DRAFT

Partial parity (g, f)-factors and subgraphs covering given vertex subsets

M. Kano, Haruhide Matsuda*
Department of Computer and Information Sciences,
Ibaraki University,
Hitachi, 316-8511 Japan

Abstract

Let G be a graph and W a subset of $V(G)$. Let $g, f : V(G) \to \mathbf{Z}$ be two integer-valued functions such that $g(x) \leq f(x)$ for all $x \in V(G)$ and $g(y) \equiv f(y) \pmod 2$ for all $y \in W$. Then a spanning subgraph F of G is called a partial parity (g, f)-factor with respect to W if $g(x) \leq \deg_F(x) \leq f(x)$ for all $x \in V(G)$ and $\deg_F(y) \equiv f(y) \pmod 2$ for all $y \in W$. We obtain a criterion for a graph G to have a partial parity (g, f)-factor with respect to W. Furthermore, by making use of this criterion, we give some necessary and sufficient conditions for a graph G to have a subgraph which covers W and has a certain given property.

1 Partial parity (g, f)-factors

We consider a finite graph G which may have multiple edges and loops, and so a graph means such a graph throughout this paper. Let $V(G)$ and $E(G)$ denote the set of vertices and that of edges of G, respectively. For two disjoint subsets S and T of $V(G)$, we write $e_G(S, T)$ for the number of edges of G joining S to T. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G, and by $N_G(v)$ the neighborhood of v. Let $\mathbf{Z}$ and $\mathbf{Z}^+$ denote the set of integers and that of non-negative integers, respectively. For a function $f : V(G) \to \mathbf{Z}^+$, a spanning subgraph F of G is called an *f-factor* if $\deg_F(x) = f(x)$ for all $x \in V(G)$. For two functions $g,\ f : V(G) \to \mathbf{Z}$ such that $g(x) \leq f(x)$ for all $x \in V(G)$, a spanning subgraph H of G is called a *(g, f)-factor* if $g(x) \leq \deg_H(x) \leq f(x)$ for all $x \in V(G)$. Note that when we consider (g, f)-factors, we allow that $g(x) < 0$ and/or $\deg_G(y) < f(y)$ for some vertices x and y of G, and this relaxation will play an important technical role.

*Present address: Department of Mathematics, Kyusyu Tokai University, Choyo, Aso, Kumamoto, 865-1404 Japan

For a given subset W of $V(G)$, let $g,\ f : V(G) \to \mathbf{Z}$ be two functions such that $g(x) \leq f(x)$ for all $x \in V(G)$ and $g(y) \equiv f(y) \pmod 2$ for all $y \in W$. Then a spanning subgraph F of G is called a *partial parity* (g, f)*-factor* with respect to W if

$$\begin{aligned} & g(x) \leq \deg_F(x) \leq f(x) \quad \text{for all } \ x \in V(G) \ \text{ and} \\ & \deg_F(y) \equiv g(y) \equiv f(y) \pmod 2 \quad \text{for all } \ y \in W. \end{aligned}$$

Note that if $W = \emptyset$, then a partial parity (g, f)-factor is a (g, f)-factor and, if $W = V(G)$, then a partial parity (g, f)-factor is briefly called a *parity* (g, f)*-factor*. The criterions for a graph to have an f-factor, a (g, f)-factor and a parity (g, f)-factor were obtained by Tutte [9], Lovász [4] and [6], respectively. Moreover, Niessen [7] recently gave a criterion for a graph G to have all (g, f)-factors, where we say that G has all (g, f)-factors if G has an h-factor for every $h : V(G) \to \mathbf{Z}^+$ such that $g(x) \leq h(x) \leq f(x)$ for all $x \in V(G)$ and $\sum_{x \in V(G)} h(x) \equiv 0 \pmod 2$, and if at least one such h exists.

We first give a necessary and sufficient condition for a graph to have a partial parity (g, f)-factor.

