, we may assume that there exist n points $P_1, P_2, \ldots, P_n$ on $\partial(S)$ such that $\ell(\text{arc}(P_iP_{i+1})) = \alpha_i\ \ell(\partial(S))$ for all $1 \leq i \leq n$, $\text{area}(\text{lune}(P_1P_2)) \geq \alpha_1 \text{area}(S)$ and $\text{area}(\text{lune}(P_rP_{r+1})) \geq \alpha_r \text{area}(S)$ for some $r,\ 2 \leq r \leq n$. We first consider the case that $3 \leq r \leq n-1$.

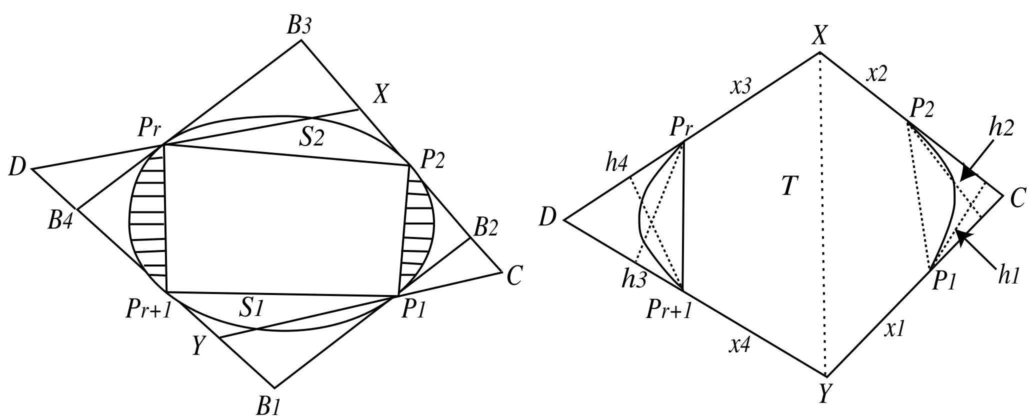

Fig. 7. $\text{quad}(B_1B_2B_3B_4)$ and the convex set T.

Since S is a convex set, we get a quadrilateral $\text{quad}(B_1B_2B_3B_4)$ whose edges $B_1B_2, B_2B_3,\ B_3B_4,\ B_4B_1$ are tangent to S at P_1, P_2, P_r and P_{r+1}, respectively, or no such a quadrilateral exists (see Figures 7 and 8). We first assume that such a quadrilateral exists (see Figures 7). Consider the triangle $\triangle P_1P_{r+1}B_1$ and the convex subset $S_1 = S \cap \triangle P_1P_{r+1}B_1$. Without loss generality, we may assume that $\angle P_1 \leq \angle P_{r+1}$ since otherwise we can apply the same argument to B_1P_1 instead of B_1P_{r+1}. Let Y be the point on B_1P_{r+1} such that $|P_1Y| +$

$|YP_{r+1}| = \ell(\text{arc}(P_1P_{r+1}))$. Then by lemma 6, we have

$$\text{area}(\triangle YP_1P_{r+1}) \leq \text{area}(S_1).$$

By the same argument as above, for the convex subset $S_2 = S \cap \triangle P_2B_3P_r$ and for the point X on P_2B_3 (or B_3P_r) with $|P_2X| + |XP_r| = \ell(\text{arc}(P_2P_r))$, we have area $\triangle\, XP_rP_2 \leq \text{area}(S_2)$.

Let

$$T := (S - (S_1 \cup S_2) \;\cup\; \triangle\, YP_1P_{r+1} \;\cup\; \triangle\, XP_rP_2 \quad (Fig.\ 7).$$

Then T is a convex with $\ell(\partial(T)) = \ell(\partial(S))$ and $\text{area}(T) \leq \text{area}(S)$. Let $\text{quad}(CXDY)$ be the quadrilateral whose edges contain segments $P_2X, XP_r, P_{r+1}Y$ and YP_1 (see Figure 7). Then it follows immediately that the four points P_1, P_2, P_r, P_{r+1} on the boundary T satisfy the following equalities and inequalities:

$$\begin{aligned}
&\ell(\text{arc}(P_1P_2)) = \alpha_1\ \ell(\partial(T)), \quad \ell(\text{arc}(P_rP_{r+1})) = \alpha_r\ \ell(\partial(T)),\\
&\text{area}(\text{lune}(P_1P_2)) \geq \alpha_1\ \text{area}(T), \quad \text{area}(\text{lune}(P_rP_{r+1})) \geq \alpha_r\ \text{area}(T).
\end{aligned}$$

