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Straight line embeddings of rooted star forests in the plane

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Abstract

For every $1 \leq i \leq n$, let T_i be a rooted star with root v_i, where v_i is not necessary to be its center. Then the union $F = T_1 \cup T_2 \cup \ldots \cup T_n$ is called a rooted star forest with roots $v_1, v_2, \ldots, v_n$. Let P be a set of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$, where $|F|$ denotes the order of F. Then we show that there exists a bijection $\phi : V(F) \rightarrow P$ such that $\phi(v_i) = p_i$ for all $1 \leq i \leq n$, $\phi(x)$ and $\phi(y)$ are joined by a straight-line segment if and only if x and y are joined by an edge of F, and such that no two straight-line segments intersect except at their common end-point.

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1 Introduction

We consider finite planar graphs without loops or multiple edges. Let G be a planar graph with vertex set $V(G)$ and edge set $E(G)$. We denote by $|G|$ the order of G, that is, $|G| = |V(G)|$. Generally, for a set X, let $|X|$ denote the cardinality of X. Given a planar graph G, let P be a set of $|G|$ points in the plane (2-dimentional Euclidean space) in general position (i.e., no three of them are collinear). Then G is said to be *line embedded onto* P or *straight-line embedded onto* P if G can be embedded in the plane so that every vertex of G corresponds to a point of P, every edge corresponds to a straight-line segment, and that no two straight-line segments intersect execpt their common end-point. Namely, G is line embedded onto P if there exists a bijection $\phi : V(G) \to P$ such that two points $\phi(x)$ and $\phi(y)$ are joined by a straight-line segment if and only if x and y are joined by an edge of G and no two distinct open straight-line segments have a point in common. We call such a bijection a *line embedding* or a *straight-line embedding* of G onto P. The following theorem is mentioned in [1].

Theorem A *An outerplanar graph G can be line embedded onto a set of $|G|$ points in the plane in general position.*

In this paper we consider a line embedding having one more property. Let G be a planar graph with n specified viertices $v_1, v_2, \ldots, v_n$, and P a set of $|G|$ points in the plane in general position containing n specified points $p_1, \ldots, p_n$. Then we say that G is *strongly line embedded onto* P if G can be line embedded onto P so that for every $1 \leq i \leq n$, v_i corrresponds to p_i, that is, if there exists a line embedding $\phi : V(G) \to P$ such that $\phi(v_i) = p_i$ for all $1 \leq i \leq n$. The line embedding mentioned above is called a *strong line embedding* of G onto P. A tree with one specified vertex v is usually called a *rooted tree with root* v. For n disjoint rooted trees T_i with root v_i, $1 \leq i \leq n$, the union $T_1 \cup T_2 \cup \ldots \cup T_n$, whose vertex set is $V(T_1) \cup \ldots \cup V(T_n)$ and whose edge set is $E(T_1) \cup \ldots \cup E(T_n)$, is called a *rooted forest* with roots $v_1, \ldots, v_n$, which are specified vertices of it.

A complete bipartite graph $K(1, k)$, $k \geq 1$, is called a *star*, which is a tree and has one center and k end-vertices. The union of stars is called a *star forest*, and the union of rooted stars, some of whose roots may be end-vertices, is called a *rooted star forest*.

We now give a known theorem on a strong line embedding, which is conjectured by Perles [3] and partially solved by Pach and Törőcsik [6], and whose another simpler proof can be found in Togunaga [4].

Theorem B (Ikebe, Perles, Tamura, and Tokunaga [5]) *A rooted tree T can be strongly line embedded onto every set P of $|T|$ points in the plane in general position containing a specified point.*

In this paper we shall prove the following theorem.

Theorem 1 *Let $n \geq 1$ be an integer, and T_i be a rooted star with root v_i for every $i, 1 \leq i \leq n$. Then a rooted star forest $F := T_1 \cup T_2 \cup \ldots \cup T_n$ with roots $v_1, \ldots, v_n$ can*

Figure 1: A rooted star forest, a set of points and its strong line embedding.

Figure 2: A rooted forest F that cannot be strongly line embedded onto P.

be strongly line embedded onto every set P of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$.

