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Radial Perfect Partitions of Convex Sets in the Plane

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Abstract. In this paper we study the following problem: how to divide a cake among the children attending a birthday party such that all the children get the same amount of cake *and* the same amount of icing. This leads us to the study of the following. A perfect k-partitioning of a convex set S is a partitioning of S into k convex pieces such that each piece has the same area and $\frac{1}{k}$ of the perimeter of S. We show that for any k, any convex set admits a perfect k-partitioning. Perfect partitionings with additional constraints are also studied.

1 Introduction

The problem we study in this paper was introduced in [1]. It arises from a simple and practical problem: how to divide a cake among the children attending a birthday party in such a way that each child gets the same amount of cake and (perhaps more important to them) the same amount of icing.

Let S be a convex set contained in the (x, y)-plane. In mathematical terms, a cake C with *base* S is a solid containing all the points with coordinates (x, y, z) such that $(x, y, 0) \in S$ and $0 \leq z \leq h$, $h > 0$; h is called the *height* of C. The exposed area of C consists of the boundary of C minus S, i.e. the base of a cake is not considered to be exposed. A cake will be called a *polygonal cake* if S is a convex polygon.

A division of a cake C into k parts by a series of vertical cuts is said to be *perfect* if:

i) Each part is convex.
ii) Each piece has the same volume and the same exposed area of S.

Our birthday cake problem can be stated as follows: given a cake C, does it have a perfect partitioning into k pieces? If a cake has such a partitioning, we will also say that C can be cut *perfectly*.

A cake whose base is a square can be cut perfectly into three pieces as follows: take any three points x, y and z that divide the perimeter of its base into three pieces of the same length. Now make vertical cuts along the line segments connecting these points to the center of the base of the cake; see Figure 1.

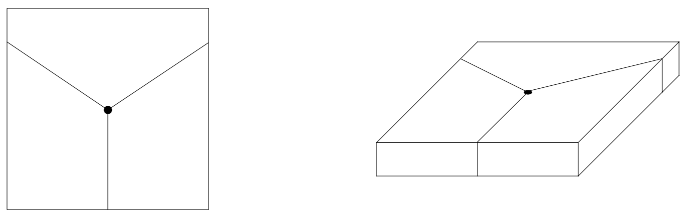

Fig. 1. Cutting a square cake into 3 pieces.

Perfect partitionings of cakes in which the vertical cuts are all along line segments concurrent at a point P are called *radial perfect partitionings*.

Notice that for any $k > 0$, any circular cake C has a radial perfect partitioning into k pieces. This motivates the following definition.

A cake C is called *graceful* if, for every k, there is a perfect radial partitioning of C into k pieces. A natural question arises here: is it true that a graceful cake must necessarily be circular? We will prove that the answer to this question is "no". We will show that there are an infinite number of graceful polygonal cakes, and give a full characterization of them.

There are perfect partitionings of rectangular cakes that are not radial. A non-radial perfect partitioning of a cake whose face is a 2-by-4 rectangle can be obtained by making vertical cuts along the line segments that divide its base into four parts each with equal area and perimeter, as shown in Figure 2.

Since we consider cakes of uniform height h, iced uniformly on the top and sides, we can model the problem of dividing the cake C with base S into pieces of both equal volume and icing, by the equivalent problem of partitioning the convex set S into subsets equal both in area and perimeter of S.

Thus in the rest of this paper, instead of perfect partitionings of cakes, we will refer to perfect partitionings of convex sets.

In Section 3, we will prove that every convex set admits a perfect radial partitioning. In Section 4, we exhibit a quadrilateral that does not admit perfect radial partititonings into four or more pieces, and give an interesting family of convex sets which admit perfect radial partititonings into four pieces.

We will conclude by showing that any cake can be perfectly partitioned into k pieces for all $k > 0$. Of course these partitionings are not necessarily radial. Some results on perfect n-partitions can be found in [3].

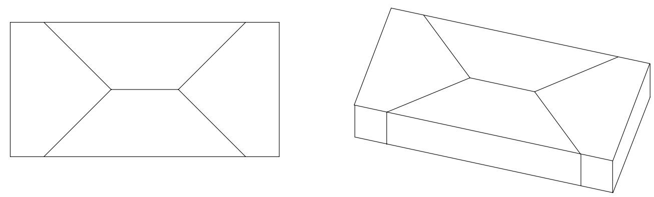

Fig. 2. A non-radial perfect partitioning of a rectangular cake into four pieces.

2 Polygonal graceful cakes

A *perfect k-partitioning* of a convex set S is a partitioning of S into k convex sets of equal area such that the boundary of each set is $\frac{1}{k}$ of the boundary of S.

We now proceed to characterize polygonal graceful cakes.