Theorem 1 *Let G be a graph and W a subset of $V(G)$. Let $g,\ f : V(G) \to \mathbf{Z}$ be two functions satisfying*

$$g(x) \leq f(x) \quad \textit{for all } \ x \in V(G), \quad \textit{and} \quad g(y) \equiv f(y) \pmod 2 \quad \textit{for all } \ y \in W.$$

Then G has a partial parity (g, f)-factor with respect to W if and only if for all disjoint subsets S and T of $V(G)$,

$$\eta_G(S, T) = \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_G(x) - g(x)) - e_G(S, T) - k_W(S, T) \geq 0, \tag{1}$$

where $k_W(S, T)$ denotes the number of components D of $G - (S \cup T)$ such that

$$g(x) = f(x) \ \textit{ for all } \ x \in V(D) \setminus W \quad \textit{and} \quad \sum_{x \in V(D)} f(x) + e_G(V(D), T) \equiv 1 \pmod 2. \tag{2}$$

In order to prove Theorem 1, we need the following (g, f)-factor theorem.

Theorem 2 (Lovász [4]) *Let G be a graph and $g, f : V(G) \to \mathbf{Z}$. Then G has a (g, f)-factor if and only if for all disjoint subsets S and T of $V(G)$,*

$$\delta_G(S, T) = \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_G(x) - g(x)) - e_G(S, T) - h_G(S, T) \geq 0, \tag{3}$$

where $h_G(S, T)$ denotes the number of components D of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(D)$ and

$$\sum_{x \in V(D)} f(x) + e_G(V(D), T) \equiv 1 \pmod 2. \tag{4}$$

We call a component of $G-(S\cup T)$ satisfying (4) a (g,f)-*odd component*.

Proof of Theorem 1 Let G' be the graph obtained from G by adding $(f(x)-g(x))/2$ loops to each vertex x in W, and define a function g' on $V(G')=V(G)$ by $g'(x)=f(x)$ for all $x\in W$ and $g'(x)=g(x)$ for all $x\in V(G)\setminus W$. Then it is easy to see that G has a partial parity (g,f)-factor with respect to W if and only if G' has a (g',f)-factor.

By Theorem 2, G' has a (g',f)-factor if and only if

$$\sum_{x\in S} f(x)+\sum_{x\in T}(\deg_{G'}(x)-g'(x))-e_G(S,T)-h_G(S,T)\geq 0, \tag{5}$$

for all disjoint subsets S and T of $V(G)$. It follows that $\deg_{G'}(x)-g'(x)=\deg_G(x)-g(x)$ for all $x\in V(G)$, and a component D of $G'-(S\cup T)$ is a (g',f)-odd component if and only if $g'(x)=f(x)$ for all $x\in V(D)$ and $\sum_{x\in V(D)} f(x)+e_{G'}(V(D),T)\equiv 1 \pmod 2$. However $g'(x)=f(x)$ for all $x\in V(D)$ if and only if $g(x)=f(x)$ for all $x\in V(D)\setminus W$, and $e_{G'}(V(D),T)=e_G(V(D),T)$. Hence D is a (g,f)-odd component of $G'-(S\cup T)$ if and only if D is a component of $G-(S\cup T)$ satisfying (2). Consequently the theorem is proved. $\square$

2 Subgraphs covering given vertex subsets

If a subgraph H of a graph G contains all the vertices of a given subset W of $V(G)$, then we briefly say that H *covers* W. On the other hand, if the vertex set of a subgraph K is contained in W, then we briefly say that K is *covered by* W.

We define a *cycle* as a connected subgraph with all degree two. In particular, a loop and two multiple edges joining the same two vertices are cycles. A *matching* is a subgraph with all degree one. Let $f: V(G)\to\{1,3,5,\ldots\}$. Then a subgraph H of G is called a $(1,f)$-*odd subgraph* if $\deg_H(x)\in\{1,3,5,\ldots,f(x)\}$ for all $x\in V(H)$, and a spanning $(1,f)$-odd subgraph is called a $(1,f)$-*odd factor*. A component of odd order is called an *odd component*, and the number of odd components of a graph G is denoted by $o(G)$.

Lovász proved the following theorem, which is an extension of Tutte's 1-factor theorem [8].