Before giving a proof, let us give an example of our theorem (see Figure 1) and an example which shows the necessity of the condition that every T_i is a star (see Figure 2). Namely, a rooted forest given in Figure 2 contains a component not beeing a star and cannot be strongly line embedded onto a set P of 22 points given in Figure 2.

We conclude this section with another related result and conjectures.

Theorem C ([2]) *A rooted forest F consisting of two rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing two specified points (see Figure 1).*

Conjecture D *A rooted forest F consisting of three rooted trees can be strongly line embedded onto every set of $|F|$ points in the plane in general position containing three specified points.*

Conjecture E *Let $F := T_1 \cup T_2 \cup \cdots \cup T_n$ be a rooted forest with roots $v_1, v_2, \ldots, v_n$, and P a set of $|F|$ points in the plane in general position containing n specified points $p_1, p_2, \ldots, p_n$. Then there exists a line embedding $\phi : V(G) \to P$ such that $\{\phi(v_i) \mid 1 \le i \le n\} = \{p_1, \ldots, p_n\}$.*

2 Proof of Theorem

In order to prove our theorem, we need some notation and definitions. For a set X of points in the plane, we denote by $\mathrm{conv}(X)$ the *convex hull* of X, which is the smallest

Figure 3: Wedges wdg(xpy), wdg(xpr) and wdg(rpr').

convex set containing X. Let G be a planar graph and P a set of $|G|$ points in the plane in general position. For a subset U of $V(G)$, $G-U$ denotes the graph obtained from G by deleting the vertices in U together with their incident edges. If $U = \{v\}$, then we write $G - v$ for $G - \{v\}$.

For a region R in the plane, we essentially consider only the set Q of points of P contained in R, and state that which points of P lying on the boundary of R are contained in Q one by one. Thus when we introduce a new notaion on region, we do not mention about its boundary. For three non-collinear points x, y and p in the plane, the plane is partitioned into two regions by two rays emanating from p and passing through x and y, respectively. We denote by wdg(xpy) one of such regions whose induced angle is less than π, that is, wdg(xpy) denotes the *wedge* with top p. Similary, for a point x and rays r and r' emanating from p, wdg(xpr) and wdg(rpr') denote the similar wedges (see Figure 3). If we consider a region including all its boundary, then we call it a *closed region*, and if we consider a region without its boundary, then we call it an *open region*.

Proof of Theorem 1 If $n = 1$, then the theorem follows immediately since T_1 is a star and P is a set of points in general position. Thus we may assume $n \geq 2$. We prove the theorem by induction on $|F|$.

For convenience, we call a non-specified point of P an *ordinary point*, and denote the set of specified points and that of ordinary points of P by $S(P)$ and $O(P)$, respectively, that is, $S(P) = \{p_1, p_2, \cdots, p_n\}$ and $O(P) = P \setminus S(P)$. We define $n_i := |T_i| - 1$, which is equal to the number of ordinary points of P being added to the specified point p_i to draw a rooted star T_i. Then $\sum n_i = |O(P)|$. Let X be a subset of P. Then we say that X is *balanced* if the number of ordinary points in X is equal to $\sum n_i$, where the summation is taken over all i such that $p_i \in X$. Moreover, since P is a set of points in general position, it is immedate that every point of X lying on the boundary of conv(X) must be a vertex of conv(X). We consider the following three cases.

Case 1. At least two specified points lie on the boundary of conv(P).

Without loss of generality, we may assume that two specified points p_1 and p_2 lie on the boundary of conv(P). We first show that there exists a point x in the plane such that $x \notin$ conv(P), a closed wedge wdg(p_1xp_2) contains conv(P), and such that every line passing through x contains at most one point of P. If p_1 and p_2 are the two end-vertices of an edge of conv(P), then we can easily find such a point x. If p_1 and p_2 do not lie on the same edge of conv(P), then conv(P) has two non-parallel edges p_1x_1 and p_2x_2, and we can find such a point x near the intersection of two lines containing edges p_1x_1 and p_2x_2, respectively.