A convex polygon $\mathcal{P}$ is called *co-circular* if there is a circle R inscribed in $\mathcal{P}$, such that R is contained in $\mathcal{P}$ and tangent to all the edges of $\mathcal{P}$. The center of R will be called the *center* of $\mathcal{P}$. We prove:

Theorem 1. *A polygon $\mathcal{P}$ with n sides is graceful if and only if it is a co-circular polygon. Moreover, all perfect partitionings of $\mathcal{P}$ are radial.*

Proof. Assume that we have a perfect division of $\mathcal{P}$ into k parts produced by cutting along lines radiating from a point C in its interior. Then the perimeter of $\mathcal{P}$ is divided equally among these parts. If $k > n$, then at least $k - n$ of these parts are triangles. Each of these triangles has a side along the perimeter of $\mathcal{P}$; call this its *base*. Since these triangles have equal base lengths and equal areas, their heights must all be the same, i.e. the distances from C to all the sides of $\mathcal{P}$ containing the base of a triangle in our partitioning must be all the same. If we take k sufficiently large, we may assume that on each side of $\mathcal{P}$ there is always a triangle whose base lies entirely on that side. That is, C is equidistant to all the sides of $\mathcal{P}$, i.e. $\mathcal{P}$ is cocircular. Sufficiency is obvious.

We now proceed to prove the second part of our result, i.e. that all perfect partitionings of $\mathcal{P}$ are radial. Let C be the center of $\mathcal{P}$, and Θ a perfect partitioning of $\mathcal{P}$ into k convex pieces $C_1, \ldots, C_k$, $k \geq 3$. Let C_j be any element of Θ that contains C in its interior or boundary.

Suppose that C_j contains several disjoint arcs $A_1, \ldots, A_m$ of the boundary of $\mathcal{P}$. Since Θ is a perfect partitioning of $\mathcal{P}$, the sum of the lengths of $A_1, \ldots, A_m$ is $\frac{1}{k}$ of the perimeter of $\mathcal{P}$. Let D_i be the set bounded by A_i and by the line segments joining the endpoints of A_i to C, $i = 1 \ldots, m$. Since C is equidistant from all the sides of $\mathcal{P}$, it follows that the area of $D_1 \cup \ldots \cup D_m$ is $\frac{1}{k}$ of the area of $\mathcal{P}$, and thus $C_j = D_1 \cup \ldots \cup D_m$. However since C_j is convex, m must be equal to 1, i.e. the intersection of the boundary of $\mathcal{P}$ with the boundary of C_j is connected. Let S_j denote the arc of the boundary of $\mathcal{P}$ contained in C_j, and let P_j and P_{j+1} be the endpoints of S_j. It follows that the boundary of C_j is S_j, together with the line segments joining P_j and P_{j+1} to C. We now prove that the set S' obtained by joining all the elements C_j of Θ that contain C in their boundary covers $\mathcal{P}$. Suppose then that S' does not cover all of $\mathcal{P}$. Let S'' be one of the components of $S - S'$. Since Θ partitions $\mathcal{P}$, it induces a partitioning P'' of S''. Since C belongs to the boundary of S'', it also belongs to the boundary of one of the elements of P'', which is a contradiction. Hence Θ is radial. □

3 Radial perfect 3-partitionings of convex sets

It is easy to see, using the Ham-Sandwich Theorem ([2] p.212), that any convex set can be partitioned into two convex subsets, each with equal area and perimeter. In this section we prove that every convex set has a perfect 3-partitioning. Some terminology will be needed in the rest of this paper. Given two points A and B on the plane, $|AB|$ will denote the distance from A to B, and AB the line segment joining them. A triangle with vertices A, B and C will be denoted by $\Delta(ABC)$. The internal angles of $\Delta(ABC)$ at vertices A, B, and C will be denoted by $\angle CAB$, $\angle ABC$, and $\angle BCA$ respectively. The area of a set S will be denoted by $\mathcal{A}(\mathcal{S})$. Given a convex set S, and an arc S_i of its boundary with endpoints A and B, the *lune* $\mathcal{L}(S_i)$ is the convex set bounded by S_i and the line segment AB. Our objective in this section is to prove the following result:

Theorem 2. *Any convex set S admits a perfect radial partitioning into three sets.*

We will need to prove some preliminary results before we prove Theorem 2.

