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ALTERNATING HAMILTON CYCLES WITH MINIMUM NUMBER OF CROSSINGS IN THE PLANE

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ABSTRACT

Let X and Y be two disjoint sets of points in the plane such that $|X| = |Y|$ and no three points of $X \cup Y$ are on the same line. Then we can draw an alternating Hamilton cycle on $X \cup Y$ in the plane which passes through alternately points of X and those of Y, whose edges are straight-line segments, and which contains at most $|X| - 1$ crossings. Our proof gives an $O(n^2 \log n)$ time algorithm for finding such an alternating Hamilton cycle, where $n = |X|$. Moreover we show that the above upper bound $|X| - 1$ on crossing number is best possible for some configurations. [a] .

Keywords: two point sets, plane, crossing number, Hamilton cycle, line embedding.

1. Introduction

Let X and Y be two disjoint sets of points in the plane such that no three points of $X \cup Y$ are on the same line. Then there are some results on drawing graphs on $X \cup Y$ whose edges are straight-line segments in the plane. For example, Tokunaga [3] showed that the minimum number of crossings of two spanning trees on X and on Y is determined by the number of edges of $\mathrm{conv}(X \cup Y)$ joining points of X to those of Y. Abellanas et al. [1] considered some problems on alternating spanning trees on $X \cup Y$ with no crossings, whose edges join points of X to those of Y, and proved that there exists an alternating spanning tree on $X \cup Y$ with no crossings whose maximum degree is at most $O(|Y|/|X| + \log |X|)$, where $|Y| \geq |X|$. Kankeo has recently showed that if $|Y| = |X|$, then there exists an alternating spanning tree on $X \cup Y$ with no crossings whose maximum degree is at most three. Moreover, Akiyama and Urrutia [2] gave an algorithm that determines if $X \cup Y$ admits an alternating Hamilton path with no crossings under the assumption that $|X| = |Y|$ and $X \cup Y$ are the set of vertices of $\text{conv}(X \cup Y)$.

In this paper, we deal with an *alternating Hamilton cycle* on $X \cup Y$, which passes through all the points of $X \cup Y$ such that points of X and Y are alternating and whose edges are straight-line segments. Our aim is to draw an alternating Hamilton cycle on $X \cup Y$ in the plane so that it has a small number of crossings. In particular, we have the following question: What is the minimum number of crossings in such alternating Hamilton cycles? An answer to this question is given in the following theorem.

Theorem 1 *Let X and Y be two disjoint sets of points in the plane such that $|X| = |Y|$ and no three points of $X \cup Y$ are on the same line. Then we can draw an alternating Hamilton cycle on $X \cup Y$ which has at most $|X| - 1$ crossings (see Figure 1). Moreover there exist configurations $X \cup Y$ for which this upper bound $|X| - 1$ is best possible.*

Since our proof of the above theorem is constructive, it provides an algorithm for drawing an alternating Hamilton cycle given in the theorem. Actually, by a direct analysis of the proof, we can draw it in $O(n^2 \log n)$ time, where $n = |X|$. Moreover, by the proof we can draw it so that no three edges of it intersect at the same point.

We first show that the upper bound $|X| - 1$ on crossing number is best possible for some configurations. Let P be a convex polygon with $2n$ vertices such that no three line segments joining two points of P intersect at the same point except their common end point, and let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_n, y_{n-1}, \ldots, y_1\}$ be disjoint sets of vertices of P such that $x_1, x_2, \ldots, x_n, y_n, y_{n-1}, \ldots\ y_1$ are consecutive vertices of P, in particular, X and Y can be separated by a certain straight-line in the plane (see Figure 1). Then we now show that every alternating Hamilton cycle on $X \cup Y$ has at least $n - 1$ crossings. Let C be any alternating Hamilton cycle on $X \cup Y$. Then C contains at least $2n - 2$ edges $x_i y_j$ which is neither $x_1 y_1$ nor $x_n y_n$. Let $x_i y_j$ be one of such edges. Since the path $C - x_i y_j$ connects x_i and y_j, and P is a convex polygon, the edge $x_i y_j$ must intersect at least one edge of $C - x_i y_j$, say $x_r y_s$, and we get a pair $(x_i y_j, x_s y_t)$ of intersecting edges. Because there exist at least $n - 1$ distinct such pairs of edges, C contains at least $n - 1$ crossings. It is obvious that there exists an alternating Hamilton cycle on $X \cup Y$ with $n - 1$ crossings (see Figure 1). Therefore the upper bound in Theorem 1 is best possible for the above configurations.