Here we consider only rays emanating from x, and so a ray means such a ray. Let r_1 and r_2 be rays passing through p_1 and p_2, respectively. For a ray r, here let $\mathrm{wdg}(r_1xr)$ denote a closed wedge. We define a function f of a ray r by

$$f(r) := |O(P) \cap \mathrm{wdg}(r_1xr)| \; - \sum_{p_i \in \mathrm{wdg}(r_1xr)} n_i.$$

Namely, $f(r)$ denotes the number of remaining ordinary points if $f(r) > 0$ or that of lacking ordinary points if $f(r) < 0$ when we try to embedd $\cup_{p_i \in \mathrm{wdg}(r_1xr)} T_i$ onto $P \cap \mathrm{wdg}(r_1xr)$.

Let r_2' denote a ray in $\mathrm{wdg}(r_1xr_2)$ which is obtained from r_2 by a very small rotation around x. Then we may assume that r_2' passes through no point of P and that an open wedge $\mathrm{wdg}(r_2'xr_2)$ contains no point P. When we rotate a ray r continuously from r_1 to r_2' around x, if r passes through a new ordinary point, then the value of f increases by one, and if it passes through a new specified point p_j, then it decreases by n_j. Since $f(r_1) = -n_1 < 0$ and $f(r_2') = n_2 > 0$ (as $f(r_2) = 0$), there exists a ray r_3 such that $f(r_3) = 0$. Then P is partitioned into two balanced subset $P_1 = P \cap \mathrm{wdg}(r_1xr_3)$ and $P_2 = P \setminus P_1$. By the inductive hypothesis, for each $j \in \{1, 2\}$, the union $\bigcup T_i$, where the union is taken over all i such that $p_i \in P_j$, can be strongly line embedded onto P_j. Therefore the rooted star forest F can be strongly line embedded onto P.

Case 2. Exactly one specified point lies on the boundary of conv(P).

Without loss of generality, we may assume that p_1 lies on the boundary of $\mathrm{conv}(P)$. Let q be a vertex of $\mathrm{conv}(P)$ such that p_1q is an edge of $\mathrm{conv}(P)$. Then q is an ordinary point of P. Suppose that the root v_1 of T_1 is the center of T_1. Let u be an end-vertex of T_1. Then $F - u$ can be strongly line embedded onto $P \setminus \{q\}$ by the inductive hypothesis. Hence by adding a straight-line segment p_1q to this embedding, we can obtain a strong line embedding of F onto P. Next assume that the root v_1 is an end-vertex of T_1. In this case, let w denote the center of T_1. Then by considering $T_1 - v_1$ as a rooted star with root w, we can regard $F - v_1$ as a rooted star forest with roots $w, v_2, \ldots, v_n$. Then by inductive hypothesis, $F - v_1$ can be strongly line embedded onto $P \setminus \{p_1\}$ with specified points $q, p_2, \ldots, p_n$. Therefore by adding a straight-line segment p_1q to this embedding, we can get a strong line embedding of F onto P.

Case 3. No specified point lies on the boundary of conv(P).

We first choose a pair (l_1, l_2) of parallel lines tangent to $\mathrm{conv}(P)$ such that P lies betweeen l_1 and l_2, and every line parallel to these two lines passes through at most one point of P. In particular, each of l_1 and l_2 passes through exactly one ordinary point of P, which is a vetex of $\mathrm{conv}(P)$. By a suitable rotation of the plane, we may assume that both l_1 and l_2 are vertical, and l_2 lies to the right of l_1. For a vertical line l, we denote by l' a vertical line that is obtained from l by a very small parallel transformation and lies to the left of l. Then we may assume that l' does not pass any point of P, and no point of P lies between l' and l except a point lying on l.

Figure 4:

For a vertical line l, let l^- and l^+ denote the two closed regions determined by l such that l^- lies to the left of l and l^+ lies to the right of l. We define a function f of a vertical line l by

$$f(l) := |O(P) \cap l^-| \; - \sum_{p_i \in l^-} n_i.$$

Then we may assume that $f(l) \neq 0$ for every vertical line l between l_1 and l_2 since otherwise P can be partitioned into two balanced subsets, and thus F can be strongly line embedded onto P by induction.