Put $\ell^* = \ell(\partial(T))$, $x_1 = |YP_1|, x_2 = |XP_2|, x_3 = |XP_r|, x_4 = |YP_{r+1}|$ and $a = \ell(\text{arc}(P_1P_2)) = \alpha_1\ell^*$, $b = \ell(\text{arc}(P_rP_{r+1})) = \alpha_r\ell^*$. Let h_1 and h_2 the heights of $\triangle CP_2P_1$ with bases CP_2 and CP_1, respectively, and h_3 and h_4 the heights of $\triangle DP_{r+1}P_r$ with bases DP_{r+1} and DP_r, respectively (see Figure 7). Then we obtain the following inequalities by lemma 7.

$$\begin{aligned}
&\text{area}(\text{lune}(P_1P_2)) < \frac{1}{2}ah_1, \quad \text{area}(\text{lune}(P_1P_2)) < \frac{1}{2}ah_2 \qquad (2)\\
&\text{area}(\text{lune}(P_rP_{r+1})) < \frac{1}{2}bh_3, \quad \text{area}(\text{lune}(P_rP_{r+1})) < \frac{1}{2}bh_4\\
&\text{area}(\text{quad}(XYP_1P_2)) \geq \text{area}(\triangle XP_1P_2) + \text{area}(\triangle YP_1P_2)\\
&= \frac{1}{2}(x_1h_2 + x_2h_1),\\
&\text{area}(\text{quad}(XP_rP_{r+1}Y)) \geq \text{area}(\triangle XP_rP_{r+1}) + \text{area}(\triangle YP_rP_{r+1})\\
&= \frac{1}{2}(x_3h_4 + x_4h_3).
\end{aligned}$$

By the symmetry of x_i and h_i, we may assume that h_1 is the smallest of all the h_i. Then

$$\text{area}(\text{hex}(XP_rP_{r+1}YP_1P_2)) \geq \frac{1}{2}(x_1 + x_2 + x_3 + x_4)h_1$$

$$= \frac{1}{2}(1-\alpha_1-\alpha_r)\ell^* h_1,$$
$$\text{area}(\text{hex}(XP_rP_{r+1}YP_1P_2))$$
$$= \text{area}(T) - \text{area}(\text{lune}(P_1P_2)) - \text{area}(\text{lune}(P_rP_{r+1}))$$
$$\leq (1-\alpha_1-\alpha_r)\text{area}(S).$$

Hence $\ell^* h_1/2 \leq \text{area}(S)$, and thus

$$\begin{aligned} \text{area}(\text{lune}(P_1P_2)) &\geq \alpha_1 \ \text{area}(S) \qquad \text{(by the choice of } P_1P_2) \\ &\geq \frac{\alpha_1 \ell^* h_1}{2} \end{aligned}$$

This contradicts the fact that

$$\text{area}(\text{lune}(P_1P_2)) < \frac{1}{2} a h_1 = \frac{\alpha_1 \ell^* h_1}{2}. \qquad \text{(by (2))}$$

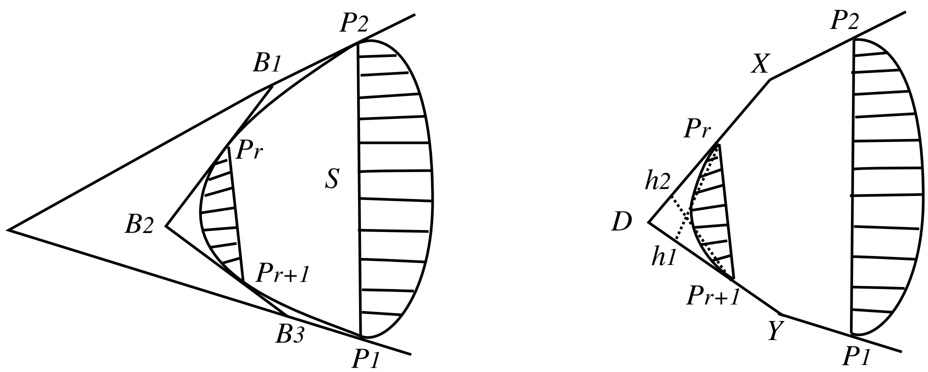

Fig. 8. The convex set S and hex$(P_2XP_rP_{r+1}YP_1)$.