Theorem 3 (Lovász [5]) *Let G be a graph and W a subset of $V(G)$. Then G has a matching which covers W if and only if*

$$o(G-S|W)\leq |S| \quad \text{for all } S\subseteq V(G),$$

where $o(G-S|W)$ denotes the number of those odd components of $G-S$ whose vertices are all contained in W.

We first give an extension of the next theorem in Theorem 5.

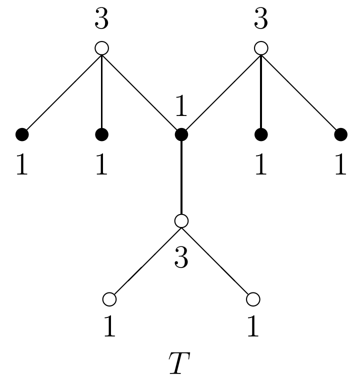

Figure 1: The tree T, where the numbers denote $f(v)$.

Theorem 4 (Cui and Kano [1]) *Let G be a graph and $f : V(G) \to \{1, 3, 5, \ldots\}$. Then G has a $(1, f)$-odd factor if and only if*

$$o(G - S) \leq \sum_{x \in S} f(x) \quad \textit{for all} \;\; S \subseteq V(G).$$

Before giving our extension, we remark that Theorem 4 cannot be directly generalized as 1-factor theorem. Consider the tree T in Figure 1, in which the numbers denote the values of f. Let W be the set of filled vertices. Then $o(T - S|W) \leq \sum_{x \in S} f(x)$ holds for all $S \subseteq V(T)$. However, T has no $(1, f)$-odd subgraph which covers W. Nevertheless, the condition characterizes another property as the following theorem shows.

Theorem 5 *Let G be a graph, W a subset of $V(G)$, and $f : V(G) \to \mathbf{Z}^+$ such that $f(y)$ is an odd integer for all $y \in W$. Then G has a subgraph H covering W such that*

$$1 \leq \deg_H(x) \leq f(x) \;\; \textit{for all} \;\; x \in V(H), \quad \textit{and} \quad \deg_H(y) \equiv 1 \pmod 2 \;\; \textit{for all} \;\; y \in W$$

if and only if

$$o(G - S|W) \leq \sum_{x \in S} f(x) \qquad \textit{for all} \;\; S \subseteq V(G), \tag{6}$$

where $o(G - S|W)$ denotes the number of those odd components of $G - S$ whose vertices are all contained in W.

Proof We first prove the necessity. Suppose that G has a subgraph H given in the theorem. Then for every odd component D of $G - S$ covered by W, at least one edge of H joins D to S. Thus we obtain

$$o(G - S|W) \leq \sum_{x \in S} \deg_H(x) \leq \sum_{x \in S} f(x).$$

We next prove the sufficiency. Let N be an odd integer greater than $\sum_{x \in V(G)} \deg_G(x)$, which is a sufficiently large integer. Define a function $g : V(G) \to \mathbf{Z}$ by $g(x) = -N$

for all $x \in V(G)$. Then a partial parity (g,f)-factor with respect to W is the desired subgraph of the theorem. So it suffices to show that G, g and f satisfy the condition (1) in Theorem 1.

Let S and T be two disjoint subsets of $V(G)$. If $T \neq \emptyset$, then $\eta_G(S,T) \geq 0$ as $\deg_G(x) - g(x) = \deg_G(x) + N$ for every $x \in T$. Thus we may assume that $T = \emptyset$. It is immediate that a component D of $G - S$ satisfying (2) is covered by W and has odd order. Therefore, it follows from (6) that

$$\eta_G(S,\emptyset) \;=\; \sum_{x\in S} f(x) - k_W(S,\emptyset) = \sum_{x\in S} f(x) - o(G-S|W) \geq 0.$$

Consequently the theorem is proved. □

Theorem 6 *Let G be a graph and $W \subseteq V(G)$ with $|W|$ even. Then G has a set of vertex disjoint paths such that W is the set of their end-vertices if and only if*

$$o_W(G-S) \leq |S\cap W| + 2|S\setminus W| \quad \textit{for all } \; S \subseteq V(G), \tag{7}$$

where $o_W(G-S)$ denotes the number of components D of $G-S$ such that $|V(D)\cap W|$ is odd.