By the same argument in the proof of Case 1 and by the fact that $f(l_1) = 1$ and $f(l_2') = -1$, there exists a vertical line l_3 such that $f(l_3) < 0$ and $f(l) > 0$ for every line l lying to the left of l_3. Then l_3 must pass through a specified point, say p_t. Set $m_1 := f(l_3') > 0$ and $m_2 := n_t - m_1$. Then $f(l_3) = f(l_3') - n_t = m_1 - n_t = -m_2 < 0$, $m_1 < n_t$ and

$$m_2 = |O(P) \cap l_3^+| \; - \sum_{p_i \in l_3^+ \setminus \{p_t\}} n_i \geq 1.$$

Suppose first the root v_t of T_t is the center of T_t. In this case we decompose T_t into two rooted stars $K(1, m_1)$ and $K(1, m_2)$ whose roots are their centers v_t. By the inductive hypothesis, the union $K(1, m_1) \cup (\bigcup T_i)$, where the union $\bigcup T_i$ is taken over all i such that $p_i \in l_3^- \setminus \{p_t\}$, is strongly line embedded onto $P \cap l_3^-$ with specified point set $S(P) \cap l_3^-$. Of course, the root v_t of $K(1, m_1)$ corresponds to the specified point p_t. Similarly, the union $K(1, m_2) \cup (\bigcup T_j)$, where the union $\bigcup T_j$ is taken over all j such that $p_j \in l_3^+ \setminus \{p_t\}$, is strongly line embedded onto $P \cap l_3^+$ with specified point set $S(P) \cap l_3^+$. By combining these two embedding, we can obtain a desired strong line embedding of F onto P. Therefore we may assume that v_t is an end-vertex of T_t.

We hereafter consider only rays emanating from p_t, and so a ray means such a ray. Let r_3 denote a ray contained in the vertical line l_3 which goes upward, and let $r_3^* := l_3 - r_3$ be a ray going downward. For every line l_i ($\neq l_3$) passing through p_t, let r_i and r_i^* denote the two rays contained in l_i which lie to the right of l_3 and to the left of l_3, respectively (see Figure 5). For two rays $r \in \{r_i, r_i^*\}$ and $r' \in \{r_j, r_j^*\}$, let $\mathrm{wdg}(r p_t r')$ denote a wedge that contains the points lying on its boundary except p_t. Then $p_t \notin \mathrm{wdg}(r p_t r')$. We define a function f of a region R as

$$f(R) := |O(P) \cap R| \; - \sum_{p_i \in R} n_i,$$

and say that R is *balanced* if $f(R) = 0$. If there exists a line l_i which passes through p_t and one more point of P and which possesses the property that at least one of wedges

Figure 5: Rays r_3, r_3^*, r_4, r_4^*, r_6^* and related wedges.

$\mathrm{wdg}(r_3 p_t r_i)$ and $\mathrm{wdg}(r_3^* p_t r_i^*)$ is balanced, then choose a line l_4 among all such l_i's so that the internal angle $\angle r_3 p_t r_i$ is as large as possible; and otherwise set $l_4 := l_3$.

For a while we assume that $l_4 \neq l_3$. By symmetry, we may assume that $\mathrm{wdg}(r_3 p_t r_4)$ is balanced. We consider two subcases.

Subcase 3.1 $\mathrm{wdg}(r_4^* p_t r_3^*)$ *contains at least one point of* P (see Figure 5).

If $f(\mathrm{wdg}(r_3^* p_t r_4^*)) < 0$, then since $f(l_3^- \setminus \{p_t\}) = m_1 > 0$, there exists a ray r_5^* between r_3 and r_4^* such that r_5^* passes through an ordinary point of P and $f(\mathrm{wdg}(r_3^* p_t r_5^*)) = 0$. This contradicts the choice of l_4. Hence $f(\mathrm{wdg}(r_3^* p_t r_4^*)) \geq 0$. This inequality together with the assumption of this case imply that $\mathrm{wdg}(r_4^* p_t r_3^*)$ contains at least one ordinary point of P.