We next assume that the quadrilateral quad$(B_1B_2B_3B_4)$ does not exist (see Figure 8). Let B_1, B_2, B_3 be the intersections of two tangent lines of S at P_2 and P_r, at P_r and P_{r+1}, and at P_1 and P_{r+1}, respectively (see Figure 8). We take two points X and Y on $P_2B_1 \cup B_1P_r$ and $P_1B_3 \cup B_3P_{r+1}$, respectively, which satisfy the conditions of lemma 6. Let D be the intersection of the two lines containing XP_r and YP_{r+1}, respectively. Let h_1 and h_2 be the heights of $\triangle P_rDP_{r+1}$ with bases BP_{r+1} and BP_r, respectively. Then by lemma 6, we have

$$\text{area}(\text{lune}(P_rP_{r+1})) < \frac{1}{2} h_1 \ \ell(\text{arc}(P_rP_{r+1}))$$
$$\text{area}(\text{lune}(P_rP_{r+1})) < \frac{1}{2} h_2 \ \ell(\text{arc}(P_rP_{r+1}))$$
$$|P_2X| + |XP_r| = \ell(\text{arc}(P_2P_r)), \quad \text{area}(\triangle P_2XP_r) \leq \text{area}(\text{lune}(P_2P_r))$$
$$|P_1Y| + |YP_{r+1}| = \ell(\text{arc}(P_1P_{r+1})),$$
$$\text{area}(\triangle P_1YP_{r+1}) \leq \text{area}(\text{lune}(P_1P_{r+1})).$$

Put $\ell^* = \ell(\partial(S))$. By the symmetry of h_1 and h_2, we may assume that $h_1 \le h_2$. Then we obtain

$$\begin{aligned}
&(1-\alpha_1-\alpha_r)\text{area}(S) \ge \text{area}(\text{hex}(P_2XP_rP_{r+1}YP_1)) \\
&> \text{area}(\triangle P_2XP_{r+1}) + \text{area}(\triangle XP_rP_{r+1}) \\
&\qquad +\text{area}(\triangle P_1YP_r) + \text{area}(\triangle YP_rP_{r+1}) \\
&\ge \frac{1}{2}|P_2X|h_2 + \frac{1}{2}|XP_r|h_2 + \frac{1}{2}|P_1Y|h_1 + \frac{1}{2}|YP_{r+1}|h_1 \\
&= \frac{1}{2}\ell(\text{arc}(P_2P_r))h_2 + \frac{1}{2}\ell(\text{arc}(P_1P_{r+1}))h_1 \ge \frac{1}{2}h_1(1-\alpha_1-\alpha_r)\ell^*.
\end{aligned}$$

Therefore $\text{area}(S) > \frac{1}{2}h_1\ell^*$. Then it follows from lemma 6 that

$$\text{area}(\text{lune}(P_rP_{r+1})) < \frac{1}{2}h_1\ \ell(\text{arc}(P_rP_{r+1})) = \frac{1}{2}h_1\alpha_r\ell^* < \alpha_r\text{area}(S).$$

This contradicts the assumption that $\text{area}(\text{lune}(P_rP_{r+1})) \ge \alpha_r\text{area}(S)$.