Lemma 1. *If we can partition the boundary of S into three arcs S_1, S_2 and S_3 of equal length such that $\mathcal{A}(\mathcal{L}(S_1))$, $\mathcal{A}(\mathcal{L}(S_2))$, and $\mathcal{A}(\mathcal{L}(S_3))$ are at most $\frac{\mathcal{A}(\mathcal{S})}{3}$, then Theorem 2 holds.*

Proof. Suppose that there are three arcs S_1, S_2, and S_3 that satisfy the conditions of our lemma, and let A, B be the endpoints of S_1; B and C the endpoints of S_2; and C and A the endpoints of S_3. Let us assume that the area of S_1 is $\frac{\mathcal{A}(\mathcal{S})}{3} - x$. Then if we take any point on the line $\mathcal{L}_1$ parallel to AB, and at distance $\frac{2x}{AB}$ from S_1, then for any point Y on $\mathcal{L}_1$, the area of $\Delta(ABY)$ equals x, and thus the area of the convex set bounded by S_1 and the line segments BY and YA has area $\frac{\mathcal{A}(\mathcal{S})}{3}$. Let $\mathcal{L}_2$ be defined in a similar way w.r.t. S_2. Then it is easy to see that $\mathcal{L}_1$ and $\mathcal{L}_2$ intersect at a point X in the interior of S. It follows that the radial partitioning of S obtained by cutting along the line segments joining X to A, B, and C is a perfect radial partitioning. □

Next we prove:

Lemma 2. *Consider two triangles $\Delta(ABC)$, and $\Delta(BCX)$ such that:*

i) $\angle BCA \leq \angle BCX$ *and* $\angle ABC \leq \angle BCX$
ii) $|AB| + |AC| = |XB| + |XC|$.

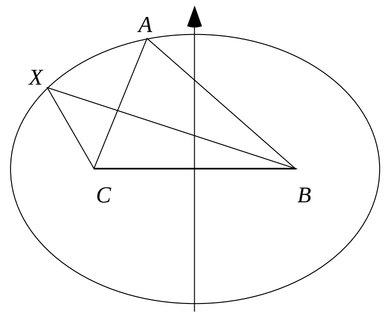

Fig. 3.

Then $\mathcal{A}(\Delta(ABC)) \geq \mathcal{A}(\Delta(BCX))$

Proof. Consider the ellipse $\mathcal{E}$ with *foci* A and B, such that for any point Y in $\mathcal{E}$, $|YA|+|YB| = |XB|+|XC|$. Assume further that the line segment BC lies on the x-axis, and that the origin is its mid-point. Suppose without loss of generality that $\angle ABC \leq \angle BCA$. Since $\angle ABC \leq \angle BCX$ it follows that the distance h_1 from A to x-axis is greater than the distance h_2 from X to the x-axis. And since $\mathcal{A}(\Delta(ABC))=\frac{h_1|BC|}{2}$, and $\mathcal{A}(\Delta(BCX))=\frac{h_2|BC|}{2}$ our result follows see Figure 3. □

We now prove:

Lemma 3. *Let triangle* $\Delta(ABC)$ *be such that* $\angle BCA \geq \angle ABC$. *Let* X *and* D *be points on* CA, *and* E *a point on* AB *that satisfy:*

$$|BE| + |ED| + |DC| = |XB| + |XC|.$$

Then the area $\mathcal{A}(\mathcal{C}(BCDE))$ *of the quadrilateral* $\mathcal{C}(BCDE)$ *with vertices* B, C, D, E *is greater than* $\mathcal{A}(\Delta(XBC))$.

Proof. Let $\alpha = \angle BCA$, $\gamma = \angle BDA$, $\theta = \angle BDE$, $\beta_1 = \angle ABD$, and $\beta_2 = \angle DBC$. Since $\angle DAB + \beta_1 + \gamma = \pi = \angle DAB + \alpha + \beta_1 + \beta_2$, it follows that $\gamma = \alpha + \beta_2$; see Figure 4.

Since $\gamma = \alpha + \beta_2 > \alpha > \beta > \beta_1$, and $\alpha > \theta$, and $|BE|+|ED| = |BX|+|XD|$, it follows from our previous lemma that the area of $\Delta(BDE)$ is greater than that of $\Delta(XBD)$. □

We now prove the following result:

Lemma 4. *Consider a triangle* $\Delta(ABC)$ *such that* $\angle BCA \geq \angle ABC$, *and let* Q *be a convex polygon with vertices* $Q_1 = C, Q_2, \ldots, Q_{n-1}, Q_n = B$ *contained in* $\Delta(ABC)$ *such that* $Q_2 \in CA$, $Q_{n-1} \in AB$, $n \geq 3$. *Let* $X \in CA$ *be such that* $|CX| + |XB| = |Q_1Q_2| + \ldots |Q_{n-1}Q_n|$. *Then the area of* Q *is greater than the area of the triangle* $\Delta(XBC)$.

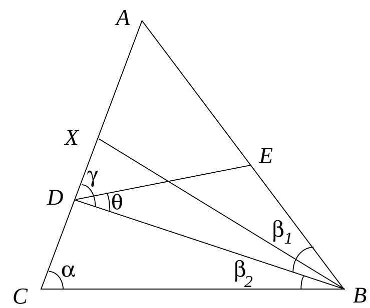

Fig. 4.

