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On Paths in a Complete Bipartite Geometric Graph

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Abstract. Let A and B be two disjoint sets of points in the plane such that no three points of $A \cup B$ are collinear, and let n be the number of points in A. A geometric complete bipartite graph $K(A, B)$ is a complete bipartite graph with partite sets A and B which is drawn in the plane such that each edge of $K(A, B)$ is a straight-line segment. We prove that (i) If $|B| \geq (n+1)(2n-4)+1$, then the geometric complete bipartite graph $K(A, B)$ contains a path that passes through all the points in A and has no crossings; and (ii) There exists a configuration of $A \cup B$ with $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ such that in $K(A, B)$ every path containing the set A has at least one crossing.

1 Introduction

Let G be a finite graph without loops or multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G. For a set X, we denote by $|X|$ the cardinality of X. *A geometric graph* $G = (V(G), E(G))$ is a graph drawn in the plane such that $V(G)$ is a set of points in the plane, no three of which are collinear, and $E(G)$ is a set of (possibly crossing) straight-line segments whose endpoints belong to $V(G)$. If a geometric graph G is a complete bipartite graph with partite sets A and B, i.e., $V(G) = A \cup B$, then G is denoted by $K(A, B)$, which may be called *a geometric complete bipartite graph*.

In 1996, M. Abellanas, J. García, G. Hernández, M. Noy and P. Ramos [1] showed the following result.

Theorem A (Abellanas et al. [1]) *Let A and B be two disjoint sets of points in the plane such that $|A| = |B|$ and no three points of $A \cup B$ are collinear. Then the geometric complete bipartite graph $K(A, B)$ contains a spanning tree T without crossings such that the maximum degree of T is $O(\log |A|)$.*

In 1999, Kaneko [3] improved their result and proved the following theorem.

Theorem B (Kaneko [3]) *Let A and B be two disjoint sets of points in the plane such that $|A| = |B|$ and no three points of $A \cup B$ are collinear. Then the geometric complete bipartite graph $K(A, B)$ contains a spanning tree T without crossings such that the maximum degree of T is at most 3.*

It is well-known that under the same condition in Theorem B, there are configurations of $A \cup B$ such that $K(A, B)$ does not contain a hamiltonian path without crossings [2]. Note that the upper bound of the number of crossings of hamiltonian cycles in $K(A, B)$ is given in [4]. So we are led to the following problem. Given two disjoint sets A and B of points in the plane such that no three points of $A \cup B$ are collinear, if $|B|$ is large compared with $|A|$, then does $K(A, B)$ contain a path P without crossings such that $V(P)$ contains the set A? The answer to the above question is in the affirmative, as we shall see now. We prove the following theorem.

Theorem 1. *Let A and B be two disjoint sets of points in the plane such that no three points of $A \cup B$ are collinear, and let n be the number of points in A.*
(i) If $|B| \geq (n+1)(2n-4)+1$, then the geometric complete bipartite graph $K(A, B)$ contains a path P without crossings such that $V(P)$ contains the set A.
(ii) There exists a configuration of $A \cup B$ with $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ such that in $K(A, B)$ every path containing the set A has at least one crossing.

In order to prove Theorem 1, we need some notation and definitions. For a set X of points in the plane, we denote by $\mathrm{conv}(X)$ the convex hull of X. The boundary of $\mathrm{conv}(X)$ is a polygon whose segments and extremes are called *the edges and the vertices* of $\mathrm{conv}(X)$, respectively. For two points x and y in the plane, we denote by xy the straight line segment joining x to y, which may be an edge of a geometric graph containing both x and y as it vertices. Let A be a set of point in the plane, let y be a vertex of $\mathrm{conv}(A)$ and let x be a point exterior to $\mathrm{conv}(A)$. Then we say that x *sees* y on $\mathrm{conv}(A)$ if the line segment xy intersects $\mathrm{conv}(A)$ only at y.

Lemma 1. *Let R and S be disjoint sets of points in the plane with $|R| \geq |S|$ such that no three points of $R \cup S$ are collinear. Suppose that there exists a line in the plane that separates R and S. Let x and y be two vertices of $\mathrm{conv}(R \cup S)$ such that $x \in S$, $y \in R$, and xy is an edge of $\mathrm{conv}(R \cup S)$. Then in $K(R, S)$, there exists a path P without crossings such that*
(i) the vertex x is an end of P, and
(ii) P passes through all the points in A.

Proof. We prove the lemma by induction on $|R \cup S|$. If $|S| = 1$ or $|S| = 2$, then the lemma follows immediately, and so we may assume $|R| \geq |S| \geq 3$.