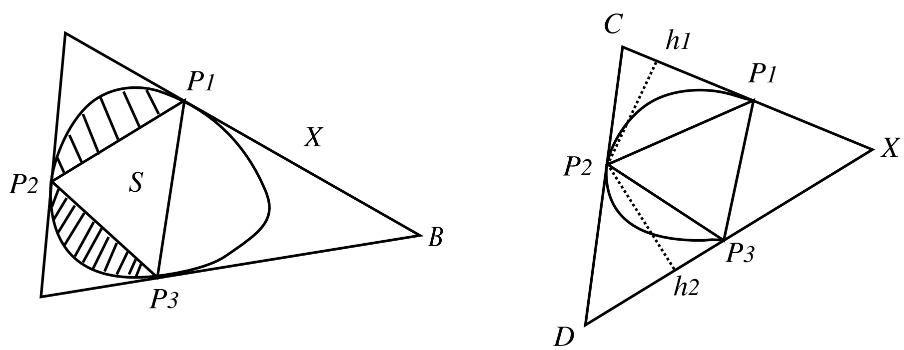

Fig. 9. The convex set S and $\text{hex}(P_2XP_rP_{r+1}YP_1)$.

We next consider the case that $r = 2$. Let B be the intersection of two tangent lines of S at P_1 and P_3 (see Figure 9). Fist suppose that P_2 and B lie on the opposite sides of the line passing through P_1 and P_3 (see Figure 9). By Lemma 6, we take a point X on $P_3B \cup BP_1$, and obtain the following equalities and inequalities. where

$$\begin{aligned}
&\ell(\text{arc}(P_1P_2)) = \alpha_1\ \ell(\partial(S)), \quad \ell(\text{arc}(P_2P_3)) = \alpha_2\ \ell(\partial(S)), \\
&\ell(\text{arc}(P_3P_1)) = (1-\alpha_1-\alpha_2)\ell(\partial(S)), \\
&\text{area}(\text{lune}(P_1P_2)) \ge \alpha_1\text{area}(S), \quad \text{area}(\text{lune}(P_2P_3)) \ge \alpha_2\text{area}(S), \\
&\ell(\text{arc}(P_3P_1)) = |P_3X| + |XP_1|, \quad \text{area}(\text{lune}(P_3P_1)) \ge \text{area}(\triangle P_3XP_1).
\end{aligned}$$

In this case, we can similarly derive a contradiction as above from the following inequalities.

$$\text{area}(\text{lune}(P_1P_2)) < \frac{1}{2}h_1\ \ell(\text{arc}(P_1P_2)),$$

$$\text{area}(\text{lune}(P_2P_3)) < \frac{1}{2}h_2\ \ell(\text{arc}(P_2P_3))$$

$$\text{area}(\triangle P_1P_2X) = \frac{1}{2}h_1|P_1X|, \quad \text{area}(\triangle P_2P_3X) = \frac{1}{2}h_2|P_2X|.$$

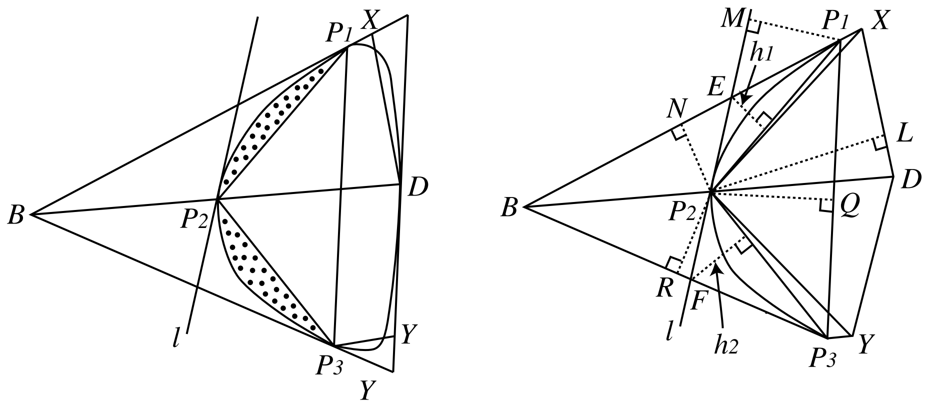

Fig. 10. The convex set S and hex$(DXP_1P_2P_3Y)$.