2. Proof of Theorem 1

Let S be a set of points in the plane. Then we denote by $\text{conv}(S)$ the *convex hull* of S, which is the smallest convex set containing S. We briefly say that S is *in general position* if no three points of S lie on the same line. The following lemma plays an important role.

Lemma 1 *Let R and B be disjoint sets of points in the plane and x be a point in the plane not contained in* $\text{conv}(R \cup B)$ *such that* $2 \leq |R| \leq |B|$ *and* $R \cup B \cup \{x\}$ *is in*

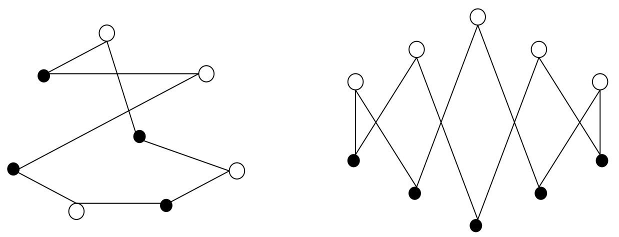

Fig. 1. Alternating Hamilton cycles with two and four crossings.

general position. If the two vertices of conv$(R \cup B \cup \{x\})$ *adjacent to* x *are contained in* R*, then we can partition* $R \cup B$ *into* $D_1 \cup D_2$ *which satisfies the following three conditions: (i)* $|D_1 \cap R| = |D_1 \cap B|$*, (ii)* conv$(D_1 \cup \{x\})$ *and* conv$(D_2 \cup \{x\})$ *intersect only at* x*, and (iii) for each* $i \in \{1, 2\}$*, one vertex of* conv$(D_i \cup \{x\})$ *adjacent to* x *is contained in* B *(see Figure 2).*

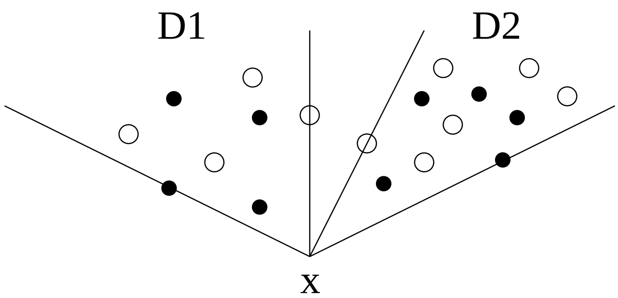

Fig. 2. A partition $D_1 \cup D_2$ of $R \cup B$, where $R = \{\bullet\}$ and $B = \{\circ\}$.

Proof We prove the lemma by induction on $|R|$. If $|R| = 2$ then the lemma follows immediately, and so we may assume $|R| \geq 3$. By a suitable rotation of the plane, we may assume that x lies on the bottom of conv$(R \cup B \cup \{x\})$. Moreover in this case, we shall prove the lemma with the additional property that D_1 lies to the left of D_2. We now consider rays emanating from x and going upward, and so a ray means such a ray in this proof. We define a function f of a ray r by

$f(r) = \{$the number of points of R lying on or to the left of $r\}$
- $\{$the number of points of B lying on or to the left of $r\}$.