Let x_1 be the vertex of $\mathrm{conv}(R \cup S)$ such that $x_1 \in S$ and xx_1 is an edge of $\mathrm{conv}(R \cup S)$ (see Figure 1(b)).

Then we can find two points $z_1 \in S - \{x\}$ and $z \in R$ such that x can see both z_1 and z, and $z_1 z$ is an edge of $\mathrm{conv}(R \cup S - \{x\})$ (see Figure 1(b)). Note that it may occur that $z_1 = x_1$ and/or $z = y$. Similarly, we can find two more points $w_1 \in S - \{x\}$ and $w \in R - \{z\}$ such that z can see both w_1 and w, and $w_1 w$ is an edge of $\mathrm{conv}(R \cup S - \{x, z\})$ (see Figure 1(b)). Note that it may occur that $w_1 = z_1$ (and/or $w = y$ if $z \neq y$).

We now apply the inductive hypothesis to $S - \{x\}, R - \{z\}, w_1$ and w. Then there exists a path P' in $K(S - \{x\},\ R - \{z\})$ without crossings that starts with

w_1 and contains $S-\{x\}$. By adding two edges w_1z and zx to P', we obtain the desired path in the lemma.

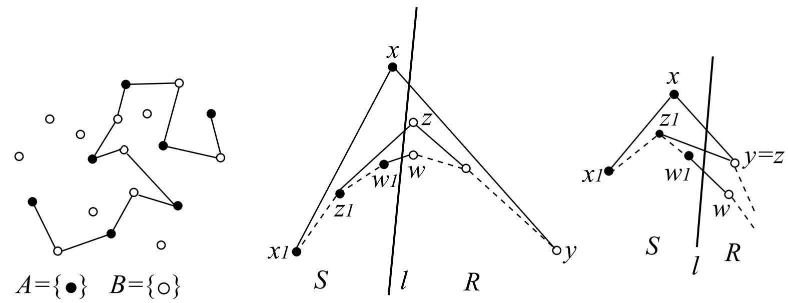

Fig. 1. (a) A path given in Theorem 1. (b) Figure of proof of Lemma 1.

Now we proceed to prove part (i) of Theorem 1. We may assume that no two points of $A \cup B$ have the same x-coordinate. Let $a_1, a_2, \ \dots \ a_n$ be points of S sorted by their x-coordinate and let l_i be the vertical line which passes through the point a_i, $1 \leq i \leq n$. These n lines separate the plane into $n+1$ regions and hence they separate the set B into $n+1$ disjoint subsets. Assume that these lines are directed upward. By the assumption, at least one subset contains at least $2n-3$ points of B. We may assume that one of the regions which contains at least $2n-3$ points of B is bounded by the lines l_j and l_{j+1}, $1 \leq j \leq n-1$. (The leftmost and rightmost unbounded regions can be treated similarly.) Let B_j be the subset of B between l_j and l_{j+1}, i.e., $|B_j| \geq 2n-3$. Let l_0 be the line between l_j and l_{j+1} satisfying the following conditions:

(i) l_0 passes through a point b_0 of B_j and is directed upward,
(ii) The number of points in $B_j - \{b_0\}$ to the left of l_0 is $2j-2$.

Let B_l be the subset of $B_j - \{b_0\}$ to the left of l_0, and $B_r = B_j - \{b_0\} - B_l$. Then $|B_l| = 2j-2$ and $|B_r| \geq 2n-2j-2$. Let A_l be the subset of A to the left of l_0 and let A_r be the subset of A to the right of l_0. Trivially $|A_l| = j$ and $|A_r| = n-j$. Let t_1 and t_2 be the two rays emanating from b_0 such that t_i is tangent to $\mathrm{conv}(A_l)$ at w_i, $1 \leq i \leq 2$, and t_1 is above t_2. Also let t_3 and t_4 be the two rays emanating from b_0 such that t_i is tangent to $\mathrm{conv}(A_r)$ at w_i, $3 \leq i \leq 4$, and t_3 is above t_4. (Notice that since no three points of $A \cup B$ are collinear, each ray contains no point of $B_l \cup B_r$.) Let B_l^+ be the subset of B_l above the ray t_2 and B_l^- the subset of B_l under the ray t_1. Also let B_r^+ be the subset of B_r above the ray t_4 and B_r^- the subset of B_r under the ray t_3. Since $|B_l| = 2j-2$, we have either $|B_l^+| \geq j-1$ or $|B_l^-| \geq j-1$, say $|B_l^+| \geq j-1$. Similarly we have either $|B_r^+| \geq n-j-1$ or $|B_r^-| \geq n-j-1$, say $|B_r^+| \geq n-j-1$. Consider now $K(B_l^+ \cup \{b_0\},\ A_l)$. Since $|B_l^+ \cup \{b_0\}| \geq j = |A_l|$, applying Lemma 1 and letting