Nexst assume that P_2 and B lie on the same side of the line passing through P_1 and P_3. Then we obtain Figure 10, where $\ell(\text{arc}(P_1P_2)) = \alpha_1\ \ell(\partial(S))$, $\ell(\text{arc}(P_2P_3)) = \alpha_2\ \ell(\partial(S))$, $\ell(\text{arc}(P_3D)\cup\text{arc}(DP_1)) = (1-\alpha_1-\alpha_2)\ell(\partial(S))$, $\text{area}(\text{lune}(P_1P_2)) \geq \alpha_1\text{area}(S)$, $\text{area}(\text{lune}(P_2P_3)) \geq \alpha_2\text{area}(S)$.

By lemma 6, we can take two points X and Y on $DE \cup EP_1$ and $DF \cup FP_3$, respectively, so that $|DX| + |XP_1| = \ell(\text{arc}(DP_1))$, $\text{area}(\triangle DXP_1) \leq \text{area}(\text{lune}(DP_1))$, $|DY| + |YP_3| = \ell(\text{arc}(DP_3))$ and $\text{area}(\triangle DYP_3) \leq \text{area}(\text{lune}(DP_3))$.

Let l denote the tangent line of S at P_2. Without loss of generality, we may assume that L and the line passing through P_1 and P_2 are parallel or intersects above the tangent line of S at P_1. Then

$$\text{area}(\text{lune}(P_1P_2)) < \text{area}\triangle\ P_1EP_2 \leq \frac{1}{2}h_1\ \ell(\text{arc}(P_1P_2)),$$

$$\text{area}(\text{lune}(P_2P_3)) < \text{area}\triangle\ P_2FP_3\frac{1}{2}h_2\ \ell(\text{arc}(P_2P_3))$$

$$\text{area}(\triangle XP_1P_2) = \frac{1}{2}|P_2N||P_1X| \geq \frac{1}{2}h_1|P_1X|$$

$$\text{area}(\triangle P_2P_3Y) \geq \frac{1}{2}|P_2R||P_3Y| \geq \frac{1}{2}h_2|P_3Y|$$

$$\text{area}(\triangle XP_2D) \geq \frac{1}{2}|P_2Q||XD| \geq \frac{1}{2}|MP_1||XD| \geq \frac{1}{2}h_1|XD|$$

$$\text{area}(\triangle DP_2Y) \geq \frac{1}{2}|P_2Q||XD| \geq \frac{1}{2}|MP_1||XD| \geq \frac{1}{2}h_1|XD|$$

Hence we obtain

$$\text{area}(\text{hex}(XP_1P_2P_3YD)) \geq \frac{1}{2}h_1(|XD| + XP_1| + |YD|) + \frac{1}{2}h_2|P_3Y|.$$

Let $\ell^* = \ell(\partial(S))$. If $h_1 \leq h_2$, then

$$(1-\alpha_1-\alpha_r)\text{area}(S) \geq \text{area}(\text{hex}(XP_1P_2P_3YD)) > \frac{1}{2}h_1(|XD| + XP_1| + |YD| + |P_3Y|)$$
$$\geq \frac{1}{2}h_1(1-\alpha_1-\alpha_2)\ell^*$$

Hence $\text{area}(S) \geq (1/2)h_1\ell^*$, and thus we get the following contradiction.

$$\frac{\alpha_1\ell^*h_1}{\leq}\alpha_1\text{area}(S) < \text{area}(\text{lune}(P_1P_2) \leq \frac{1}{2}\ell(\text{arc}(P_1P_2))h_1 = \frac{1}{2}\alpha_1\ell^*$$

If $h_2 \leq h_1$, then

$$(1-\alpha_1-\alpha_2)\text{area}(S) \geq \text{area}(\text{hex}(XP_1P_2P_3YD)) > \frac{1}{2}h_2(|XD| + XP_1| + |YD| + |P_3Y|)$$
$$\geq \frac{1}{2}h_2(1-\alpha_1-\alpha_2)\ell^*$$

Hence $\text{area}(S) \geq (1/2)h_2\ell^*$, and thus we get the following contradiction.

$$\frac{\alpha_1\ell^*h_2}{\leq}\alpha_1\text{area}(S) < \text{area}(\text{lune}(P_2P_3) \leq \frac{1}{2}\ell(\text{arc}(P_2P_3))h_2 = \frac{1}{2}\alpha_2\ell^*$$