Let r_1 and r_2 denote the rays that passes through the left vertex and the right vertex of conv$(R \cup B \cup \{x\})$ adjacent to x, respectively, and let r'_2 be a ray obtained from r_2 by a very small counterclockwise rotation around x. Then $f(r_1) = 1$ and $f(r'_2) = |R| - 1 - |B| \leq -1$. Since the value of f changes ± 1 and since $f(r_1) > 0 > f(r'_2)$, there exists a ray r_a such that $f(r_a) = 0$ and r_a passes through a point of B. We next rotate r_a clockwise around x until it meets a new point of $R \cup B$, and denote the ray by r_b. If r_b passes through a point of B, then let D_1 and D_2 be the set of points of $R \cup B$ lying on or to the left of r_a and that of those points lying on or to the right of r_b, respectively. Then $D_1 \cup D_2$ is the desired partition.

If r_b passes through a point of R, then let R_1 and B_1 be the subsets of R and

B, respectively, whose points lie on or to the right of r_b. Then R_1, B_1 and x satisfy the assumption of the lemma, and hence by the inductive hypothesis, there exists a partition $D_3 \cup D_4$ of $R_1 \cup B_1$ that satisfies the conditions of this lemma together with the condition that D_3 lies to the left of D_4. Consequently we can obtain the desired partition $(D_1 \cup D_3) \cup D_4$ of $R \cup B$. $\square$

For two disjoint sets S and T of points in the plane, we can define an *alternating Hamilton path* on $S \cup T$ in the same way as an alternating Hamilton cycle on $S \cup T$. Of course, if there exists an alternating Hamilton path on $S \cup T$ which begins with a vertex in S, then $|T| \leq |S| \leq |T| + 1$. An alternating Hamilton path connecting two points x and y is called an alternating Hamilton $x - y$ path and denoted by $P(x, y)$. In order to prove Theorem 1, we need the following lemma, which will be proved latter.

Lemma 2 *Let R and B be disjoint sets of points in the plane such that $R \cup B$ is in general position. Suppose that a point, say r, of R is a vertex of* $\mathrm{conv}(R \cup B)$*. Then the following two statements hold.*

(i) If $|R| = |B|$, then there exists an alternating Hamilton path on $R \cup B$ which begins with r and ends with a vertex of B, and which has at most $|R| - 1$ crossings. Furthermore, if a point, say b, of B is a vertex of $\mathrm{conv}(R \cup B)$*, then there exists an alternating Hamilton $r - b$ path which has at most $|R| - 1$ crossings.*

(ii) If $|R| = |B| + 1$, then there exists an alternating Hamilton path on $R \cup B$ which begins with r and ends with another point of R, and which has at most $|R| - 2 = |B| - 1$ crossings. Furthermore, if a point, say s, of $R \setminus \{r\}$ is a vertex of $\mathrm{conv}((R \cup B) \setminus \{r\})$*, then there exists an alternating Hamilton $r - s$ path which has at most $|R| - 2 = |B| - 1$ crossings. Note that s is not necessary to be a vertex of* $\mathrm{conv}(R \cup B)$.

We now prove our main theorem using the above lemma.

Proof of Theorem 1. Suppose that one vertex of $\mathrm{conv}(X \cup Y)$ is contained in X and another vertex of $\mathrm{conv}(X \cup Y)$ is contained in Y. Then we can choose two consecutive vertices x and y of $\mathrm{conv}(X \cup Y)$ so that $x \in X$ and $y \in Y$. By Lemma 2, there exists an alternating Hamilton $x - y$ path $P(x, y)$ which has at most $|X| - 1$ crossings. Therefore $P(x, y) + xy$ is the desired alternating Hamilton cycle.