Choose an ordinary point x of $P \cap \mathrm{wdg}(r_4^* p_t r_3^*)$ so that $\mathrm{wdg}(r_4^* p_t x)$ contains no other ordinary point than x, and let r_6^* denote a ray passing through x. Note that if r_4^* passes through an ordinary point, then x must be this ordinary point and $r_6^* = r_4^*$.

If $f(\mathrm{wdg}(r_3 p_t r_6^*)) \leq 0$, then there exists a ray r_7^* between r_6^* and r_3^* for which $f(\mathrm{wdg}(r_3 p_t r_7^*)) = 0$ because $f(l_3^- \setminus \{p_t\}) = m_1 > 0$. Then P can be partitioned into three disjoint balanced subsets $P \cap \mathrm{wdg}(r_3 p_t r_4)$, $P \cap \mathrm{wdg}(r_3 p_t r_7^*)$, and $(P \cap \mathrm{wdg}(r_7^* p_t r_4)) \cup \{p_t\}$, and thus by the inductive hypothesis, F can be strongly line embedded onto P. Therefore we may assume $f(\mathrm{wdg}(r_3 p_t r_6^*)) > 0$.

Put $m := f(\mathrm{wdg}(r_3 p_t r_6^*)) > 0$. If $m \leq n_t - 1$, then we decompose a star $T_t - v_t = K(1, n_t - 1)$ into two rooted stars $K(1, m)$ and $K(1, n_t - 1 - m)$, whose roots are their centers and will correspond to x. Then by the inductive hypothesis, rooted star forests

$$\Big(\bigcup_{p_i \in \mathrm{wdg}(r_3 p_t r_6^*)} T_i\Big) \cup K(1, m) \quad \text{and} \quad \Big(\bigcup_{p_i \in \mathrm{wdg}(r_4 p_t r_6^*)} T_i\Big) \cup K(1, n_t - 1 - m)$$

can be strongly line embedded onto $P \cap \mathrm{wdg}(r_3 p_t r_6^*)$ with specified point set $(S(P) \cap \mathrm{wdg}(r_3 p_t r_6^*)) \cup \{x\}$ and onto $P \cap \mathrm{wdg}(r_4 p_t r_6^*)$ with specified point set $(S(P) \cap \mathrm{wdg}(r_4 p_t r_6^*)) \cup \{x\}$, respectively. Moreover, a rooted star sub-forest of F corresponding to a balanced wedge $\mathrm{wdg}(r_3 p_t r_4)$ is strongly line embedded onto $P \cap \mathrm{wdg}(r_3 p_t r_4)$. By adding a straight-line segment $x p_t$ to these strong line embeddings, we can obtain a desired line embedding of F onto P.

If $m = n_t$, then $\mathrm{wdg}(r_3 p_t r_6^*) \cup \{p_t\}$ is balanced, and so P can be partitioned into three balanced subsets $(P \cap \mathrm{wdg}(r_3 p_t r_6^*)) \cup \{p_t\}$, $P \cap \mathrm{wdg}(r_3 p_t r_4)$, and $P \cap \mathrm{wdg}(r_6^* p_t r_4)$. Thus F can be strongly line embedded onto P by induction.

We finally assume $m > n_t$. Since $x \in \mathrm{wdg}(r_3^* p_t r_6^*) \cap \mathrm{wdg}(r_3 p_t r_6^*)$, we have $f(\mathrm{wdg}(r_3^* p_t r_6^*)) = f(l_3^- \setminus \{p_t\}) - f(\mathrm{wdg}(r_3 p_t r_6^*)) + 1 = m_1 - m + 1 \leq m_1 - n_t < 0$. Furthermore since $f(l_3^- \setminus \{p_t\}) = m_1 > 0$, there exists a ray r_8^* between r_6^* and r_3 such that $f(\mathrm{wdg}(r_3^* p_t r_8^*)) = 0$ and r_8^* passes through an ordinary point x' of P. By the choice of x, we have $x' \neq x$ and r_8^* must lie above r_4^*, which contradicts the choice of l_4.