Proof. Our proof proceeds by induction on the number of vertices of Q. For $n = 4$, our problem reduces to Lemma 3. Suppose then that our result is true for polygons with $n-1$ vertices, and let Q be a polygon with n vertices $\{Q_1 = C, Q_2, \ldots, Q_{n-1}, Q_n = B\}$, $n \geq 5$. Consider now the quadrilateral with vertices $\{Q_{n-3}, Q_{n-2}, Q_{n-1}, Q_n\}$ and the triangle with vertices $\{Z,\ Q_n,\ Q_{n-3}\}$, where Z is the point of intersection of the lines containing $Q_{n-3}Q_{n-2}$ and $Q_{n-1}Q_n$; see Figure 5. By Lemma 3 there is a point Y either on $Q_{n-3}Z$ or on ZQ_n, (depending on which of $\angle Q'Q_nQ_{n-3}$ and $\angle Q_nQ_{n-3}Q'$ is smaller) such that $|Q_{n-3}Y| + |YQ_n| = |Q_{n-3}Q_{n-2}| + |Q_{n-2}Q_{n-1}| + |Q_{n-1}Q_n|$, and the area of the quadrilateral $\mathcal{C}(Q_{n-3}\ Q_{n-2}\ Q_{n-1}\ Q_n)$ is greater than the area of triangle $\Delta(Q_nQ_{n-3}Y)$.

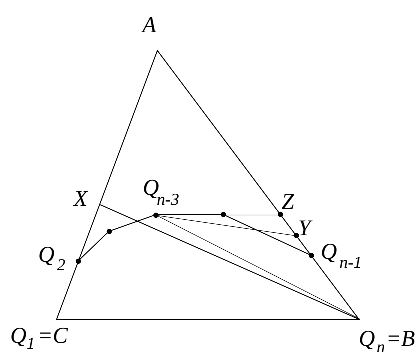

Fig. 5.

Two cases arise:

Suppose first that $Y \in ZB$. By the previous paragraph, the area of Q is greater than the area of the polygon Q'' with vertices $Q_1, \ldots, Q_{n-3}, Y, Q_n$. Moreover Q'' has the same perimeter as Q. By induction there is a point $X \in CA$ such that $|CX| + |XB| = |Q_1Q_2| + \ldots + |Q_{n-3}Y| + |YQ_n|$ and the area of $\Delta(XBC)$ is smaller than the area of Q'', which is smaller than the area of Q. A similar analysis is done when $Y \in Q_{n-3}Z$. □

We now prove:

Lemma 5. *Let $\Delta(ABC)$ be such that $\angle BCA \geq \angle ABC$, and let ψ be a convex curve contained in $\Delta(ABC)$ joining C to B. Let Y be the point on CA such that $|CY| + |YB|$ equals the length of ψ. Then the area of the convex set $\mathcal{Q}$ bounded by ψ and the line segment BC is greater than $\mathcal{A}(\Delta(ABC))$.*

Proof. Let $Q_1 = C, Q_2, \ldots, Q_{n-1}, Q_n = B$ be n equidistant points on ψ, and R and S be points on CA and AB such that the lines through R and Q_1, and Q_{n-1} and S are tangent to ψ. By the previous lemma, there is a point X_n on CA such that $|CX_n| + |X_nB|$ equals $|Q_1R| + |RQ_1| + \ldots + |SQ_n|$, and the area of $\Delta(CX_nB)$ is smaller than that of the polygon $\mathcal{Q}_n$ with vertices $Q_1 = C,\ R,\ Q_2, \ldots, Q_{n-1},\ S,\ Q_n = B$. As n increases, $\mathcal{Q}_n$ converges to $\mathcal{Q}$, and X_n converges to a point $X \in CA$. □

We now prove our main lemma in this section, namely:

Lemma 6. *Let S be a convex set, and let S_1, S_2, and S_3 be a partitioning of the boundary of S into three arcs of equal length. Then at most one of $\mathcal{L}(S_1)$, $\mathcal{L}(S_2)$, and $\mathcal{L}(S_3)$ has area greater than or equal to $\frac{\mathcal{A}(\mathcal{S})}{3}$.*

Proof. Suppose that the endpoints of $\mathcal{L}(S_1)$, $\mathcal{L}(S_2)$, and $\mathcal{L}(S_3)$ are A and B, B and C, and C and A respectively. Let l_A, l_B, and l_C be tangent lines to S at A, B, and C respectively. Suppose that $\mathcal{L}(S_1)$, $\mathcal{L}(S_2)$ have area greater than $\frac{\mathcal{A}(\mathcal{S})}{3}$. Two cases arise:

i) In the first case, l_A, l_B and l_C determine a triangle that contains S. Let D be the intersection point of l_A and l_C, E the intersection point of l_A and l_B, and X the point on DA such that $|CX| + |XA| = t$, where t is the length of S_1, and finally, let F be the point at which l_B intersects the line through X and C. See Figure 7(a). Observe now that the area of $\mathcal{L}(S_1)$ is less than $\frac{th_1}{2}$, and that the area of $\mathcal{L}(S_2)$ is also less than $\frac{th_2}{2}$, where h_1 and h_2 are the distances from B to l_A and l_B respectively. Let $t_1 = |XA|$, and $t_2 = |CX|$. Then by definition, $t_1 + t_2 = t$. Notice now that the sum of the areas of triangles $\Delta(DAC)$ and $\Delta(ABC)$ equals $\frac{t_1h_1+t_2h_2}{2}$ which is less than the areas of both $\mathcal{L}(S_1)$and $\mathcal{L}(S_2)$. We may assume that $h_1 \leq h_2$. Thus we obtain that

$$\frac{th_1}{2} = \frac{(t_1 + t_2)h_1}{2} \leq \frac{t_1h_1}{2} + \frac{t_2h_2}{2}$$

That is, the area of $\mathcal{L}(S_1)$ is less than the sum of the areas of $\Delta(DAX)$ and $\Delta(ABC)$, which is a contradiction.

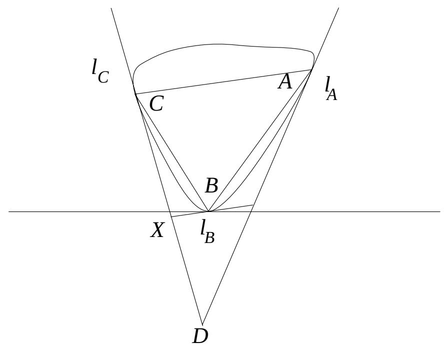

Fig. 6.

ii) Two subcases arise when the triangle determined by l_A, l_B and l_C does not contain S.
   (a) The intersection point of l_A, l_C (call it D), together with A and C, determine a triangle $\Delta(ADC)$ that contains B in its interior; see Figure 6. Suppose that l_B is horizontal, and that the line through A and C intersects it to the left of B as shown in Figure 6. Consider the line parallel to AC through B, and let X be the point at which this line intersects CD. Observe that $|XB| < |AC|$, and thus the area of triangle $\Delta(CXB)$ is smaller than the area of $\Delta(ABC)$. However $\mathcal{L}(S_2) \subset \Delta(CEB) \subset \Delta(CXB)$ which is a contradiction, as the area of $\Delta(ABC)$ is less than $\frac{\mathcal{A}(\mathcal{S})}{3}$.
   (b) A similar argument solves the remaining cases when the intersection point D' of l_A and l_B (respectively l_B and l_C) determines a triangle $\Delta(D'AB)$ (resp. $\Delta(D'BC)$) that contains C (resp. A) in its interior. □

Lemma 7. *There are three points A, B, and C on the boundary of S which divide S into three sectors S_1, S_2, and S_3 of equal length such that the areas of $\mathcal{L}(S_1)$, $\mathcal{L}(S_2)$, and $\mathcal{L}(S_3)$ are are smaller than one third of the area of S.*

Proof. Choose A, B, and C on the boundary of S, and assume by Lemma 6 that $\mathcal{A}(\mathcal{L}(S_2)) \geq \frac{\mathcal{A}(\mathcal{S})}{3}$, $\mathcal{A}(\mathcal{L}(S_2)) < \frac{\mathcal{A}(\mathcal{S})}{3}$, and $\mathcal{A}(\mathcal{L}(S_3)) < \frac{\mathcal{A}(\mathcal{S})}{3}$. Simultaneously rotate A, B and C counter-clockwise along the boundary of S, keeping the lengths of S_1, S_2, and S_3 equal, until we reach the first position in which either $\mathcal{A}(\mathcal{L}(S_3)) = \frac{\mathcal{A}(\mathcal{S})}{3}$ or $\mathcal{A}(\mathcal{L}(S_2)) = \frac{\mathcal{A}(\mathcal{S})}{3}$. Suppose that the second case arises. This must happen before B reaches the original position of A. At this point, we know by Lemma 6 that $\mathcal{A}(\mathcal{L}(S_1)) < \frac{\mathcal{A}(\mathcal{S})}{3}$, and $\mathcal{A}(\mathcal{L}(S_3)) < \frac{\mathcal{A}(\mathcal{S})}{3}$. It now follows that if we rotate A, B, and C in the clockwise direction by a sufficiently small amount, we reach a final position for A, B and C in which $\mathcal{A}(\mathcal{L}(S_2)) < \frac{\mathcal{A}(\mathcal{S})}{3}$, $\mathcal{A}(\mathcal{L}(S_2)) < \frac{\mathcal{A}(\mathcal{S})}{3}$, and $\mathcal{A}(\mathcal{L}(S_3)) < \frac{\mathcal{A}(\mathcal{S})}{3}$. The case in which $\mathcal{A}(\mathcal{L}(S_3))$ reaches the value $\frac{\mathcal{A}(\mathcal{S})}{3}$ first is solved in a similar way. □

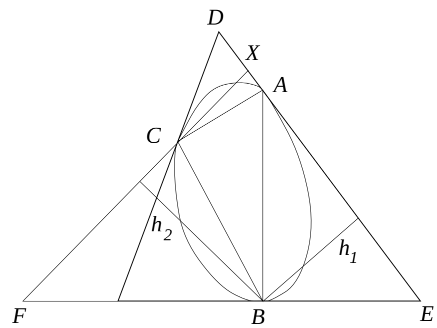

Fig. 7.