Consequently we say that the lemma is proved. □

Lemma 9 *Let S, n and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the same as Theorem 1. Then there exist n points $P_1, P_2, \ldots, P_n$ on $\partial(S)$ such that for every $1 \leq i \leq n$, it follows that $\ell(\text{arc}(P_iP_{i+1})) = \alpha_i \times \ell(\partial(S))$ and $\text{area}(\text{lune}(P_iP_{i+1})) \leq \alpha_i$ area(S), where $P_{n+1} = P_1$.*

Proof By lemma 8 and by a new labeling of $\{P_i\}$, we may assume that there exist n points $Q_1, \ldots, Q_n$ on $\partial(S)$ such that $\ell(\text{arc}(Q_iQ_{i+1})) = \alpha_i\ \ell(\partial(S))$ for all $1 \leq i \leq n$, $\text{area}(\text{lune}(Q_1Q_2)) > \alpha_1$ area(S) and $\text{area}(\text{lune}(Q_jQ_{j+1})) \leq \alpha_j$ area(S) for all $2 \leq j \leq n$. If there exist n points $R_1, \ldots, R_n$ for which $\ell(\text{arc}(R_iR_{i+1})) = \alpha_i\ell(\partial(S))$ for all $1 \leq i \leq n$ and $\text{area}(\text{lune}(R_1R_2)) \leq \alpha_1$ area(S), then by lemma 8, when we continuously move $\{Q_i\}$ to $\{R_i\}$, we obtain the desired n points $\{P_i\}$, which satisfy the conditions of the lemma. So it is sufficient to show the existence such n points $R_1, \ldots, R_n$. Moreover, if there exists two points Y_1 and Y_2 on $\partial(S)$ for which $\ell(\mathrm{arc}(Y_1Y_2)) = \alpha_1\ \ell(\partial(S))$ and $\mathrm{area}(\mathrm{lune}(Y_1Y_2)) \leq \alpha_1 \mathrm{area}(S)$, then by suitably adding $n-2$ points on $\partial(S) - \mathrm{arc}(Y_1Y_2)$, we can obtain the desired n points $R_1, \ldots, R_n$.

If $\alpha_1 < \frac{1}{2}$, then the existence of n points $\{R_i\}$ mentioned above is easily shown by applying lemma 8 to the three points X_1, X_2, X_3 on $\partial(S)$ such that $\ell(X_1X_2) = \ell(X_2X_3) = \alpha_1 \ell(\partial(S))$ and $\ell(X_3X_1) = (1-2\alpha_1)\ell(\partial(S))$. If $\alpha_1 = \frac{1}{2}$, then we divide S into two half convex subsets with the same arc length, and then one of them satisfies the required condition. □

Proof of Theorem 2 By lemma 9, there exists n points $\{P_i\}$ on $\partial(S)$ such that $\ell(\mathrm{arc}(P_iP_{i+1})) = \alpha_i\ \ell(\partial(S))$ and $\mathrm{area}(\mathrm{lune}(P_iP_{i+1})) \leq \alpha_i \mathrm{area}(S)$ for all $1 \leq i \leq n$. Let P^* be the polygon with vertex set $\{P_1, P_2, \ldots, P_n\}$, and let $\{e_1, \ldots, e_m\}$ and $\{\beta_1, \ldots, \beta_m\}$ be the sets of edges $e_k = P_jP_{j+1}$ and positive real numbers

$$\beta_k = \alpha_j\ \mathrm{area}(S) - \mathrm{area}(\mathrm{lune}(P_jP_{j+1}))$$

for which $\mathrm{area}(\mathrm{lune}(P_jP_{j+1})) < \alpha_j \mathrm{area}(S)$. Then by Theorem 5, P^* can be partitioned into m convex subsets $\{Q_k\}$ with area $\{\beta_k\}$. Then it is clear that $\mathrm{lune}(P_jP_{j+1}) \cup Q_k$ is a convex subset which has area $\alpha_j \mathrm{area}(S)$ and has one continuous part of $\partial(S)$ with length $\ell(\mathrm{arc}(P_jP_{j+1})) = \alpha_i\ \ell(\partial(S))$. Consequently the theorem is proved. □

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