Proof Let H be a subgraph of G that satisfies

$$\deg_H(x) = 1 \;\text{ for all }\; x\in W, \;\text{and}\quad \deg_H(y) = 2 \;\text{ for all }\; y \in V(H)\setminus W. \tag{8}$$

Then it is obvious that a subgraph H with minimal vertex set satisfying (8) consists of vertex disjoint paths whose end-vertices coincide with W. Thus G has the desired set of paths if and only if G has a subgraph H satisfying (8).

Suppose that G has such a subgraph H. For each component D of $G-S$ such that $|V(D)\cap W|$ is odd, at least one edge of H joins D to S. Thus we obtain

$$o_W(G-S) \leq \sum_{x\in S} \deg_H(x) = |S\cap W| + 2|S\setminus W|.$$

We now prove the sufficiency. We use Theorem 1 with $W = V(G)$, that is, we apply the parity (g,f)-factor theorem to the graph G given in the theorem.

Let N be a sufficiently large integer, and define two functions $g, f : V(G) \to \mathbf{Z}$ by

$$g(x) = \begin{cases} -2N-1 & \text{if } x\in W,\\ -2N & \text{otherwise;}\end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 1 & \text{if } x\in W,\\ 2 & \text{otherwise.}\end{cases}$$

Then if G has a parity (g,f)-factor F, which may have degree 0 for some vertices of $V(G)\setminus W$ but must have degree 1 for every vertex of W, then the subgraph of G induced by the edge set $E(F)$ is the desired subgraph H. Hence it suffices to show that G, g and f satisfies (1).

By the assumption that $o_W(G) = 0$, any component of G contains even number of vertices in W. Thus it follows that $\eta_G(\emptyset,\emptyset) = -k_{V(G)}(\emptyset,\emptyset) = 0$. Let S and T be disjoint subsets of $V(G)$ such that $S \cup T \neq \emptyset$. If $T \neq \emptyset$, then since $-g(x) \geq 2N$ is sufficiently large, we have

$$\eta_G(S,T) = \sum_{x\in S} f(x) + \sum_{x\in T}(\deg_G(x) - g(x)) - e_G(S,T) - k_{V(G)}(S,T) \geq 0.$$

Hence we may assume that $T = \emptyset$. Then we obtain

$$\eta_G(S,\emptyset) = \sum_{x\in S} f(x) - k_{V(G)}(S,\emptyset) = 2|S \setminus W| + |S \cap W| - o_W(G-S) \geq 0,$$

as desired. □

Theorem 7 *Let G be a graph, W a set of vertices of G. Then G has a set of vertex disjoint cycles that cover W if and only if for all disjoint subsets $S \subseteq V(G)$ and $T \subseteq W$, it follows that*

$$2|S| + \sum_{x\in T}(\deg_G(x) - 2) - e_G(S,T) - k^*(S,T) \geq 0, \tag{9}$$

where $k^*(S,T)$ denotes the number of component D of $G-(S\cup T)$ such that $V(D) \subseteq W$ and $e_G(T,V(D)) \equiv 1 \pmod 2$.

Proof Let N be a sufficiently large number, and define two functions $g, f : V(G) \to \mathbf{Z}$ by

$$g(x) = \begin{cases} 2 & \text{if } x \in W, \\ -2N & \text{otherwise;} \end{cases} \qquad \text{and} \qquad f(x) = 2 \ \text{ for all } \ x \in V(G).$$

Then G has a parity (g,f)-factor, which may have degree 0 for some vertices of $V(G)\setminus W$, if and only if G has the desired set of disjoint cycles. Thus it suffices to show that (1) and (9) are equivalent.