Using Lemma 1 and Lemma 7, Theorem 2 follows. □

3.1 Perfect 4-partitionings of convex sets

We now show that that we cannot extend Theorem 2 to radial perfect partitionings with four or more convex subsets.

Theorem 3. *Let a and b be positive real numbers such that $a > 4b$, and let $n \geq 5$ be an integer. Then every $a \times b$ rectangle R cannot be radially perfectly partitioned into five or more convex subsets. Moreover there are convex quadrilaterals that admit no perfect radial four partititonings.*

Proof. Let V_1, V_2, V_3, V_4 be the vertices of R such that $|V_1V_2| = |V_3V_4| = a$ and $|V_2V_3| = |V_4V_1| = b$.

Suppose that R can be radially perfectly partitioned into n convex subsets by n line segments $CX_1, CX_2, \ldots, CX_n$, where C is a point in R and the X_i's are points on the boundary of R. Since $n \geq 5$ and $a > 4b$, both of the arc V_1V_2 and the arc V_3V_4 contain at least two points X_i and X_{i+1}. Thus C must lie on the line passing through the midpoints of V_2V_3 and V_1V_4, respectively. Hence the area of the triangle with vertices X_i, X_{i+1}, and C equals:

$$\frac{1}{2}\frac{(2a+2b)}{n} \times \frac{b}{2} < \frac{ab}{n} = \frac{\mathcal{A}(\mathcal{R})}{n}.$$

This is a contradiction, and the first part of our result is proved.

To prove the second part of our result, let us consider the quadrilateral Q with vertices P_1, P_2, P_3, P_4 such that $|P_1P_2| = |P_1P_4| = 40$, $|P_2P_3| = 4$, $|P_3P_4| = 2$ and $|P_2P_4| = 5$; see Figure 8. Suppose that Q can be radially perfectly partitioned into four convex subsets by line segments DY_1, DY_2, DY_3, DY_4, where D is a point of Q and Y_i is a point on the boundary of Q, $i = 1, \ldots, 4$. Since at least three elements of $\{Y_i; i = 1, \ldots 4\}$, say Y_1, Y_2 and Y_3, lie on $P_1P_2 \cup P_1P_4$, we can easily show that D must lie on the bisector of the angle of Q at P_1.

We may assume that $Y_1, Y_2 \in P_1P_4$ and $Y_3 \in P_1P_2$. If $Y_4 \in P_2P_3$, then since the quadrilateral whose vertices are $\{D, Y_3, P_2, Y_4\}$ is the union of $\Delta(DY_3P_2)$ and $\Delta(DP_2Y_4)$, it follows that D must be lie on the bisector of the angle of Q at P_2. Then since the height of $\Delta(DP_3P_4)$ with base P_3P_4 is greater than that of $\Delta(DP_4P_1)$ with base P_4P_1, it follows that the area of the pentagon with vertex set $\{D,\ Y_4,\ P_3,\ P_4,\ Y_1\}$ is greater than that of the quadrilateral with vertex set $\{D,\ Y_3,\ P_2,\ Y_4\}$, a contradiction. If $Y_4 \in P_3P_4$, then we get a contradiction as above. Hence we may assume that $Y_4 \in P_1P_2$. Since the pentagon defined by D, Y_4, P_2, P_3, P_4, and Y_1 is a convex set, we can prove that D must be at distance at least 7 from the lines containing P_2P_3 and P_3P_4, and thus the area of the sector containing P_2P_3 and P_3P_4 on its boundary is bigger than that of the remaining sectors, which is a contradiction. □

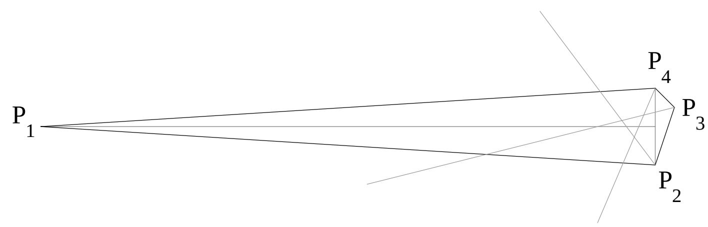

Fig. 8. A quadrilateral that has no perfect radial partitioning into four pieces.