We next assume that all the vertices of $\mathrm{conv}(X \cup Y)$ are contained in X. Let $x \in X$ be a vertex of $\mathrm{conv}(X \cup Y)$. Then $X \setminus \{x\}$, Y and x satisfy the assumption of Lemma 1, and so there exists a partition $D_1 \cup D_2$ of $(X \setminus \{x\}) \cup Y$ which satisfies the conditions in Lemma 1. In particular, for every $i \in \{1, 2\}$, we can take a vertex d_i of $\mathrm{conv}(D_i)$ which is contained in Y and adjacent to x. Then by applying Lemma 2 to $D_2 \cup \{x\}$, we can draw an alternating Hamilton path $P(x, d_2)$ on $D_2 \cup \{x\}$ which has at most $|D_2 \cap Y| - 1$ crossings. Similarly by applying Lemma 2 to $D_1 \cup \{d_2\}$, in which d_1 is a vertex of $\mathrm{conv}((D_1 \cup \{d_2\}) \setminus \{d_2\})$, we can obtain an alternating Hamilton path $P(d_2, d_1)$ on $D_1 \cup \{d_2\}$ that has at most $|D_1 \cap Y| - 1$ crossings. Therefore we can obtain the desired alternating Hamilton cycle $P(x, d_2) + P(d_2, d_1) + d_1 x$, which has at most $|D_2 \cap Y| - 1 + |D_1 \cap Y| - 1 + 1 = |Y| - 1$ crossings. $\square$

Proof of Lemma 2 We prove the two statements of Lemma 2 at the same time by induction on $|R \cup B|$. The cases in which $|R \cup B| \leq 4$ are trivial. So we may assume $|R \cup B| \geq 5$. For convenience, we color all the points of R with *red* and those of B with *blue*.

We now prove Statement (i). First suppose that all the vertices of $\mathrm{conv}(R \cup B)$ are red. Then by Lemma 1, there exists a partition $D_1 \cup D_2$ of $(R \setminus \{r\}) \cup B$ with respect to r. We denote by $d_1 \in D_1 \cap B$ and $d_2 \in D_2 \cap B$ the two blue points satisfying the condition (iii) of Lemma 1. By applying the inductive hypothesis of (ii) to $D_1 \cup \{d_2\}$, we can draw an alternating Hamilton path $P(d_2, d_1)$ on $D_1 \cup \{d_2\}$ which has at most $|D_1 \cap R| - 1$ crossings. Similarly by the inductive hypothesis of (i), there exists an alternating Hamilton path $P(r, d_2)$ on $D_2 \cup \{r\}$ which has at most $|D_2 \cap R|$ crossings. Thus we can obtain an alternating Hamilton $r - d_1$ path $P(r, d_2) + P(d_2, d_1)$, which has at most $|D_1 \cap R| - 1 + |D_2 \cap R| = |R| - 2$ crossings.

We next assume that at least one vertex of $\mathrm{conv}(R \cup B)$ is blue, and let b be such a blue vertex. It suffices to show that we can draw an alternating Hamilton $r - b$ path which has at most $|R| - 1$ crossings.

Suppose that there exists a blue vertex, say b_1, of $\mathrm{conv}((R \cup B) \setminus \{r\})$ which is different from b and is visible from r (i.e., the straight-line segment rb_1 intersects $\mathrm{conv}((R \cup B) \setminus \{r\})$ only at b_1). Then by the inductive hypothesis, we can draw an alternating Hamilton path $P(b_1, b)$ on $(R \cup B) \setminus \{r\}$ which has at most $|B| - 2$ crossings. Therefore we obtain the desired alternating Hamilton $r - b$ path $P(b_1, b) + rb_1$ on $R \cup B$. Similarly, if there exists a red vertex of $\mathrm{conv}((R \cup B) \setminus \{b\})$ which is not r and is visible from b, then we can get the desired alternating Hamilton $r - b$ path. Therefore we may assume that all the vertices of $\mathrm{conv}((R \cup B) \setminus \{r\})$ visible from r are red except b, and that a similar situation occurs for b and $(R \cup B) \setminus \{r\}$.

We consider two cases.

Case 1. *r and b are consecutive vertices of* $\mathrm{conv}(R \cup B)$.

Since r and b are adjacent, we can find a red vertex r_1 and a blue vertex b_1 of $\mathrm{conv}((R \cup B) \setminus \{r, b\})$ which are visible from both b and r. Then by induction, we can draw an alternating Hamilton path $P(b_1, r_1)$ on $(R \cup B) \setminus \{r, b\}$ which has at most $|R| - 2$ crossings,. Consequently we can get an alternating Hamilton $r - b$ path $P(b_1, r_1) + b_1 r + r_1 b$, which has at most $|R| - 1$ crossings since $b_1 r$ and $r_1 b$ may intersect.