Suppose that (1) holds. Let $S \subseteq V(G)$ and $T \subseteq W$. Then by (1), we have

$$\eta_G(S,T) = 2|S| + \sum_{x\in T}(\deg_G(x) - 2) - e_G(S,T) - k_{V(G)}(S,T) \geq 0.$$

Moreover, every component D of $G-(S\cup T)$ satisfying (2) is covered by W since $g(x) < f(x)$ for all $x \in V(G)\setminus W$, and satisfies $e_G(T,V(D)) \equiv 1 \pmod 2$. Thus $k_{V(G)}(S,T) = k^*(S,T)$, and hence (9) holds.

Conversely, assume that (9) holds. It follows that $\eta_G(\emptyset,\emptyset) = -k_{V(G)}(\emptyset,\emptyset) = 0$ since $f(x) = 2$ for all $x \in V(G)$. Let S and T be disjoint subsets of $V(G)$ such that $S \cup T \neq \emptyset$. If $T \setminus W \neq \emptyset$, then since $-g(x) = 2N$ is sufficiently large for $x \in T \setminus W$, we have $\eta_G(S,T) \geq 0$. Thus we may assume that $T \subseteq W$. It follows that $k_{V(G)}(S,T) = k^*(S,T)$ as we showed above. Hence by (9), we obtain

$$\eta_G(S,T) = 2|S| + \sum_{x\in T}(\deg_G(x) - 2) - e_G(S,T) - k_{V(G)}(S,T) \geq 0.$$

Therefore (1) holds. Consequently, the theorem is proved. □

Theorem 8 *Let G be a graph and W a set of vertices of G. Then G has a set of vertex disjoint cycles and paths such that it covers W and only the end-vertices of the paths are contained in $V(G) \setminus W$ if and only if for all disjoint subsets $S \subseteq V(G)$ and $T \subseteq W$, it follows that*

$$|S \setminus W| + 2|S \cap W| + \sum_{x \in T}(\deg_G(x) - 2) - e_G(S,T) - k^*(S,T) \geq 0, \tag{10}$$

where $k^*(S,T)$ denotes the number of component D of $G - (S \cup T)$ such that $V(D) \subseteq W$ and $e_G(T, V(D)) \equiv 1 \pmod 2$.

Proof Let N be a sufficiently large integer, and define two functions $g, f : V(G) \to \mathbf{Z}$ by

$$g(x) = \begin{cases} 2 & \text{if } x \in W, \\ -N & \text{otherwise;} \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 2 & \text{if } x \in W, \\ 1 & \text{otherwise.} \end{cases}$$

Then G has a (g,f)-factor, which may have degree 0 for some vertices of $V(G) \setminus W$, if and only if G has the desired set of cycles and paths. Hence it suffices to show that (10) and (3) are equivalent. It follows that $\delta_G(\emptyset, \emptyset) = -h_G(\emptyset, \emptyset) = 0$ since $g(x) < f(x)$ for all $x \in V(G) \setminus W$ and $f(y) = 2$ for all $y \in W$. Let S and T be disjoint subsets of $V(G)$ such that $S \cup T \neq \emptyset$. If $T \setminus W \neq \emptyset$, then $\delta_G(S,T) \geq 0$ since $-g(x) = N$ is sufficiently large for $x \in T \setminus W$. Thus we may assume that $T \subseteq W$. It follows immediately that $h_G(S,T) = k^*(S,T)$ and

$$\delta_G(S,T) = |S \setminus W| + 2|S \cap W| + \sum_{x \in T}(\deg_G(x) - 2) - e_G(S,T) - h_G(S,T).$$

Hence (10) and (3) are equivalent. Therefore the theorem is proved. □

Theorem 9 *Let G be a graph and W a set of vertices of G. Then G has a subgraph H such that*

$$\deg_H(x) \in \{2,4,6,\ldots\} \text{ for all } x \in W, \text{ and } \deg_H(y) = 1 \text{ for all } y \in V(G) \setminus W \tag{11}$$

if and only if for all disjoint subsets $S \subseteq V(G) \setminus W$ and $T \subseteq W$, it follows that

$$|S| + \sum_{x \in T}(\deg_G(x) - 2) - e_G(S,T) - k^*(S,T) \geq 0, \tag{12}$$

where $k^*(S,T)$ denotes the number of component D of $G - (S \cup T)$ such that $V(D) \subseteq W$ and $e_G(T, V(D)) \equiv 1 \pmod 2$.