We now give a weaker but interesting generalization of Theorem 2. A convex set $\mathcal{S}$ is called *normal* if for every arc S_i of its boundary with length equal to one quarter of the length of the perimeter of $\mathcal{S}$, $\mathcal{A}(\mathcal{L}(S_i)) \leq \frac{\mathcal{A}(\mathcal{S})}{4}$

Theorem 4. *Let S be a normal convex set. Then S admits a perfect radial partitioning into four convex subsets.*

Proof. Let P_1, P_2, P_3, P_4 be four points on the boundary of S that divide its boundary into four arcs $S_1, \ldots, S_4$ of equal length. Assume without loss of generality that S_i has endpoints P_1 and P_{i+1}, $i = 1, \ldots, 4$, where $P_5 = P_1$. Given a point Q in the interior of $\mathcal{S}$, let $Wed(S_i, Q)$ be the subset of $\mathcal{S}$ bounded by S_i and the line segments joining the endpoints of S_i to Q, $i = 1, \ldots, 4$.

Clearly there is a unique point Y in the interior of S such that:

$$\mathcal{A}(Wed(S_1, Y)) = \mathcal{A}((Wed(S_2), Y) = \frac{\mathcal{A}(\mathcal{S})}{4}$$

See Figure 9.

Since $\mathcal{S}$ is normal, Y belongs to the quadrilateral with vertices $P_1, \ldots, P_4$. It is clear that the point Y is uniquely determined by (P_1, P_2, P_3, P_4), and that as P_1, P_2, P_3, and P_4 move continuously on the boundary of S splitting its boundary into equal length segments, Y moves continuously within S.

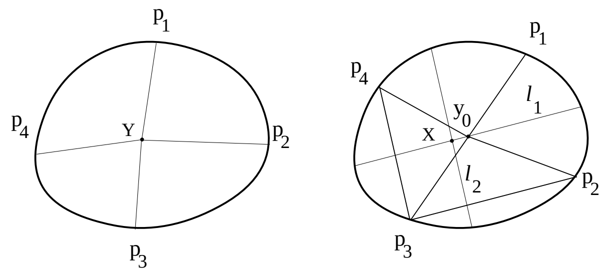

Fig. 9. Finding a perfect radial partitioning of a normal convex set.

By the Ham-Sandwich Theorem on the plane ([2] p.212), we can choose initial positions of P_1 and P_3 on $\partial(S)$ such that P_1P_3 bisects both the area and the boundary of S. For this choice, let Y_0 be the initial position of Y. Clearly Y_0 lies on the line segment joining P_1 to P_3.

Assume without loss of generality that $\mathcal{A}(Wed(S_3)) \geq \mathcal{A}(Wed(S_4))$. Consider the line L_1 parallel to the segment joining P_3 to P_4 such that the area of any wedge $Wed(S_4, Q)$ equals $\frac{\mathcal{A}(\mathcal{S})}{4}$; $Q \in L_1$.

Similarly, consider the line L_2 parallel to the line segment joining P_2 to P_3, such that the area of any wedge $Wed(S_2, Q)$ equals $\frac{\mathcal{A}(\mathcal{S})}{4}$; $Q \in L_2$, and let X be the point of intersection of L_1 and L_2. Observe that $Wed(S_1, Y_0) \subset Wed(S_1, X)$, and thus $\mathcal{A}(Wed(S_1, X)) \geq \frac{\mathcal{A}(\mathcal{S})}{4}$.

Notice that when we slide $P_1, \ldots, P_4$ continuously in the clockwise direction until P_1 reaches the original position of P_2, Y moves from Y_0 to X, and $Wed(S_4, y_0)$ becomes $Wed(S_1, X)$. Then its area moves from an initial value smaller than or equal to $\frac{\mathcal{A}(\mathcal{S})}{4}$ to a final value greater than or equal to the same value. Thus at some point in time equality holds, and our theorem is proved. □

4 Perfect partitionings of convex sets

In this section we prove the following result:

Theorem 5. *For every k, any convex set S has a perfect k-partitioning.*

To prove our result, we will prove the following:

Theorem 6. *Let S be a convex set such that its boundary is divided into an even number of alternately-coloured arcs, say red and blue. Then for every k there is a convex partitioning of S such that each piece has $\frac{1}{k}$ of the red boundary of S.*

Proof. The result is true for $k = 1$. Notice that for $k = 2$ it follows directly from the ham-sandwich theorem [2]. Suppose then that the result is true for

$0 < k' < k$. We show that it also holds for k. The case when $k = 2m$ can be solved as follows: by the ham-sandwich theorem, there is a line segment l that splits S into two pieces, C_1 and C_2, each with half the area of S, and half of its red boundary. Color l blue to produce two convex sets, each with half of the red boundary of S. By induction, both C_1 and C_2 can be partitioned into m convex sets, each with $\frac{1}{m}$ of the red perimeter of C_1 and C_2, i.e. $\frac{1}{k}$ of the boundary of S.