Case 2. *r and b are not consecutive vertices of* $\mathrm{conv}(R \cup B)$.

In this case, there exists a partition $D_1 \cup D_2$ of $(R \setminus \{r\}) \cup B$ with respect to r which satisfies the conditions of Lemma 1. For every $i \in \{1, 2\}$, let d_i denote the blue vertex of D_i satisfying (iii) of Lemma 1.

If $b \in D_1$, then we can obtain the desired alternating Hamilton $r - b$ path from two alternating Hamilton paths $P(d_2, b)$ on $D_1 \cup \{d_2\}$ and $P(r, d_2)$ on $D_2 \cup \{r\}$. Hence we may assume that $b \in D_2$.

If $b \neq d_2$, then we can obtain the desired alternating Hamilton $r - b$ path from two alternating Hamilton paths $P(d_2, d_1)$ on $D_1 \cup \{d_2\}$ and $P(d_2, b)$ on D_2 by adding rd_1. Suppose next $b = d_2 \in D_2$. Then both d_1 and r are vertices of $\mathrm{conv}((D_2 \setminus \{b\}) \cup \{d_1, r\})$, and so there exists an alternating Hamilton path $P(r, d_1)$ on $(D_2 \setminus \{b\}) \cup \{d_1, r\}$ which has at most $|D_2 \cap B| - 1$ crossings. Similarly there exists an alternating Hamilton path $P(b, d_1)$ on $D_1 \cup \{b\}$, which has at most $|D_1 \cap B| - 1$ crossings. Therefore we obtain the desired alternating Hamilton $r - b$ path $P(r, d_1) + P(b, d_1)$, which has at most $|D_2 \cap B| - 1 + |D_1 \cap B| - 1 + 1 = |B| - 1$ crossings since $P(r, d_1)$ and $P(b, d_1)$ intersect at most once. Consequently the proof of (i) is complete.

Next we prove Statement (ii). Note that if the red point s exists, then it suffices to show that there exists an alternating Hamilton $r - s$ path. Assume first that a blue point, say b', is a vertex of $\text{conv}(R \cup B)$ and is adjacent to r. Then by the inductive hypothesis on $(R \cup B) \setminus \{r\}$, we can draw an alternating Hamilton path on $(R \cup B) \setminus \{r\}$ which begins with b' and ends with the red point s if any, or a red point of $R \setminus \{r\}$ and which has at most $|B| - 1$ crossings. By adding a straight-line segment rb' to this path, we obtain the desired alternating Hamilton $r - s$ path or the desired alternating Hamilton path connecting r and a red point. Hence we may assume that the two vertices of $\text{conv}(R \cup B)$ adjacent to r are red points.

By Lemma 1, there exists a partition (D_1, D_2) of $(R \cup B) \setminus \{r\}$. Let d_1 and d_2 are blue points of D_1 and D_2, respectively, satisfying the condition (iii) of Lemma 1. Note that an equality $|D_2 \cap R| = |D_2 \cap B|$ also holds, and thus without loss of generality, we may assume that $s \in D_1$ if any. Then we can draw an alternating Hamilton path P_1 on D_1 which begins with d_1 and ends with s if any, or a red point of $(D_1 \cap R) \setminus \{r\}$ and which has at most $|D_1 \cap B| - 1$ crossings. Similarly, there exists an alternating Hamilton path $P(d_1, d_2)$ on $D_2 \cup \{d_1\}$ which has at most $|D_2 \cap B| - 1$ crossings. Therefore we can get the desired alternating Hamilton path $P_1 + P(d_1, d_2) + d_2 r$, which has at most $|D_1 \cap B| - 1 + |D_2 \cap B| - 1 + 1 = |B| - 1$ crossings.

Consequently the proof of Theorem 1 is complete. □

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