Proof Let N be a sufficiently large integer, and define two functions $g, f : V(G) \to \mathbf{Z}$ by

$$g(x) = \begin{cases} 2 & \text{if } x \in W, \\ -N & \text{otherwise;} \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 2N & \text{if } x \in W, \\ 1 & \text{otherwise.} \end{cases}$$

Then G has a partial parity (g,f)-factor with respect to W, which may have degree 0 for some vertices of $V(G) \setminus W$, if and only if G has the desired subgraph H. Hence it suffices to show that (12) and (1) are equivalent. This can be done by a similar argument as in the previous proofs. □

For two integers a and b, a subgraph H of a graph G is called an $[a,b]$-*subgraph* if $a \le \deg_H(x) \le b$ for all $x \in V(H)$. A spanning $[a,b]$-subgraph is called an $[a,b]$-*factor*. The following theorem is an extension of the result by Las Vergnas [3].

Theorem 10 *Let G be a graph and W a subset of $V(G)$ and $n \ge 2$ an integer. Then G has a $[1,n]$-subgraph covering W if and only if*

$$i(G-S|W) \le n|S| \quad \textit{for all} \;\; S \subseteq V(G), \tag{13}$$

where $i(G-S|W)$ denotes the number of isolated vertices of $G-S$ contained in W.

Proof Suppose that G has a $[1,n]$-subgraph H which covers W. Then for every isolated vertex x of $G-S$ contained in W, H has at least one edge which joins x to S. Hence

$$i(G-S|W) \le \sum_{x\in S} \deg_H(x) \le n|S|.$$

We next prove the sufficiency. Let N be a sufficiently large integer. We define two functions f and g as follows:

$$g(x) = \begin{cases} 1 & \text{if } x \in W, \\ -N & \text{otherwise;} \end{cases} \qquad \text{and} \qquad f(x) = n \quad \text{for all } x \in V(G).$$

Then a (g,f)-factor of G is the desired subgraph. So it suffices to show that G satisfies the condition (3).

Let S and T be two disjoint subsets of $V(G)$. If $T \setminus W \ne \emptyset$, then $\delta_G(S,T) \ge 0$ since $-g(x) = N$ for every $x \in T \setminus W$. Thus we may assume that $T \subseteq W$. Note that $h(S,T) = 0$ since $g(x) < f(x)$ for all $x \in V(G)$. By (13), we obtain

$$\begin{aligned} \delta_G(S,T) &= n|S| + \sum_{x\in T}(\deg_G(x) - 1) - e_G(S,T) \\ &= n|S| + \sum_{x\in T}(\deg_{G-S}(x) - 1) \\ &\ge n|S| - i(G-S|W) \ge 0. \end{aligned}$$

Therefore the theorem is proved. □

Acknowledgments

The authors wish to thank the referees for their valuable comments and suggestions, which shortened the proof of Theorem 1 and improved some other parts of the paper.

References

[1] Y. Cui and M. Kano, Some results on odd factors of graphs, *J. Graph Theory* **12** (1988) 327-333.

[2] M. Kano and G. Y. Katona, Odd subgraphs and matchings, submitted.

[3] M. Las Vergnas, An extension of Tutte's 1-factor theorem, *Discrete Math.* **23** (1978) 241-255.

[4] L. Lovász, Subgraphs with prescribed valencies, *J. Combin. Theory.* **8** (1970), 391-416.

[5] L. Lovász, On graphs not containing independent circuits (in Hungarian), *Mat. Lapok* **16** (1965) 289 - 299.

[6] L. Lovász, The factorization of graphs II. *Acta Math. Acad. Sci. Hungar.* **23** (1972) 223-246.

[7] T. Niessen, A characterization of graphs having all (g, f)-factors, *J. Combin. Theory Ser. B* **72** (1998), 152-156.

[8] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22** (1947) 107-111.

[9] W. T. Tutte, The factors of graphs, *Canad. J. Math.* **4** (1952) 314 - 328.