Consequently the proof of this case is complete.

In order to make the proof of the following Subcase 3.2 simpler and clearer, we prove the following claim, in which a line is not necessary to be a vertical line.

Claim *Suppose that the root v_t of T_t is an end-vertex of T_t. If there exists a line l passing through p_t and one ordinary point u of P such that the two open half-planes R_1 and R_2 determined by l satisfy the following inequality, then F can be strongly line embedded onto P.*

$$k_j := |O(P) \cap R_j| \; - \sum_{p_i \in R_j} n_i \geq 0 \qquad \text{for every} \quad j \in \{1, 2\}.$$

Proof It is clear that $P \setminus (R_1 \cup R_2) = \{p_t, u\}$ and $k_1 + k_2 = n_t - 1$. We decompose $T_t - v_t$ into two rooted stars $K(1, k_1)$ and $K(1, k_2)$, whose roots are their centers and will correspond to u. Then by the inductive hypothesis, for every $j \in \{1, 2\}$, a rooted star forest

$$\Big(\bigcup_{p_i \in R_j} T_i\Big) \cup K(1, k_j)$$

can be strongly line embedded onto $(R_j \cap P) \cup \{u\}$ with specifed point set $(R_j \cap S(P)) \cup \{u\}$. By combining these two strong line embedding and by adding a straight-line segment $p_t u$ to this, we can get a strong line embedding of F onto P. $\Box$

Subcase 3.2 $\mathrm{wdg}(r_3^* p_t r_4^*)$ *contains no point of P.*

Note that if $l_3 = l_4$, then this case occurs. Rotate l_4 clockwise around p_t until it passes through another first point of P, say y, and let l_9 denote this line. Of course l_9 passes through y, and the two open wedges between l_4 and l_9 sweeped by l_4 contains no point of P.

Assume that y is an ordinary point and lies to the right of l_3, which implies that y lies on r_9. Then l_9 satisfies the assumption of Claim since $f(\mathrm{wdg}(r_9 p_t r_3)) + f(\mathrm{wdg}(r_3 p_t r_9^*)) = 1 + f(l_3^- \setminus \{p_t\}) = 1 + m_1 \geq 1$, and $f(\mathrm{wdg}(r_9 p_t r_3^*)) + f(\mathrm{wdg}(r_3^* p_t r_9^*)) = f(l_3^+ \setminus \{p_t\}) + 0 = m_2 \geq 1$ by the assumption of this case. Hence F can be straight-line embedded onto P by applying Claim to l_9, y, $\mathrm{wdg}(r_9 p_t r_3)) \cup \mathrm{wdg}(r_3 p_t r_9^*)$ and $\mathrm{wdg}(r_9 p_t r_3^*)) \cup \mathrm{wdg}(r_3^* p_t r_9^*)$. Next assume that y is a specifed point, say p_s, and lies to the right of l_3. Then $f(\mathrm{wdg}(r_3 p_t r_9)) = -n_s < 0$ and so there exists a ray r_{10} between r_9 and r_3^* such that $f(\mathrm{wdg}(r_3 p_t r_{10})) = 0$, which contradicts the choice of l_4.

If y is an ordinary point and lies to the left of l_3, then we can apply Claim to l_9 since $f(\mathrm{wdg}(r_9 p_t r_3)) + f(\mathrm{wdg}(r_3 p_t r_9^*)) = 0 + f(l_3^-) = m_1 \geq 1$, and $f(\mathrm{wdg}(r_9 p_t r_3^*)) + f(\mathrm{wdg}(r_3^* p_t r_9^*)) = 1 + f(l_3^+) = 1 + m_2 \geq 1$. Hence F can be strongly line embedded onto F. If y is a specified point, say p_s, and lies to the left of l_3, then $f(\mathrm{wdg}(r_3^* p_t r_9^*)) = -n_s < 0$

and so we can find a ray r_{11}^* between r_3 and r_9^* such that $f((r_3^* p_t r_{11}^*)) = 0$, which contradicts the choice of l_4.

Consequently the the theorem is proved. □

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