Now suppose that $k = 2m+1$. Choose k points $P_1, \ldots, P_k$ in clockwise order along the boundary of S that divide it into k sectors $S_1, \ldots, S_k$ (where the endpoints of S_i are the points P_i and P_{i+1}, with $P_{n+1} = P_1$) each containing $\frac{1}{k}$ of the red boundary of S.

If for some i the area of the lune $\mathcal{L}(S_i)$ equals $\frac{\mathcal{A}(\mathcal{C})}{k}$, then by cutting $\mathcal{L}(S_i)$ away from S and coloring the line segment used in this cut blue, we reduce the problem to that of cutting $C - \mathcal{L}(S_i)$ into $k-1$ pieces, and our result follows.

Suppose now that the area of at least one lune $\mathcal{L}(S_i)$ of S is strictly greater than $\frac{A(C)}{k}$. Notice that there is at least one $\mathcal{L}(S_j)$ such that its area is strictly less than $\frac{A(C)}{k}$, $i \neq j$. It now follows that if we continuously rotate P_j and P_{j+1} in the clockwise direction along the boundary of S in such a way that the red sector of the boundary of S contained in S_i is $\frac{1}{k}$ of the red perimeter of C at all times, then at some point before P_j reaches P_i, the area of $\mathcal{L}(S_j)$ will equal $\frac{A(C)}{k}$, and our result follows.

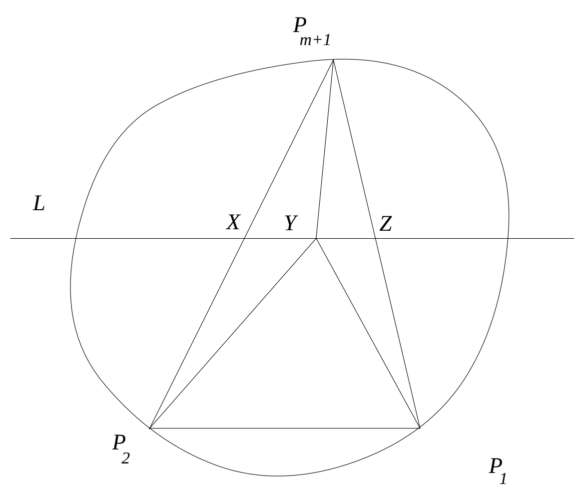

Fig. 10. $A(B_1) > \frac{A(C)}{k}$

Thus we can assume that the areas of all the lunes of S are smaller than $\frac{\mathcal{A}(\mathcal{S})}{k}$. Given two points P_i, P_{i+m}, we define $S_{i,m}$ to be the sector of the boundary of S

connecting P_i to P_{i+m}, and containing $P_{i+1}, \ldots, P_{i+m-1}$ addition taken mod n. Using similar arguments, we can prove that for every i the area of the lune $\mathcal{L}(S_{i,m})$ is smaller than $\frac{m\mathcal{A}(\mathcal{S})}{k}$.

Let B_i be the set bounded by S_i and the line segments joining P_i and P_{i+1} to the point P_{i+m}, $i = 1, \ldots, k$. Notice that the union of all B_i's covers S, and thus the area of one of them, say B_1, is strictly greater than $\frac{\mathcal{A}(\mathcal{S})}{k}$; see Figure 10.

Let L be the line parallel to P_1P_2 such that the area of any triangle with base P_iP_{i+1} and having its third point on L has area $\frac{\mathcal{A}(\mathcal{S})}{k} - \mathcal{A}(\mathcal{L}(S_1)$. By the assumption that the area of B_1 is greater than $\frac{\mathcal{A}(\mathcal{C})}{k}$, L intersects P_2P_{m+1} and P_1P_{m+1}. Call the points of intersection of L with P_2P_{m+1} and P_1P_{m+1} X and Y respectively.

Then there is a point Z on L between X and Y such that the rays connecting Z to P_1, P_2 and P_{m+1} divide S into three sectors; the one bounded by the segments connecting Z with P_1 and P_2 and $\mathcal{L}(S_1)$ having area $\frac{\mathcal{A}(\mathcal{S})}{k}$–***???, and the others having area $\frac{m\mathcal{A}(\mathcal{S})}{k}$–***???; each having $\frac{m}{k}$ of the red boundary of S. Color the line segments connecting Z with P_1, P_2 and P_{m+1} blue. Our result now follows by induction on k. □

We now observe that Theorem 5 follows directly from Theorem 6 by coloring the entire boundary of C red. As in the proof of Theorem 6, the cuts used to partition S will be colored blue. □

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