$x = b_0$, we can find a path R_l in $K(B_l^+ \cup \{b_0\},\ A_l)$ without crossings such that (i) the vertex b_0 is an end of R_l and (ii) $V(R_l)$ contains A_l. In a similar manner, we can find a path R_r in $K(B_r^+ \cup \{b_0\},\ A_r)$ without crossings such that (i) the vertex b_0 is an end of R_r and (ii) $V(R_r)$ contains A_r. Set $P = R_l \cup R_r$. Clearly P is a path in $K(A, B)$ without crossings such that $V(P)$ contains the set A.

In order to show part (ii) of Theorem 1, suppose that $n = 4k$ and all points of $A = \{a_j^i\}$ and $B = \{b_j^i\}$ lie on a cycle in the following order:

$a_1^0,\ a_2^0,\ \ldots,\ a_{k+2}^0,\ b_1^0,\ b_2^0,\ \ldots,\ b_k^0,\ a_1^1,\ a_2^1,\ b_1^1,\ b_2^1,\ \ldots,\ b_k^1,$

$a_1^2,\ a_2^2,\ b_1^2,\ b_2^2,\ \ldots,\ b_k^2,\ \ \ldots\ \ldots\ \ldots,\ a_1^{k-2},\ a_2^{k-2},\ b_1^{k-2},\ b_2^{k-2},\ \ldots,\ b_k^{k-2},$

$a_1^{k-1},\ a_2^{k-1},\ \ldots,\ a_{k+2}^{k-1},\ b_1^{k-1},\ b_2^{k-1},\ \ldots,\ b_{3k-1}^{k-1}$ (see Figure 2).

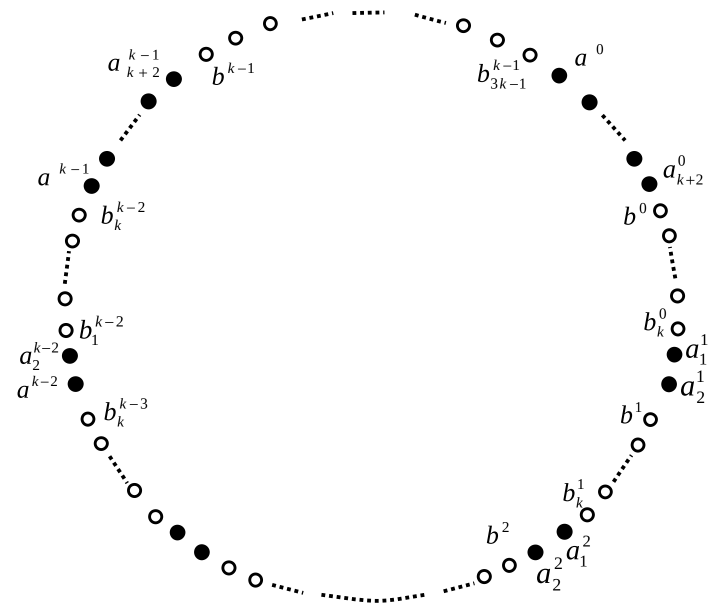

Fig. 2.

It is not difficult to show that $|A| = n$ and $|B| = \frac{n^2}{16} + \frac{n}{2} - 1$ and that in $K(A, B)$ every path containing the set A has at least one crossing.

This completes the proof of Theorem 1.

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References

1. M. Abellanas, J. García, G. Hernández, M. Noy and P. Ramos, Bipartite embeddings of trees in the plane, *Graph Drawing'96* , Springer Verlag LNCS **1190**, (1996), 1-10
2. J. Akiyama and J. Urrutia, Simple alternating path problem, Discrete Mathematics **84** no. 1, (1990), 101-103
3. A. Kaneko, On the maximum degree of bipartite embeddings of trees in the plane, *Discrete and Computational Geometry* Springer Verlag LNCS **1763**, (2000), 166-171.
4. A. Kaneko, M. Kano and K. Yoshimoto, Alternating hamilton cycles with minimum number of crossings in the plane, International Journal of Computational Geometry & Applications, **10** no.1, (2000), 73-78